Advanced Linear Algebra
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Lecture – 46
Best Approximation in I.P.S

So welcome to lecture series of Advanced Linear Algebra. We have already learned inner product space if a linearly independent set of vectors are given over a vector space that is also inner product space. Then one can have corresponded a set of orthogonal vectors. This has been proved by construction by Gram-Schmidt or theological procedures. In that construction procedure we have adopted and principle.

I mean a geometrical concept which is called basically orthogonal projection repeatedly we have with that one. This concept will be more clear and this concept is most important in approximate theorems. We will see soon but before that let me define what is the meaning of best approximation of a vector over a subspace of that vector space.

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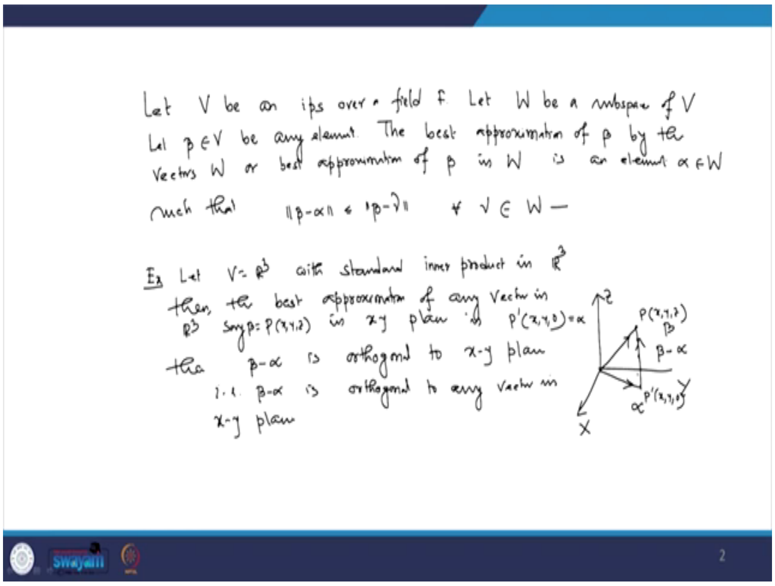

So, this concept, let me introduce like this, Let V be an inner product space(ips) over a field either real or complex that is my F. Let W be a subspace of V. Let $\beta \in$ V be any element. The best approximation of β by the vectors W or best approximation of β in W is an element $\alpha \in$ W such that, $||\beta - \alpha|| \leq ||\beta - \gamma|| \ \forall \ \gamma \in$ W.

So, α is said to be best approximation to β over this subspace W provided, $||\beta\text{-}\ \alpha|| \leq ||\beta - \gamma|| \ \forall \ \gamma \in$ W. Now, if I go to 11-th, 12-th concept and go to this say us over the space R^3 with standard inner product on R^3, let V = R^3 with standard inner product in R^3. And if I consider say this is my say, x-axis, y-axis and z-axis now over this equivalent space.

If I take a point, say this, a point p(x, y, z) = β and when you are talking the best approximation by the vectors over the x-y plane. What I see that we used to project this vector over the x-y plane, suppose it is here this vector that is basically having say p'(x, y, z) = α. So, we see that suppose this is the α and this is your β. Then I see that this is vector β.

So, you see that β- α which is equal to this p' to p is β- α, p' to p is a orthogonal or perpendicular to the x-y plane. So, let thing then, the best approximation of any vector in R^3 say β = p(x, y, z) in x-y plane is p'(x, y, 0) = α then β- α is orthogonal to x-y plane.

i.e. β- α is orthogonal to any vector in x-y plane. Now, whether this concept is valid for more than 3 dimensional or not so, now you can see this concept will remain valid for finite dimensional vector space. That if α is the best approximation to the vector β then β- α will be orthogonal to the any element of W. This concept will remain valid for the finite dimensional space.

But for the infinite dimensional space it may not because in that case the existence of best approximation also questionable, in some cases. So and we need a special treatment for to understand to prove this concept. So now, we will talk in nice results which will talk about the best approximation of any vector β on V with respect to a given subspace W of V.

