Advanced Linear Algebra

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Lecture – 47
Orthogonal Projection in I.P.S

So, welcome to the lecture series on Advanced Linear Algebra. See, we have already started in the inner products space, the concept of best approximations of a vector on a subspace of vector space. We have seen if we consider vector β in an inner product space V and if I consider W be the subspace of V. If the best approximation of β on W exist say that is α then we have seen that $(\alpha - \beta)$ is orthogonal to each and every element of W.

So, this means that $(\alpha - \beta) \notin$ W. So, this motivates us to introduce one more terminology. It is called orthogonal complement.

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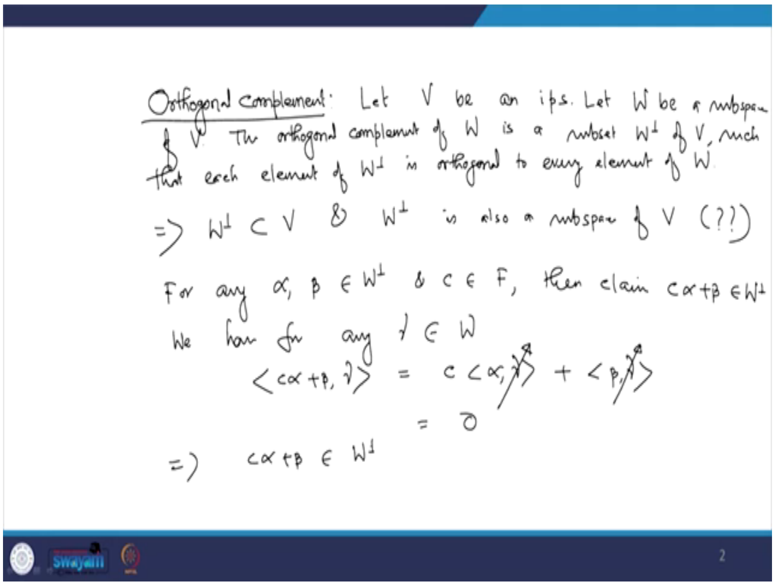

So, what is Orthogonal complement: - Let V be an inner product space. Let W be a subspace of V. The orthogonal complement of W is a subset $W^{\perp}$ of V such that each element of $W^{\perp}$ is orthogonal to every element of W $\Longrightarrow$ $W^{\perp} \subset$ V & $W^{\perp}$ is also a subspace of V.

So, $W^{\perp}$ is a subspace of V this has to be also prove it. So, for any $\alpha, \beta \in$ $W^{\perp}$ and c $\in$ F then claim $c\alpha + \beta \in$ $W^{\perp}$. So, this will be in $W^{\perp}$ provided $c\alpha + \beta \in$ is orthogonal to each element of W. We have for any $\gamma \in$ W, $< c\alpha + \beta, \gamma >$ = c $<\alpha, \gamma> + <\beta, \gamma> = 0$

So, since $\alpha, \ \beta \in \ W^{\perp}$ and $\gamma \in$ W. So, this means that this is $<\alpha,\gamma> = 0 \ \& < \beta,\gamma > \ = 0$. So, this implies $c\alpha + \beta \ \in \ W^{\perp}$. So, we see that if here the orthogonal complement of the subspace W is also in subspace of the vector space V.

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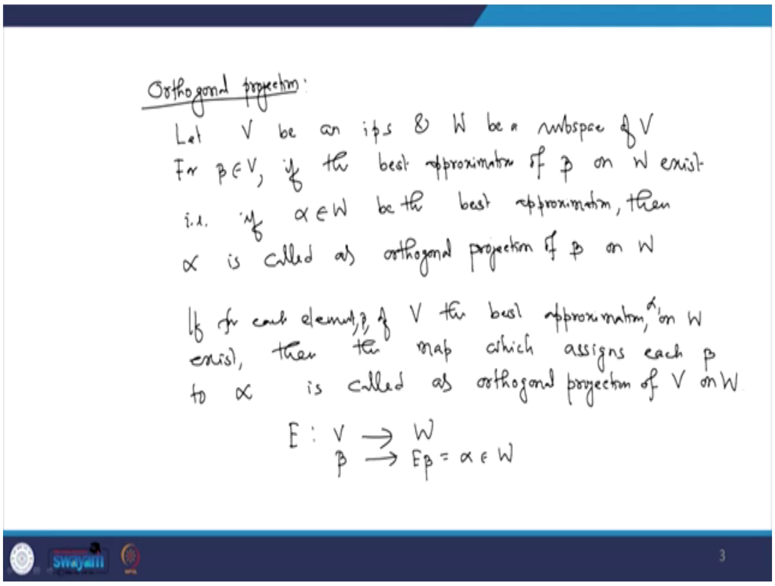

Let me introduce one more terminology this is called orthogonal projection. So, let V be an inner product space and W be a subspace of V. If for $\beta \in$ V, if the best approximation of β on W exist i.e. if $\alpha \in$ W be the best approximation, then α is called as orthogonal projection of β on W, where we know that $(\alpha - \ \beta)$ is orthogonal to each element of W.

