Advanced Linear Algebra

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Lecture – 48
Linear Functionals and Adjoints-I

Welcome to the lecture series on Advanced Linear Algebra. Today, we are going to talk about topics it is called Linear Functionals and Adjoints. In this topic, we are going to say if a vector space which is also equipped with an inner product. Then we want to show for each linear functionals defined on the finite dimensional inner product space. There exists a fixed element in V where the linear functional is inner product of the fixed vector β

And using this results will show suppose a T be the linear operator defined on the inner product is V which is also finite dimensionals will show there exists adjoint T. I mean which is another operator on the V which satisfied certain conditions like for any $\alpha, \beta \in$ V, $<T\alpha, \beta>$ = $<\alpha, T^*\beta>$

And next will be what will the matrix representations of the operator T & T^* with respect to an order basis which is also orthogonal order basis for the vector space V. So, under this section we are going to answer these three important topics. So, let me first show that for a finite dimensional inner product space, if we consider any linear functional on V, this linear functional is basically inner product of some fixed vector in the vector space V.

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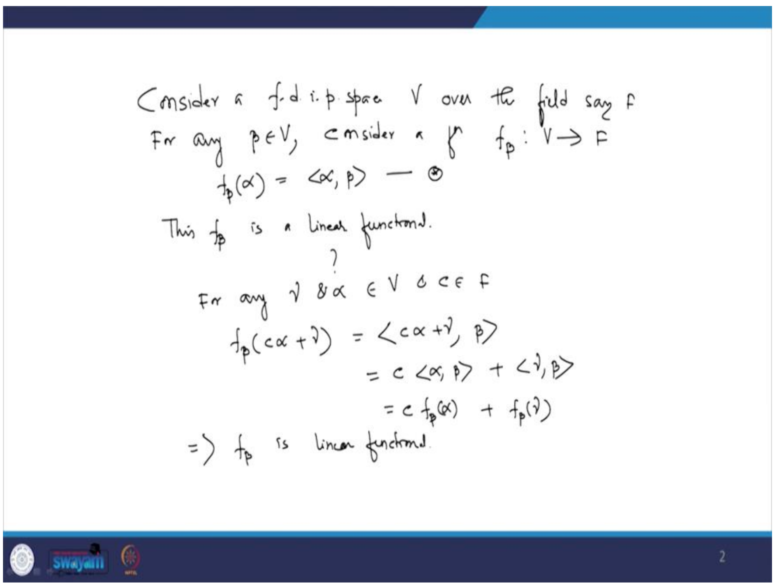

So, before answering this, let me consider a finite dimensional inner product space V over the field F. So, this field is maybe real number or if it may be complex number for any $\beta \in$ V consider a function, $f_\beta: V \rightarrow F$ defined by $f_\beta(\alpha) = <\alpha, \beta> \rightarrow (*)$

Then we can easily show that this beta is a linear functional. So, this can be checked quickly for any γ & $\alpha \in$ V and c $\in$ F. We have, $f_\beta(c\alpha + \gamma) = <c\alpha + \gamma, \beta> = c <\alpha, \beta> + <\gamma, \beta>$ $= cf_\beta(\alpha) + f_\beta(\gamma)$ according to star and definition of the inner product.

So, this implies f_β is a linear functional.

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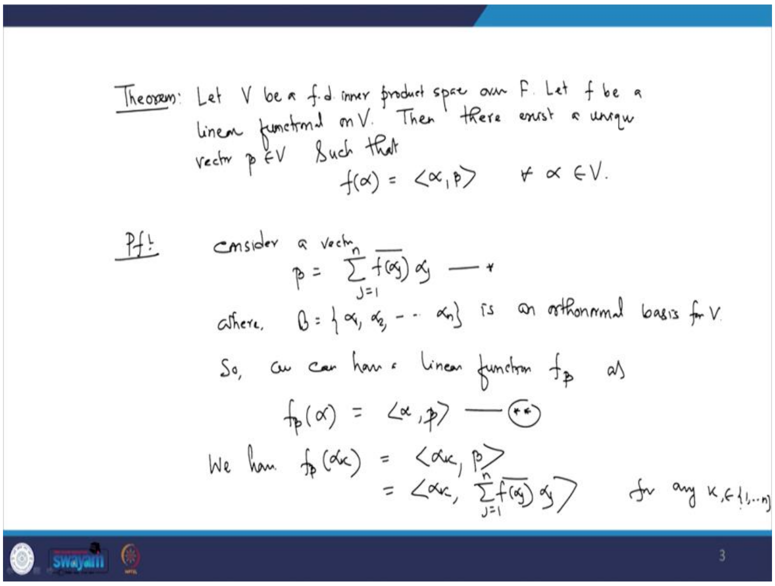