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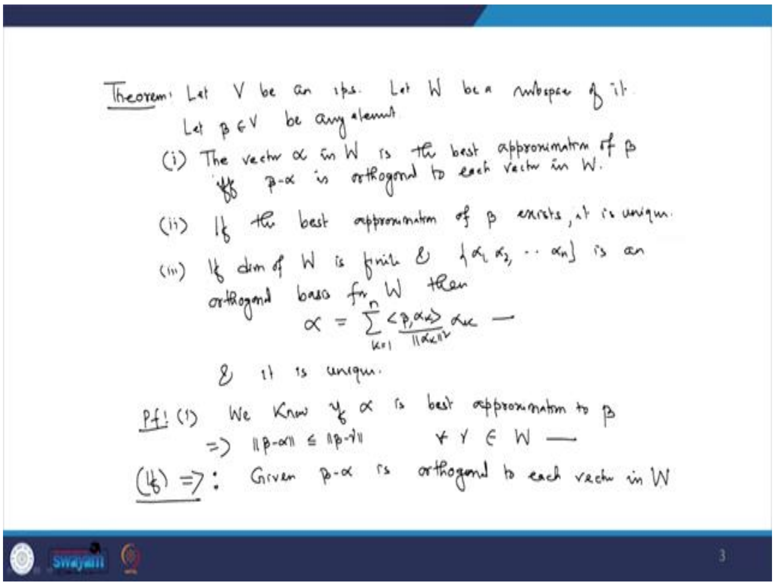

This is like this in terms of a theorem, I am writing it. Let V be an inner product space I am not saying the dimension of V is finite or infinite nothing. I am saying V be an inner product space. Let W be a subspace of it, let $\beta \in V$ be any element. Then (i) The vector α in W is the best approximation of β iff β - α is orthogonal to each vector in W.

(iii) If the best approximation of β exist then it is unique. (iii) If dimension of W is finite and $\alpha_1, \alpha_2, \dots \alpha_n$ is an orthogonal basis for W, then, $\alpha = \sum_{i=1}^{n} \frac{<\beta,\alpha_k>}{||\alpha_k||^2} \alpha_k$ and it is unique. So, let us talk about the proof of this interesting results. So, what is the theorem says? Theorem says that if V be an inner product space, the space may be finite dimensional may be infinite dimensional.

And if W be a subspace of V, here also W may be finite may be infinite dimensions of space. And we picked up β be any element of V the, (i) The vector α in W is said to be the best approximation to β with respect to W if and β - α is orthogonal to each vector in W. So, for this we do not put any restriction that whether W is a finite dimensional or infinite dimensional.

(ii) If the best approximations of β in W exist, it has to be unique. So, here also, we do not need that whether W has to be finite dimensional or infinite dimensional. (iii) If the W is a finite dimensional vector space and $\alpha_1, \alpha_2, \dots \alpha_n$ and orthogonal basis for the space subspace W. Then the best approximation to β which is given by α, $\alpha = \sum_{i=1}^{n} \frac{<\beta,\alpha_k>}{||\alpha_k||^2} \alpha_k$

So, let me Proof (i):- We know, if α is best approximations to β $\Longrightarrow$ $||\beta\text{-}\ \alpha|| \leq ||\beta - \gamma||$ $\forall$ $\gamma \in$ W. So, we have to show so, this this is known given to us. Now, this definition is known. Now, is saying that if $\beta\text{-}\ \alpha$ is orthogonal to each vector in W, I have to show that α is the best approximation to β.

So now, let me prove the (if) $\Longrightarrow$: given your $\beta\text{-}\ \alpha$ is orthogonal to each vector in W it is given to us. We have to prove that $||\beta\text{-}\ \alpha|| \leq ||\beta - \gamma||$ $\forall$ $\gamma \in$ W.