See in our last class we have seen if W is finite dimensional then for each vector β in V the best approximation of β on W exist. And that is also unique which is having is typical form which is already known to us. Now, if I say that W is may not be finite dimensional but if for each element of V the best approximation of that element on W exist. Then it will introduce a function that is called orthogonal projection of V on W.

So, what is that? If for each element of V the best approximation on W exist. If for each element, say β of V the best approximation say α on V exist. Then the map which assign each β to α is called as orthogonal projection of V on W. So, orthogonal projection of V on W is a map is a function from V to W which assign each element of V to it is best approximation on W.

It is like this, E: $V \rightarrow \ W$ is mapping, $\ \beta \ \rightarrow \ E_{\beta} = \alpha \in$ W. So, we have orthogonal projection of a vector V on a subspace W. See the existence of best approximation will be assured if W is finite dimensional. So, in that case, if W is a finite dimensional then one can also have a orthogonal projection of V on it is orthogonal complement space also subspace also.

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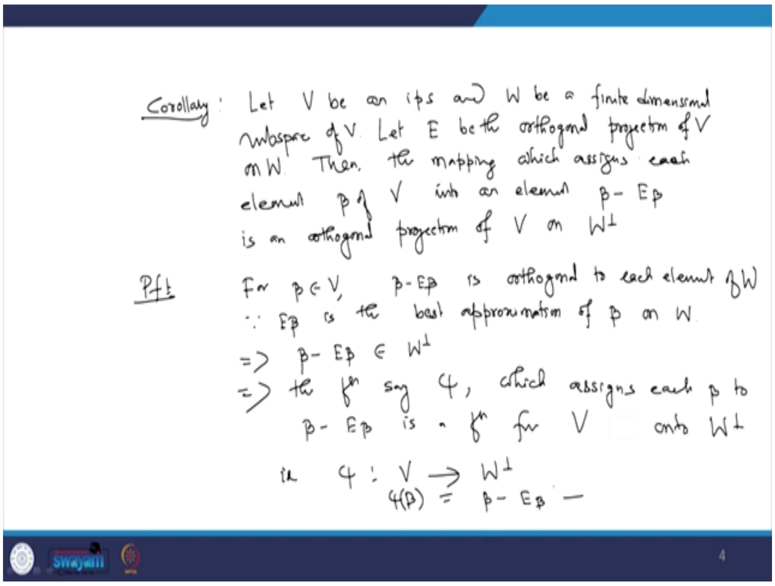

So, what I mean to say you? It is in terms of Corollary: - Let V be an inner product space and W be a finite dimensional subspace of V. Let E be the orthogonal projection of V on W. Then the mapping which assigns each element β of V into an element $\beta - E_\beta$ is a function from V onto $W^\perp$ is an orthogonal projection. Now, projection of V on $W^\perp$, I mean orthogonal complement of W.

So, this can be proved using best approximation and the concept of orthogonal projection, Proof: - For $\beta \in V$, $\beta - E_\beta$ is orthogonal to each element of W. Because since E_β is the best approximation of β or W. So, definition $\beta - E_\beta$ is orthogonal to each element of W. So $\Longrightarrow$ $\beta - E_\beta \in W^\perp \Longrightarrow$ the function say Ψ, which assign is β to $\beta - E_\beta$ is a function from V onto $W^\perp$.

i.e. $\Psi: V \rightarrow W^\perp$ defined by $\Psi(\beta) = \beta - E_\beta$. Now, claim is that this function, Ψ I mean which mapping β to $\beta - E_\beta$ is basically orthogonal projection of V on $W^\perp$.

