Advanced Linear Algebra
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Lecture – 45
Inner Products Spaces - II

Welcome to lecture series in Advance Linear Algebra. We have already defined in the inner product space, the meaning of orthogonality of two vectors. Now, let me define today another terminology called orthogonal set of vectors.

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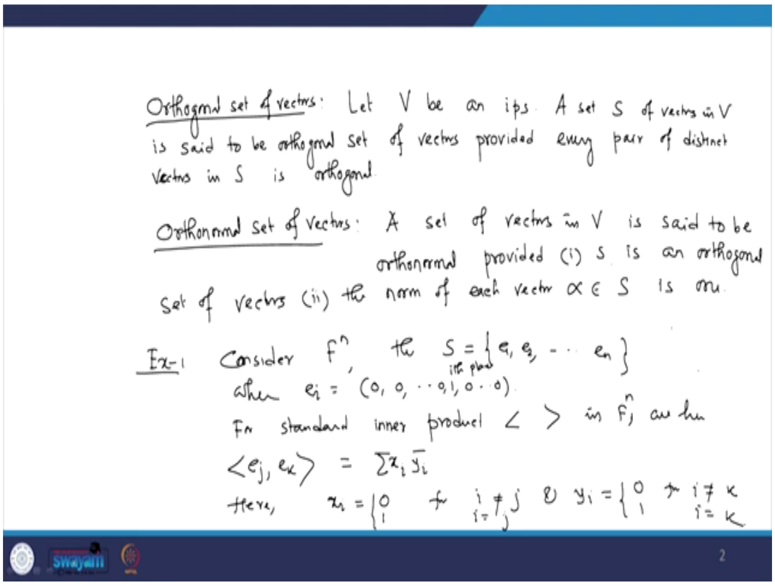

Orthogonal set of vectors: - Let V be an inner product space(ips). A set S of vectors in V is said to be orthogonal set of vectors, provided every pair of distinct vectors in S is orthogonal. So, every pair of distinct vectors in S is orthogonal. So that is the definition of an orthogonal set of vectors. Again, another terminology called a set of orthonormal vectors.

Orthonormal set of vectors: - A set of vectors in V is said to be orthonormal, provided (i) S is an orthogonal set of vectors. (ii) The norm of each vector α ∈ S is one. So, orthonormal set of vectors, is basically orthogonal set of vectors, along with the length of the or magnitude of these vector is 1. Example1: - Consider over the F^n, S = $\{e_1, e_2, \ldots e_n\}$ that is standard order of basis on F^n, where e_i= (0, 0, …0, 1, 0, … 0)

Then, for standard inner product < > in F^n, we have $< e_j, e_k > = \sum_{j=1}^{n} x_i \, \overline{y_\iota}$, here $x_i = \begin{cases} 0 \; for \; i \neq j \\ 1 \; for \; i = j \end{cases}$ & $y_i = \begin{cases} 0 \; for \; i \neq k \\ 1 \; for \; i = k \end{cases}$

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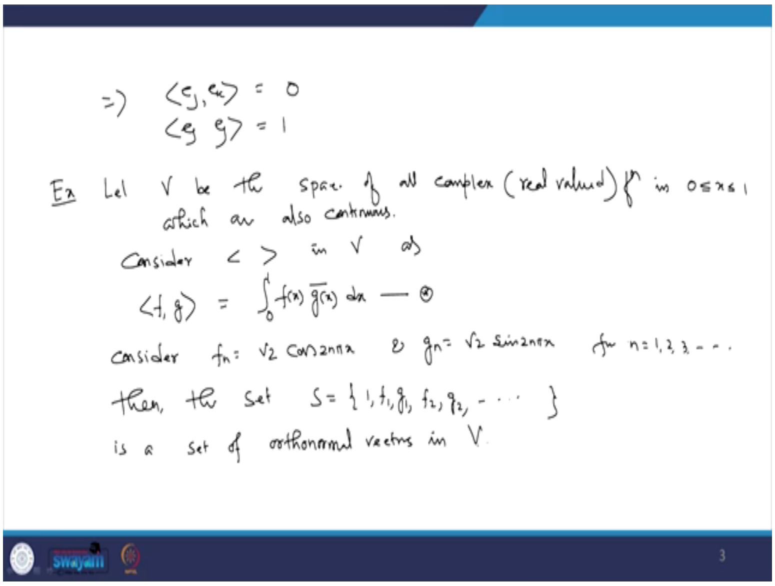

So, in this case we see that $\Longrightarrow < e_j, e_k > = 0 \,, < e_j, e_j > = 1$. Similarly, what the let me consider vector space let V be the space of all complex or real valued functions in $0 \leq x \leq 1$ which are also continuous. Consider < > in V as, < f, g > = $\int_0^1 f(x) \, \overline{g(x)} \, dx$. Now, with respect to this inner product, consider f_n = $\sqrt{2}$ cos(2nπx) & g_n = $\sqrt{2}$ sin(2nπx). So, for n = 1, 2, 3, ….