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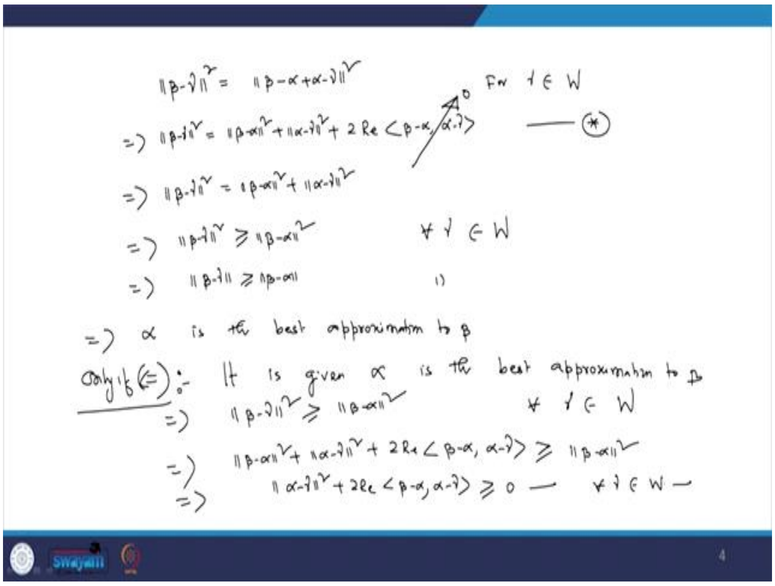

We have, $||\beta - \gamma||^{2} = ||\beta - \alpha + \alpha - \gamma||^{2}$ $\forall$ $\gamma \in$ W $\Longrightarrow$ $||\beta - \gamma||^{2} = ||\beta - \alpha||^{2} + ||\alpha - \gamma||^{2}$ + 2Re $< \beta - \alpha\ , \alpha - \gamma >$. Now, since $\beta\text{-}\ \alpha$ is orthogonal to $\alpha - \gamma$. So, this means that, $< \beta - \alpha\ , \alpha - \gamma >$ $\Longrightarrow$ $||\beta - \gamma||^{2} = ||\beta - \alpha||^{2} + ||\alpha - \gamma||^{2}$ So, in the right-hand side both the terms are positive quantity $\Longrightarrow$ $||\beta - \gamma||^{2} \geq ||\beta - \alpha||^{2}$ $\forall\, \gamma \in$ W $\Longrightarrow$ $||\beta - \gamma|| \geq ||\beta - \alpha||$ $\forall\, \gamma \in$ W, So, this implies that α is the best approximation to β.

Now, let us go for the only if ($\Leftarrow$):- part that is this one. So, here it is given α is the best approximations to β $\Longrightarrow$ $||\beta - \gamma||^{2} \geq ||\beta - \alpha||^{2}$ $\forall\, \gamma \in$ W $\Longrightarrow$ $||\beta - \alpha||^{2} + ||\alpha - \gamma||^{2}$ + 2Re $< \beta - \alpha\ , \alpha - \gamma >$ $\geq$ $||\beta - \alpha||^{2}$ $\Longrightarrow$ $||\alpha - \gamma||^{2}$ + 2Re $< \beta - \alpha\ , \alpha - \gamma > \geq 0$ $\forall\, \gamma \in$ W

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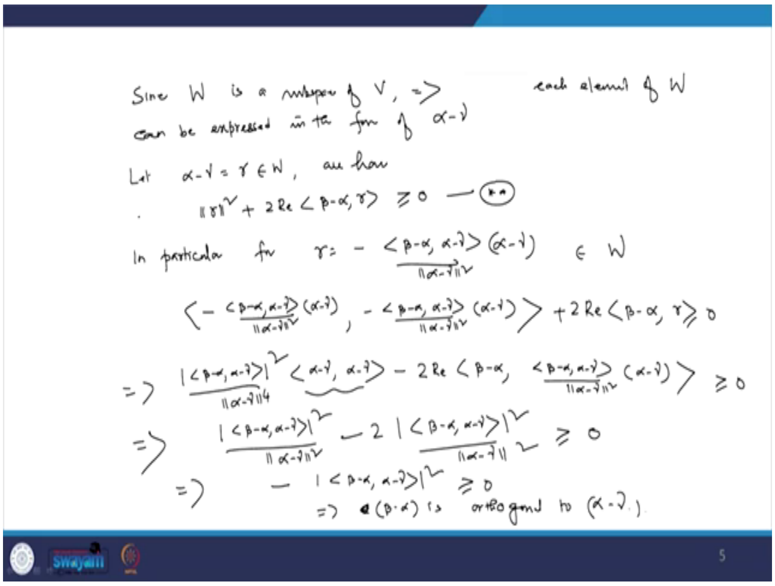