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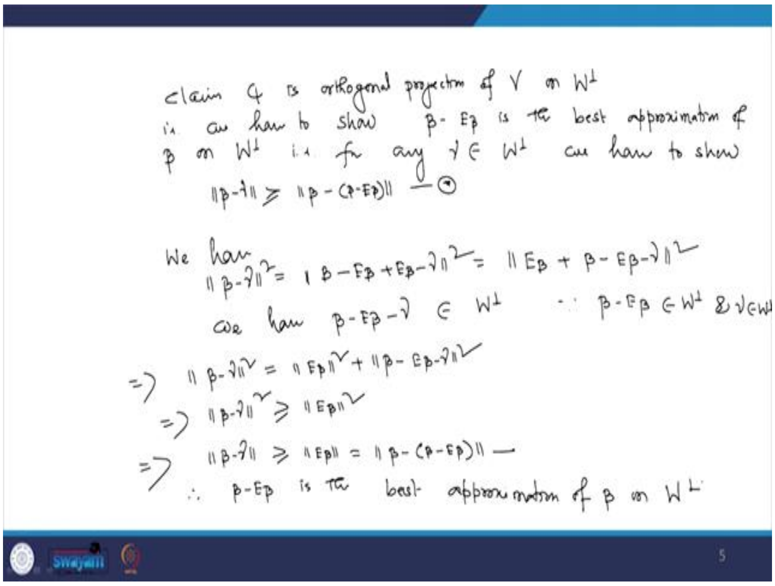

Claim Ψ is orthogonal projection of V on $W^{\perp}$. We have to show $\beta - E_{\beta}$ is the best approximation of beta on $W^{\perp}$ i.e. for any $\gamma \in W^{\perp}$. I have to show that to that, $||\beta - \gamma|| \geq ||\beta - (\beta - E_{\beta})|| \rightarrow (*)$. We have, $||\beta - \gamma||^{2} = ||\beta - E_{\beta} + E_{\beta} - \gamma||^{2} = ||E_{\beta} + \beta - E_{\beta} - \gamma||^{2}$, we have, $\beta - E_{\beta} - \gamma \in W^{\perp}$ since $\beta - E_{\beta} \in W^{\perp}$ & $\gamma \in W^{\perp}$

$\Rightarrow ||\beta - \gamma||^{2} = ||E_{\beta}||^{2} + ||\beta - E_{\beta} - \gamma||^{2} \Rightarrow ||\beta - \gamma||^{2} \geq ||E_{\beta}||^{2} \Rightarrow ||\beta - \gamma|| \geq ||E_{\beta}|| = ||\beta - (\beta - E_{\beta})||$. So this is basically our equal result. So, $(\beta - E_{\beta})$ is the best approximation of β on $W^{\perp}$. So, we see that if V be a inner product space and W is a finite dimensional subspace then one can have actually orthogonal projection of V on W as well as orthogonal projection of V on orthogonal complement.

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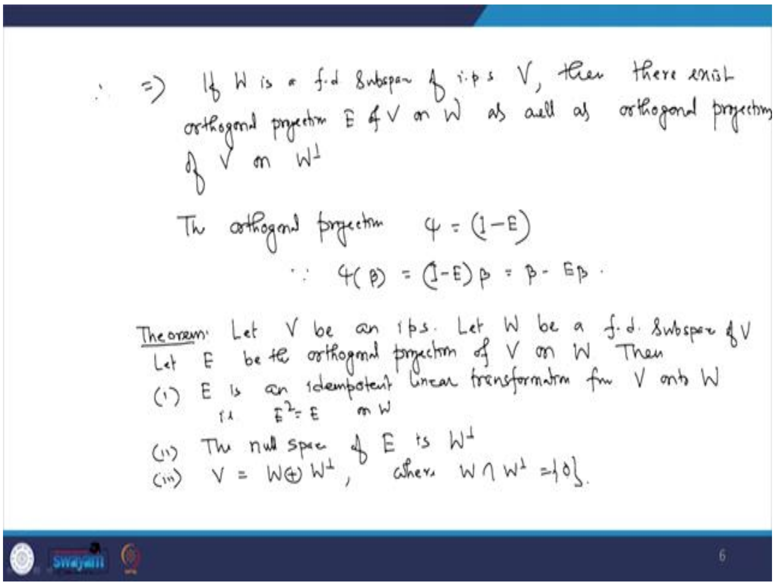

So, this implies if W is a finite dimensional subspace of inner product space V. Then there exist orthogonal projection E of V on W as well as orthogonal projection of V on $W^{\perp}$. The way the orthogonal projection Ψ defined here the orthogonal projection Ψ = (I-E), because $\Psi(\beta)$ = (I-E)β = $\beta - E_{\beta}$.