Then this set S = $\{1, f_1, g_1, f_2, g_2 \ldots\}$ is a set of orthonormal vectors in V. So, this is a set of orthonormal vectors in V. How to prove it?

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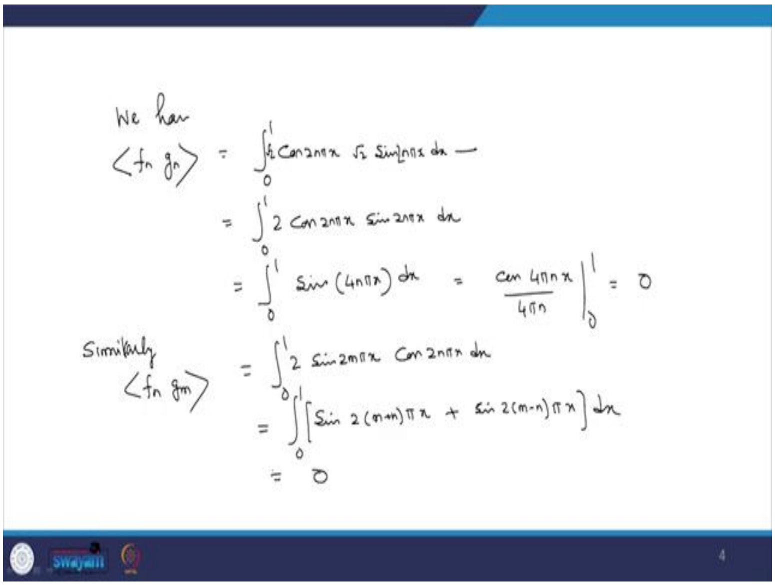

We have, $< f_n, g_n > = \int_0^1 \sqrt{2}\cos(2\text{n}\pi\text{x})\, \sqrt{2}\,\sin(2\text{n}\pi\text{x})dx = \int_0^1 2\cos(2\text{n}\pi\text{x})\,\sin(2\text{n}\pi\text{x})dx =$ $\int_0^1 \sin(4\text{n}\pi\text{x})dx = \frac{\cos(4\text{n}\pi\text{x})}{4\text{n}\pi}\Big|_0^1 = 0$, Since it is a real, so, bar is not necessary, not required. **(09:59)**

And because $\cos(4\text{n}\pi)$ where n is a positive integer. So, this is $\cos(4\text{n}\pi) = 1$ and $\cos(0) = 1$ so that is why it is equal to 0. Similarly, $< f_n, g_n > = \int_0^1 2\sin(2\text{m}\pi\text{x})\,\cos(2\text{n}\pi\text{x})dx =$ $\int_0^1 [2\sin 2(\text{m}+\text{n})\pi\text{x} + \sin 2(\text{m}-\text{n})\pi\text{x}]dx = 0$

So, if we consider any pair of distinct vectors then we see the inner product equal to 0, whether it is orthonormal or not.

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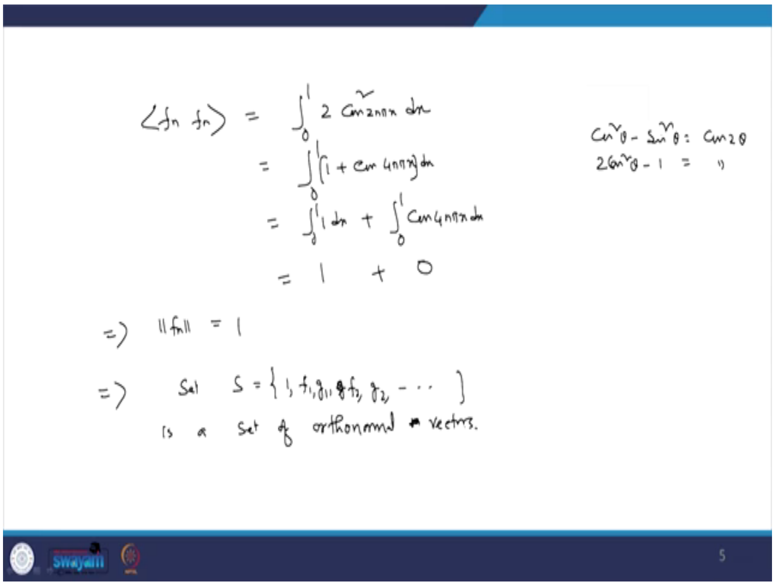