Now, using this let me introduce one nice result in terms of theorems it is like this. Let V be a finite dimensional inner product space over the field is F. Let f be a linear functional on V. Then exist a unique vector $\beta \in$ V such that f$(\alpha) = <\alpha, \beta> \forall \alpha \in V$. Proof: - So, I will prove that over a finite dimensional inner product space, any linear functionals is basically inner product of some fixed vectors in the vector space.

I have to first find out the, what is the element β when in linear functional f is given to us? Consider a vector $\beta = \sum_{j=1}^{n} \overline{f(\alpha_j)}\, \alpha_j$ where B = $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is an orthonormal basis for V. So, for a given orthonormal basis, one can define a vector β for a given function of, f like this. And the question is how to think that for a given orthonormal basis or given linear functionals?

One has to consider the element $\beta = \sum_{j=1}^{n} \overline{f(\alpha_j)}\, \alpha_j$ or what is the geometrical interpretation of β? All this we are going to answer shortly. But I can define a vector β in terms of like this when as an orthogonal ordered basis may consider V is given to us. Suppose if I choose like this then I can define a vector like (*). So, under this so, we can have a linear functionals, $f_\beta(\alpha)$ as $f_\beta(\alpha) = < \alpha, \beta >$

So, I can define like this. We have, $f_\beta(\alpha_k) = < \alpha_k, \beta > = < \alpha_k, \sum_{j=1}^{n} \overline{f(\alpha_j)}\, \alpha_j >$ for any k = 1 to n. So, I have like this.

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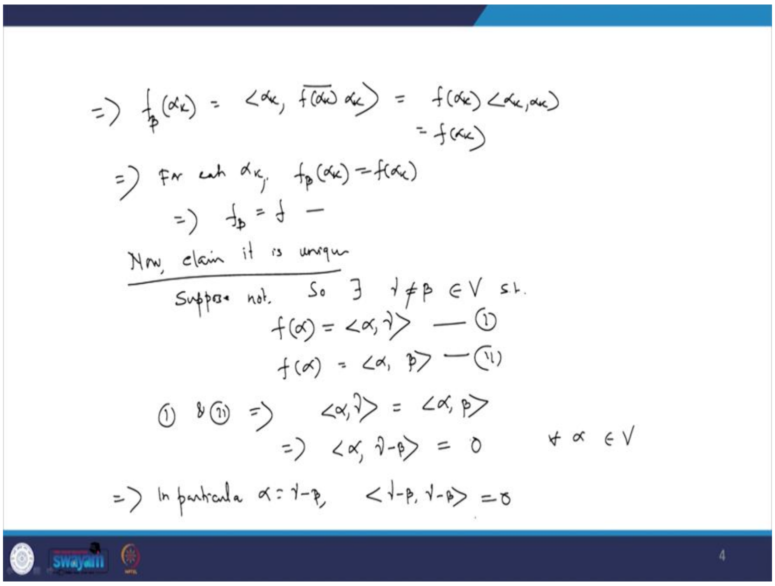

So, this implies the linear functional we have defined $\Rightarrow f_\beta(\alpha_k) = < \alpha_k, \overline{f(\alpha_k)}\, \alpha_k >$ = $f(\alpha_k) < \alpha_k, \alpha_k >$ = $f(\alpha_k)$, where all other will cancel out Because $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ this is basically orthonormal basis for the vector space V. So, therefore, $< \alpha_i, \alpha_j >$ = 0 if i ≠ j ⟹ For each α_{k_j}, $f_\beta(\alpha_k) = f(\alpha_k) \Rightarrow f_\beta = f$

So, based on, Now, claim it is unique, suppose not so, $\exists\, \gamma \neq \beta \in$ V, s.t. f(α) =< α, γ> → (i) f(α) = < α, β> → (ii), so, (i) & (ii) ⟹ < α, γ> = < α, β> ⟹ < $\alpha, \gamma - \beta$> = 0 ∀ $\alpha \in V$ ⟹ $\alpha = \gamma - \beta$, < $\gamma - \beta, \gamma - \beta$> = 0