So, we have now seen that if V is an inner product space and W is a finite dimensional subspace of V then there is a orthogonal projection of V on W as well as on orthogonal complement of W. So, these give hints that is it possible to write the space V as a direct sum of two subspaces which are independent to each other. So, these questions can be answered by these theorems, let V be an inner product space.

Let W be a finite dimensional subspace of V. Let (i) E be the orthogonal projection of V on W. Then E is an idempotent linear transformation from V onto W, i.e. E^2 = E on W. (ii) The null space of E is $W^{\perp}$. (ii) V = $W \oplus W^{\perp}$, where $W \cap W^{\perp} = \{0\}$. So, let me prove one by one these results.

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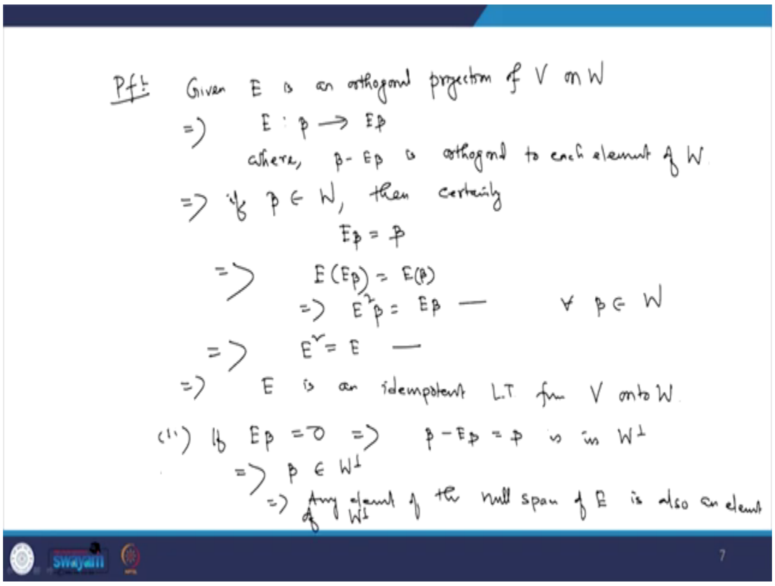

So, it is given E is an orthogonal projection of V on W $\Longrightarrow$ E: $\beta \rightarrow E_{\beta}$, where $\beta - E_{\beta}$ is orthogonal to each element of W, means Each element of this means E_{β} is the best approximation of E_{β} on W $\Longrightarrow$ if $\beta \in$ W, then certainly, $E_{\beta} = \beta \Longrightarrow E(E_{\beta}) = E(\beta) \Longrightarrow$ $E^2\beta$ = E(β) $\Longrightarrow$ E^2= E, So that the best approximation of β on W will be itself, basically. E^2= E

So, this implies E is an idempotent linear transformation from V onto W. So, this is the first part of these results. (ii), if $E_\beta = 0 \Longrightarrow \beta - E_\beta = \beta$ is in $W^\perp \Longrightarrow \beta \in W^\perp$

$\Longrightarrow$ any element of the null space of E is also an element of $W^\perp$.

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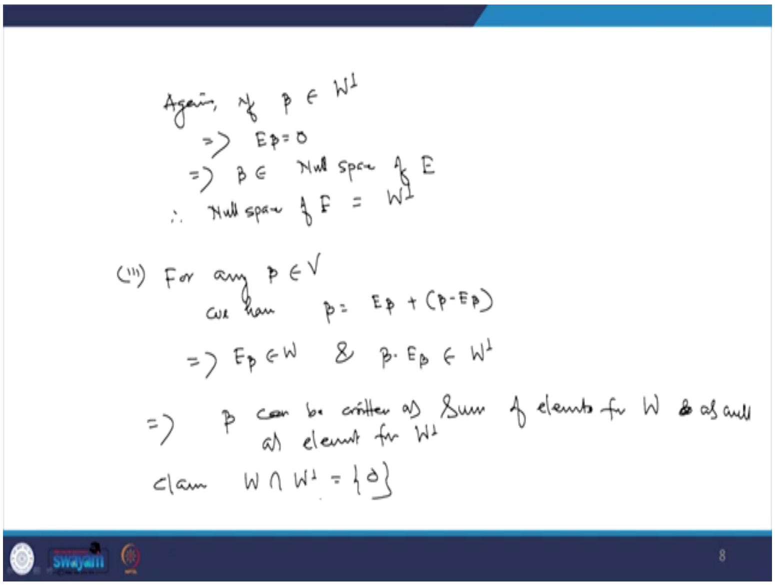

Again, if $\beta \in W^\perp \Longrightarrow E_\beta = 0 \Longrightarrow \beta \in$ Null space of E , because the best approximations of β on W will be 0, so, null space of E = $W^\perp$. (iii) For that I have to show that V is the direct sum of W and $W^\perp$ for this for any $\beta \in V$

We have, $\beta = E_\beta + (\beta - E_\beta)$. So, this implies $E_\beta \in W$ & $\beta . E_\beta \in W^\perp$. So, we see that $\beta . E_\beta$ can be written as a sum of element from the W and some as well as element of the $W^\perp$. So, this implies β can be written as sum of element from W as well as element from your $W^\perp$.