Since, W is a subspace W of vector space V. So, this implies that each element of W can be expressed in the form of $\alpha - \gamma$. I mean, if you picked up any element, say T then one can write down if $\alpha\ \in$ W then can write down your that $\alpha\ -$ t. So, one can write down this form. So, let $\alpha - \gamma\ =$ r $\in$ W.

Our last result says that so, we have, $||r||^{\,2}$ + 2Re $< \beta - \alpha\,, r\ > \geq 0\ \ \rightarrow (**)\ \forall\ r \in$ W. So, in particular for, r = $-\frac{<\beta-\alpha\,,\alpha-\gamma>}{||\alpha-\gamma|^2}\ (\alpha - \gamma)\ \in$ W, form (**), I substitute r value then, I am getting

$<-\frac{<\beta-\alpha\,,\alpha-\gamma>}{||\alpha-\gamma|^2}\ (\alpha - \gamma), -\frac{<\beta-\alpha\,,\alpha-\gamma>}{||\alpha-\gamma|^2}\ (\alpha - \gamma) >$ + 2Re $< \beta - \alpha, r> \geq 0 \qquad \Longrightarrow$

$<\frac{|<\beta-\alpha\,,\alpha-\gamma>|^{\,2}}{||\alpha-\gamma|^4} < (\alpha - \gamma), (\alpha - \gamma) >$ - 2Re $< \beta - \alpha, -\frac{<\beta-\alpha\,,\alpha-\gamma>}{||\alpha-\gamma|^2}\ (\alpha - \gamma) > \geq 0 \Longrightarrow$

$<\frac{|<\beta-\alpha\,,\alpha-\gamma>|^{\,2}}{||\alpha-\gamma|^2}$ - $2\frac{|<\beta-\alpha\,,\alpha-\gamma>|^{\,2}}{||\alpha-\gamma|^2} > \ \geq 0\ \Longrightarrow\ -|<\ \beta - \alpha\,, \alpha - \gamma\ \ > |^{\,2} \geq 0.$

So a positive number is greater than equal to 0, implies that your certainly β- α is orthogonal to $\alpha - \gamma$. So, we see, if α is the best approximation to β then β- α is orthogonal to $\alpha - \gamma$. In fact, it is orthogonal to every vector in W. Because we each vector in W we have already explained that it can be written as $\alpha - \gamma$ also in this form. So, we see that if β- α is orthogonal to every vector W.

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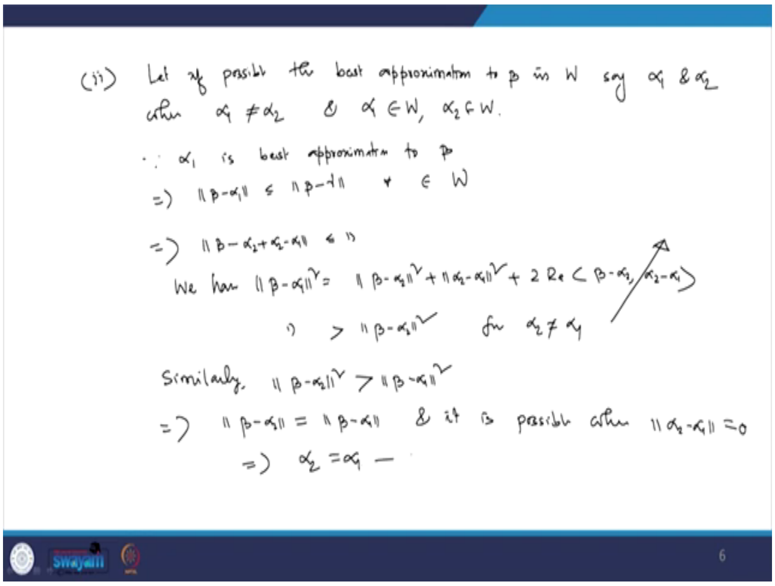