That we can again check, $< f_n, f_n > = \int_0^1 2\cos^2(2n\pi x)dx = \int_0^1[1+\cos(4n\pi x)]dx = \int_0^1 1\,dx + \cos(4n\pi x)dx = 1 + 0 = 1$ $\|f_n\| = 1$. Since we use $2\cos^2\theta$-1 = $\cos 2\theta$.

You can also check $\|f_n\| = 1$. So, it is a this implies the set $S = \{1, f_1, g_1, f_2, g_2 ...\}$ is a set of orthonormal vectors, also orthonormal vectors.

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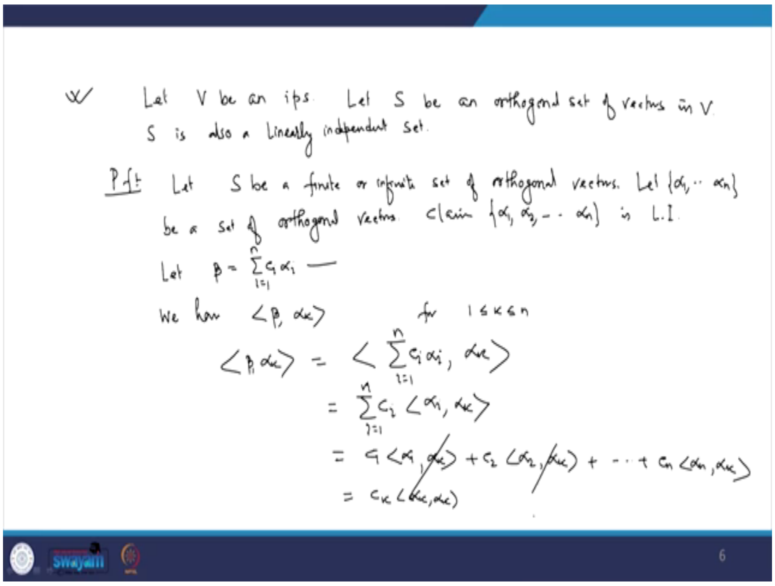

Now, let me prove on interesting results that the set of orthogonal vectors is a linearly independent set, let V be an inner product space(i.p.s.). Let S be an orthogonal set of vectors in V. Then S is also a linearly independent set prove is very simple. So, let us give a proof on these results. So, let S be a finite or infinite set of orthogonal vectors. Let $\{\alpha_1, \alpha_2, ...\ \alpha_n\}$ be a set of orthogonal vectors, claim $\{\alpha_1, \alpha_2, ...\ \alpha_n\}$ is linearly independent.

Let $\beta = \sum_{i=1}^{n} c_i\, \alpha_i$. So, I have taken a linear combination of this vector this once. I have to show that $\beta = 0 \Longrightarrow$ all $c_i = 0$. We have, $< \beta, \alpha_k>$ for I $\leq$ k $\leq$ n. So $< \beta, \alpha_k> = < \sum_{i=1}^{n} c_i\, \alpha_i, \alpha_k>$ $= \sum_{i=1}^{n} c_i < \alpha_i, \alpha_k> = c_1 < \alpha_1, \alpha_k> + c_2 < \alpha_2, \alpha_k > + \cdots + c_n < \alpha_n, \alpha_k> = c_k < \alpha_k, \alpha_k>$. This will be remaining only $c_k < \alpha_k, \alpha_k>$ because α_i are orthogonal vectors.