So, it has two component one component from W another component from the $W^\perp$.. It will be direct sum, provided if I can some also that W and $W^\perp$ their mutuality's giants. So now, I want to show $W \cap W^\perp = \{0\}$, suppose not.

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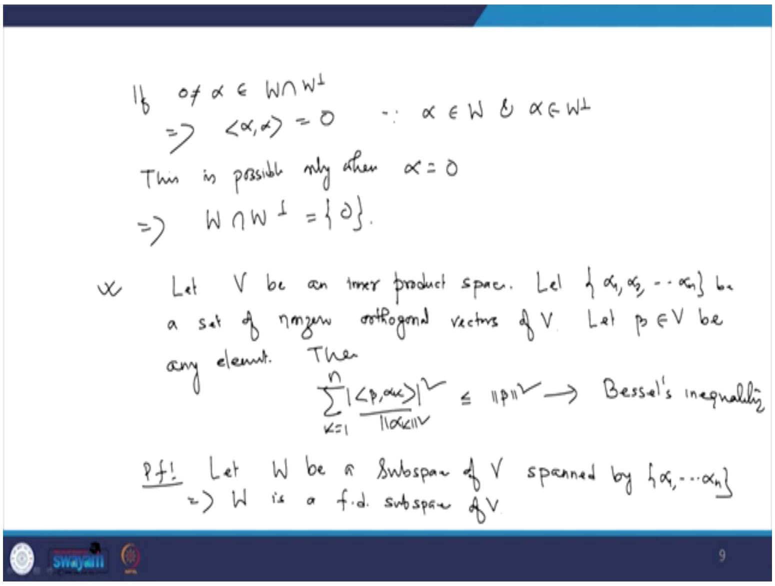

If $0 \neq \alpha \in W \cap W^{\perp}$. So, this implies $<\alpha, \alpha> = 0$, Since $\alpha \in W$ & $\alpha \in W^{\perp}$ and so, this implies α is orthogonal complement of itself. This is possible only when $\alpha = 0$, so, this implies that $W \cap W^{\perp} = \{0\}$.

So now, we have seen that if V is an inner product space and W is a finite dimensional subspace. Then V can be written as direct sum of W and it is corresponding orthogonal complement. So, this, given hints and also on interesting results that is called Bessel's inequality. What is that? Let V be an inner product space, let $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a set of non-zero orthogonal vectors of V. Let $\beta \in V$ be any element.

Then, $\sum_{k=1}^{n} \frac{<\beta, \alpha_k>}{||\alpha_k||^2} \alpha_k \leq ||\beta||^2$. This is called the Bessel's inequality. This is also coming as a consequence of a straightforward results. What have we already explained? How proof is again straight forward? Proof: - Let W be a subspace of V spanned by given orthogonal vectors $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$. So, this implies W is a finite dimensional subspace of V.

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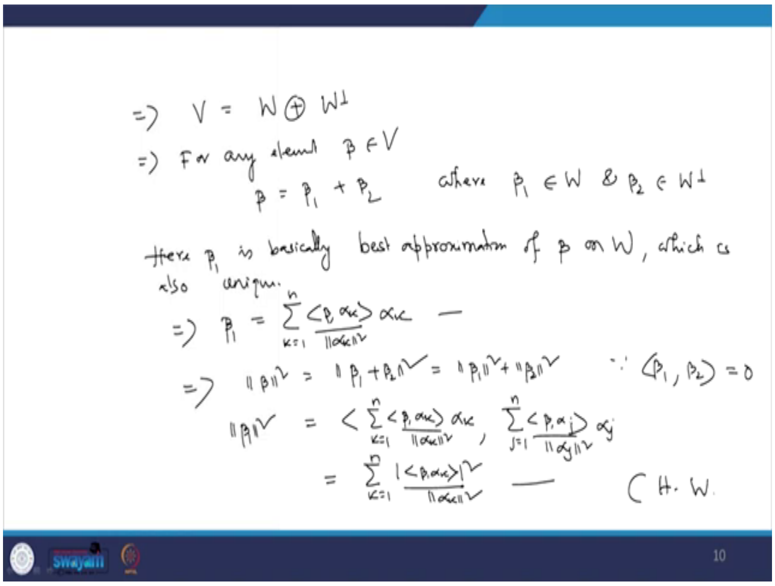