(ii) If β a best approximation to β in W exist that has to be unique and this prove it is very straightforward also. Let, if possible, the best approximation to β in W say α_1, α_2 where $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 \in$ W. We have to show that $\alpha_1, = \alpha_2$. See since, α_1 is best approximation to β implies that, $||\beta - \alpha_1|| \leq ||\beta - \gamma|| \quad \forall \ \gamma \in$ W,

So, this implies $||\beta - \alpha_2 + \alpha_2 - \alpha_1|| \leq ||\beta - \gamma|| \quad \forall \ \gamma \in$ W. But we have, $||\beta - \alpha_1||^2 = ||\beta - \alpha_2||^2 + ||\alpha_2 - \alpha_1||^2 + 2\text{Re} < \beta - \alpha_2 , \alpha_2 - \alpha_1 >$. So, this component has to be 0 because $\alpha_1, \alpha_2 \in$ W and $\beta - \alpha_2$ is also orthogonal to any element of W.

So, $\beta - \alpha_2$ is also orthogonal to $\alpha_2 - \alpha_1$ of W and since α_2 is also best approximation to β. So, there has to be 0, they are orthogonal, so, this is 0. So, this implies this quantity, $||\beta - \alpha_1||^2 > ||\beta - \alpha_2||^2$ for $\alpha_1 \neq \alpha_2$. Similarly, we can show that, $||\beta - \alpha_2||^2 > ||\beta - \alpha_1||^2$ for $\alpha_1 \neq \alpha_2$.

So, these 2 will contradict. So, this implies that, $\beta - \alpha_1 = \beta - \alpha_2$ and it is possible we will when $\| \alpha_1 - \alpha_2 \| = 0$ So, this implies $\alpha_1 = \alpha_2$.

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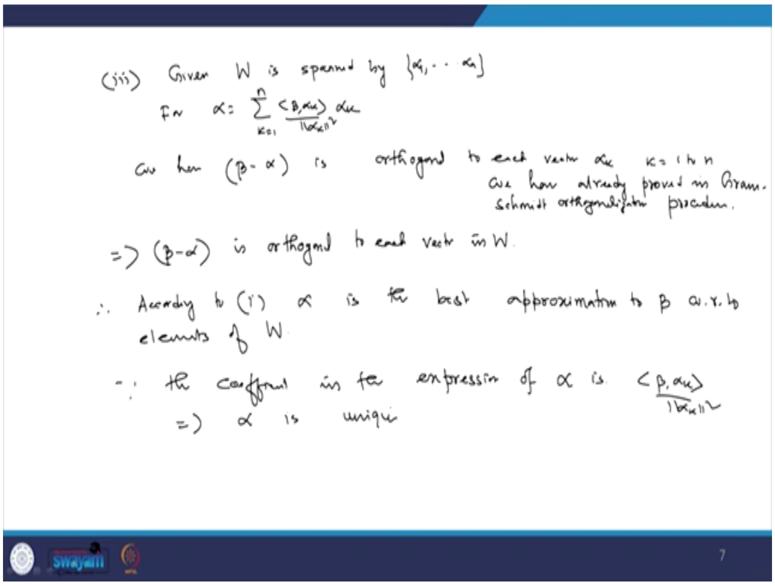