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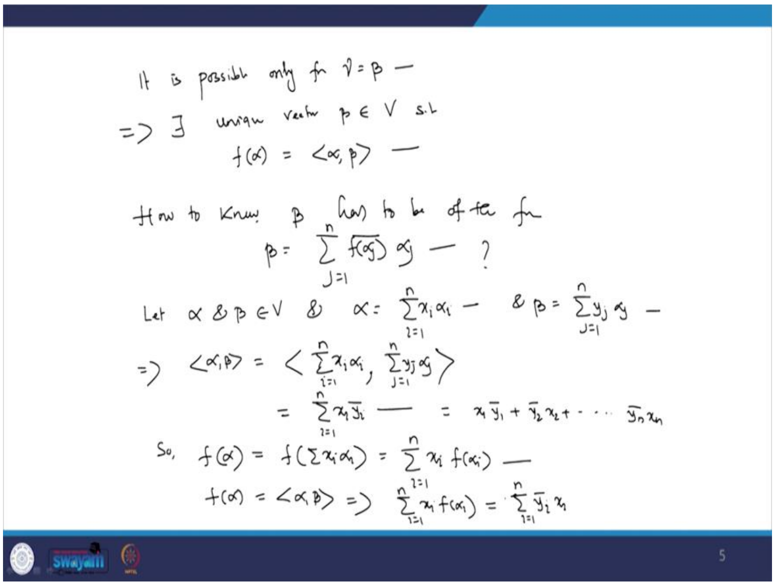

Now, this is possible only for so $\beta \Longrightarrow$ unique vector $\exists\ \beta \in$ V such that, f(α) = $< \alpha, \beta >$. Now, the question is how to think that one has to consider $\beta = \sum_{j=1}^{n} \overline{f(\alpha_j)}\, \alpha_j$? How to know? $\beta = \sum_{j=1}^{n} \overline{f(\alpha_j)}\, \alpha_j$. For a given linear function f $\exists\ \beta \in$ V such that, f(α) =$< \alpha, \beta>$ for all $\alpha \in$ V.

Let α & $\beta \in$ V & $\alpha = \sum_{i=1}^{n} x_i\, \alpha_i$ & $\beta = \sum_{j=1}^{n} y_j\, \alpha_j \implies < \alpha, \beta > = < \sum_{i=1}^{n} x_i\, \alpha_i, \sum_{j=1}^{n} y_j\, \alpha_j >$ = $\sum_{i=1}^{n} x_i\, \overline{y_i} = x_1\overline{y_1} + x_2\overline{y_2} + \ldots + x_n\overline{y_n}$. So, f($\alpha$) = f($\sum_{i=1}^{n} x_i\, \alpha_i$) = $\sum_{i=1}^{n} x_i\, f(\alpha_i)$. So, if at all there is a element $\beta \in$ V such that f(α) =$< \alpha, \beta>$ $\implies$ $\sum_{i=1}^{n} x_i\, f(\alpha_i) = \sum_{i=1}^{n} \overline{y_i} x_i$

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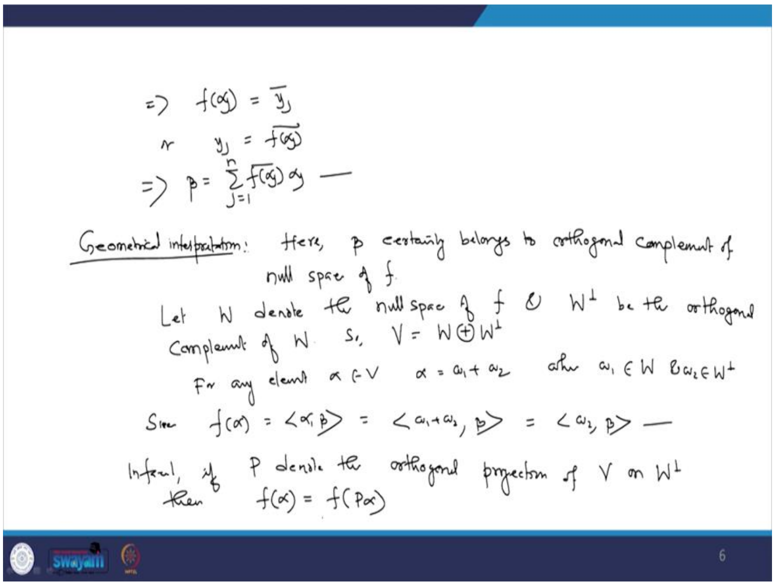

So, comparing the coefficient of the x_i, will have $\Longrightarrow$ $f(\alpha_j) = \overline{y_j}$ or $y_j = \overline{f(\alpha_j)} \Longrightarrow \beta = \sum_{j=1}^{n} \overline{f(\alpha_j)}\, \alpha_j$. So, this thing we have done without bothering about the geometrical interest of this problem. Suppose we want to know what is the geometrical interpretation of this type of concept?