This implies V = W $\oplus$ W$^{\perp}$. So, this implies for any $\beta \in$ V, I will have, $\beta = \beta_1 + \beta_2$ where $\beta_1 \in$ W & $\beta_2 \in$ W$^{\perp}$. Now, what is β_1? β_1 is basically the best approximation of β on W. Here, β_1 is basically best approximation of β on W.

Because W is a finite dimensional, so, best approximation of β on W exists and it is unique which is also unique. So, this implies one can write down $\beta_1 = \sum_{k=1}^{n} \frac{<\beta, \alpha_k>}{||\alpha_k||^2} \alpha_k \implies ||\beta||^2$, this is the best approximation of β on W. So, this implies that β_1 is orthogonal to β_2 also.

Now, $||\beta||^2 = ||\beta_1 + \beta_2||^2 = ||\beta_1||^2 + ||\beta_2||^2$, since $< \beta_1, \beta_2 > = 0$. So, we have, $||\beta_1||^2$ $= < \sum_{k=1}^{n} \frac{< \beta, \alpha_k >}{||\alpha_k||^2} \alpha_k, \sum_{j=1}^{n} \frac{< \beta, \alpha_k >}{||\alpha_k||^2} \alpha_j > = \sum_{k=1}^{n} \frac{|< \beta, \alpha_k >|^2}{||\alpha_k||^2}$. You can check as a homework that this is correct.

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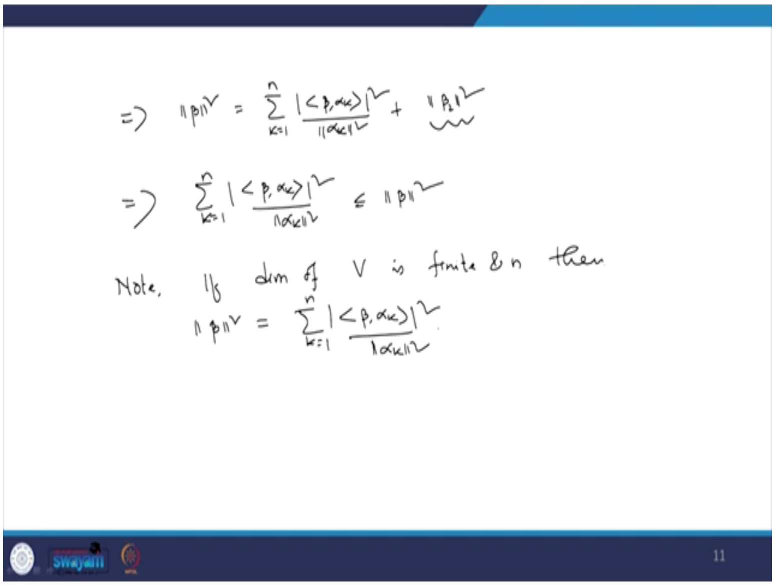

$\Rightarrow \; ||\beta||^2 = \sum_{k=1}^{n} \frac{|<\beta,\alpha_k>|^2}{||\alpha_k||^2} \; + \; ||\beta_2||^2$ So, this quantity is also positive quantity $\Rightarrow$ $\sum_{k=1}^{n} \frac{|<\beta,\alpha_k>|^2}{||\alpha_k||^2} \; \leq \; ||\beta||^2$. So, one may as a question under what condition equality will whole square?

So, as a note I can say, if dimension of V is finite n, then $||\beta||^2 = \sum_{k=1}^{n} \frac{|<\beta,\alpha_k>|^2}{||\alpha_k||^2}$. So, friends, we see that this concept of orthogonal projections gives so, nice results on this inner product space. It split the space into direct sum of two subspaces and it also talk about the best approximations of a vector in V with respect to subspace W.

And the concept of orthogonal projections will be useful in our next lectures in coming lectures. So, you will see that this call set useful in different angle. So, we will discuss more on this issue as in assignment problems. So, I hope you have understood the concept of orthogonal projections on a inner product space. Next class talks about the linear functionals and inner products and the relation.