The third one it is given to us, given W is spanned by order basis consisting of orthogonal vector number is finite. So, spanned by $\alpha_1, \alpha_2, \ldots\ \alpha_n$ which are orthogonal. We have now for $\alpha = \sum_{k=1}^{n} \frac{<\beta,\alpha_k>}{||\alpha_k||^2}\ \alpha_k$. We have $(\beta - \alpha)$ is orthogonal to each vector α_k, k = 1 to n. This is basically which is coming as a consequence of our Gram-Schmidt orthogonolization procedure.

if I consider that $(\beta - \alpha) = \left(\beta - \sum_{k=1}^{n} \frac{<\beta,\alpha_k>}{||\alpha_k||^2}\ \alpha_k\right)$ and this quantity this vector. If I do not say it is say, $(\beta - \alpha) = \gamma$, γ will be orthogonal to each vector of α_k. So that is why I can say $(\beta - \alpha)$ is orthogonal to each vector α_k, k = 1 to n which is we have already seen. We have already proved in Gram-Schmidt orthogonalization procedure.

So, this implies that $(\beta - \alpha)$ is orthogonal to each vector in W is spanned by this one. So, according to (i) α is the best approximation to β with respect to elements of W. And since the way α has been constructed defined here. So, since the coefficient in the expression of $\alpha = \sum_{k=1}^{n} \frac{<\beta,\alpha_k>}{||\alpha_k||^2}\ \alpha_k,$ implies α is unique.

If you say any other vector the definitely that vector will also have coefficient of α_k will be exactly $\left(\beta - \sum_{k=1}^{n} \frac{<\beta,\alpha_k>}{||\alpha_k||^2}\ \alpha_k\right)$. So, therefore, I can say that this α is unique. So, we will discuss more and in our next class but before completing this class. Let me give examples.

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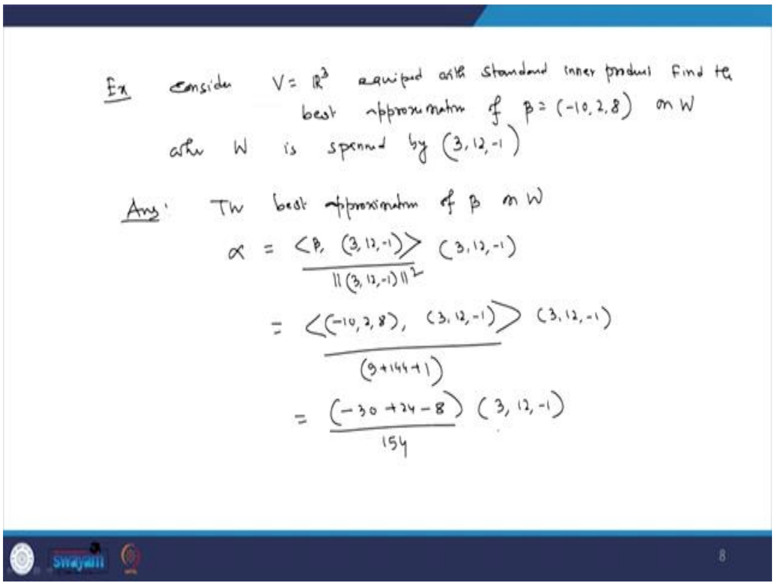

Consider V = R^3, equipped with standard inner product. So, the inner product space V = R^3, I have considered that is our standard in our product space in Euclidean space. So, let me consider the problem like this V = R^3. That is our 3-dimensional space Euclidean space and the inner product in this space is our standard inner product.

And suppose I have to get the best approximation elements have, β = (-10, 2. 8) and this best approximation with respect to subspace W which is spanned by (3, 12, – 1). So, in this case, so, I have to then according to the last theorems, the best approximation of beta on W is say $\alpha = \frac{<\beta,(3,12,-1)>}{||(3,12,-1)||^2}(3, 12, - 1) = \frac{<(-10,2.8),(3,12,-1)>}{(9+144+1)}(3, 12, - 1) = \frac{(-30+24+8)}{(194)}(3,12,-1)$

Since the dimension of the space W is 1. So, the best approximation of beta on W will be alpha of this form. So, this is the answer. So, we will continue the concept of best approximations over the inner product space V in our next class also thank you.