So, to understand that one see here β certainly belongs to orthogonal complement of null space of f. Let me clarify this once or the meaning of that one. So, let W denote the null space of f and $W^{\perp}$ is the orthogonal complement of W. I mean to say any element $\alpha \in W$ & $\beta \in W^{\perp}$ they have to be orthogonal to each other. I want to say $< \alpha, \beta >$

So, here I can write down the vector space $V = W \oplus W^{\perp}$. for any element $\alpha \in V$, $\alpha = w_1 + w_2$ where $w_1 \in W$, $w_2 \in W^{\perp}$. Since $f(\alpha) = < \alpha, \beta > = < w_1 + w_2, \beta > = < w_2, \beta >$

and $< w_1, \beta > = 0$, so that is why it is like this thing only. So, in fact, if P denote the orthogonal projection of V on $W^{\perp}$ of the W then I can say, $f(\alpha) = f(P\alpha)$. So, what is $P\alpha$?

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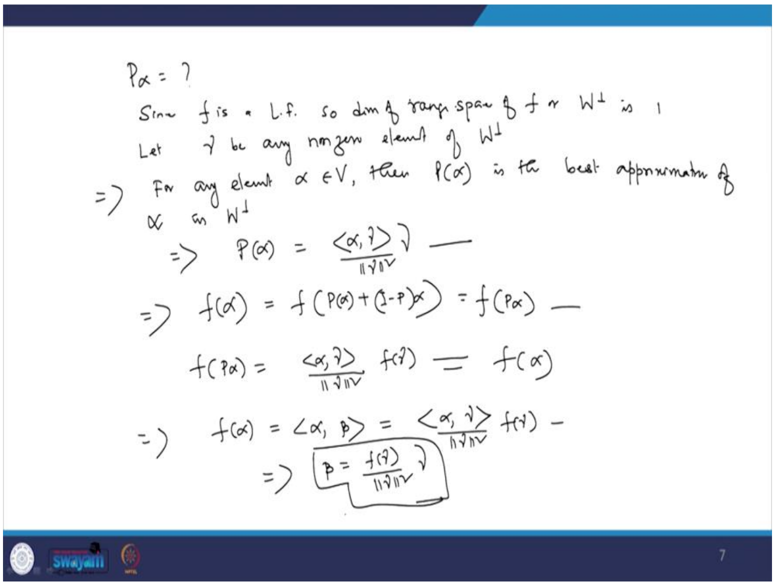

Since f is a linear functional so, dimension of range space of f or $W^{\perp}$ is 1. Let gamma be any non-zero element of $W^{\perp}$. So, I can say γ will basically basis for the $W^{\perp}$ because the dimension of the space $W^{\perp}$ is 1. So, certainly, if I take a single to non-zero element, γ will belongs to $W^{\perp}$ will add to the basis.

So, this implies for any element $\alpha \in V$, then $P(\alpha)$ will be the best approximations of α in $W^{\perp}$. So, this implies $P(\alpha) = \frac{< \alpha, \gamma >}{||\gamma||^2} \gamma \implies f(\alpha) = f(P(\alpha) + (I - P)\alpha) = f(P\alpha)$

Where (I – P)α will be in W that is null space of f orthogonal to γ. So, f(Pα) $=$ $\frac{<\alpha,\gamma>}{||\gamma||^2}$ f(γ) = f(α). So, this implies if I know what is the appropriate beta? I can also calculate from this $\Rightarrow$f(α) $= <\alpha,\beta> = \frac{<\alpha,\gamma>}{||\gamma||^2}$ f(γ) $\Rightarrow \beta = \frac{f(\gamma)}{||\gamma||^2}\gamma$

So, this is the basically geometrical interpretation of this beta element. We shall now raise a question see we have proven that existence of unique vector β in V for any linear functional f assuming that the inner product space is finite dimensions. If we take out this concept that he is finite dimensional, if I replace finite dimensional and inner product space by infinite dimensional inner product space, whether we can also have such type of relation.

I mean to say, is it possible to say that for a given linear functional on that space, there will be a unique vector in the vector space the linear functionals can be written as inner product of that space vector in the space answer is no. This we will discuss in the next class.