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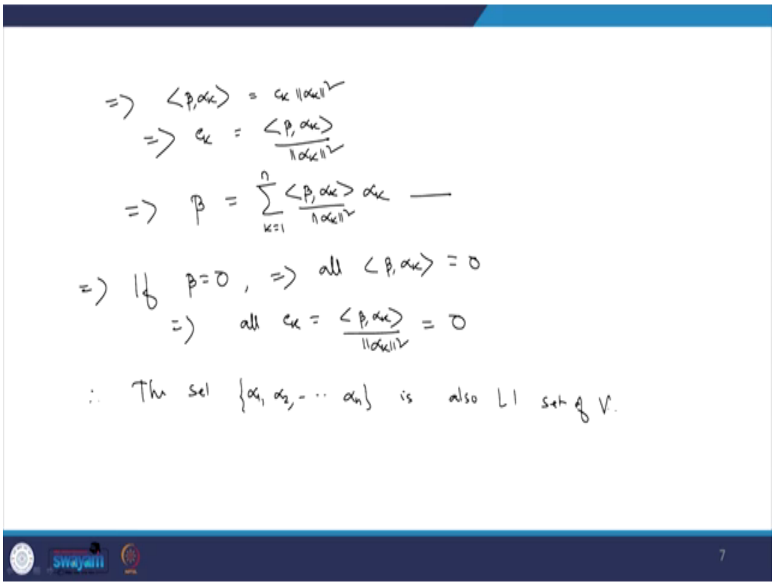

So, $\Longrightarrow < \beta, \alpha_k> = c_k \mid\mid \alpha_k \mid\mid^2 \;\; \Longrightarrow c_k = \frac{<\beta, \alpha_k>}{||\alpha_k||^2} \Longrightarrow \beta = \sum_{k=1}^{n} \frac{<\beta, \alpha_k>}{||\alpha_k||^2} \alpha_k$. If $\beta = 0, \Longrightarrow$ all $<\beta, \alpha_k> = 0 \;\; \Longrightarrow$ all, $c_k = \frac{<\beta, \alpha_k>}{||\alpha_k||^2} = 0$

So, the set of $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is also a linearly independent set of inner product space V. So, every orthogonal set of vectors is also linearly independent. Not only that interesting we see that any vector which is linear combination of the orthogonal vectors $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$. That coefficient of each α_k will basically, $\beta = \sum_{k=1}^{n} \frac{<\beta, \alpha_k>}{||\alpha_k||^2} \alpha_k$.

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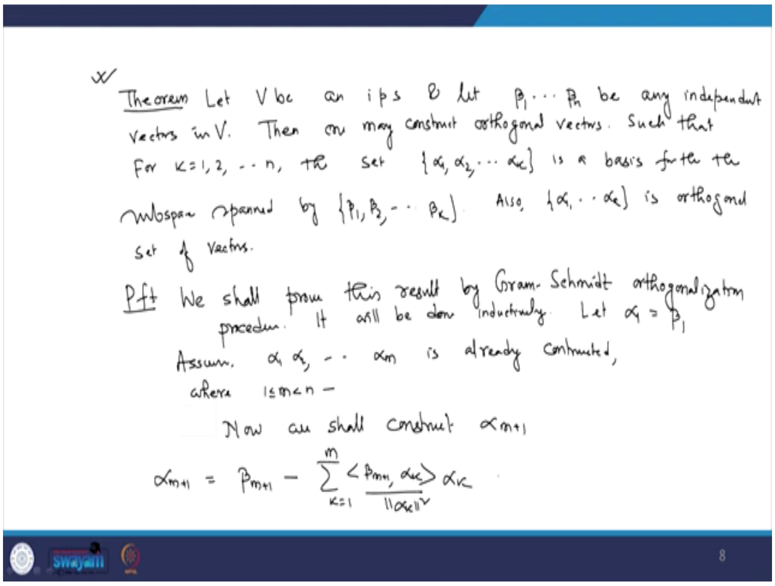

So, now an interesting question is like this, suppose a linearly independent set of vectors is given to us in an inner product space is it possible to have corresponding orthogonal vectors? So, let me put this question in this theorem form like this. So, let V be an inner product space and let $\beta_1, \beta_2, \ldots \beta_n$ be any independent vectors in V. Then one may construct orthogonal vectors, such that for k = 1 to n, this set $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ is a basis for the subspace spanned by $\beta_1, \beta_2, \ldots \beta_k$.

Also, $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ is orthogonal set of vectors. Proof: - Let me give the proof of this results, so, we shall prove this result by Gram-Schmid orthogonalization procedure. It will be done inductively. Let $\alpha_1 = \beta_1$. Assume, $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$ is already constructed by the above procedure. Where, $1 \leq m < n$.

So, now we shall construct α_{m+1}. Now, we shall construct, $\alpha_{m+1} = \beta_{m+1} - \sum_{k=1}^{m} \frac{<\beta_{m+1}, \alpha_k>}{||\alpha_k||^2} \alpha_k$.

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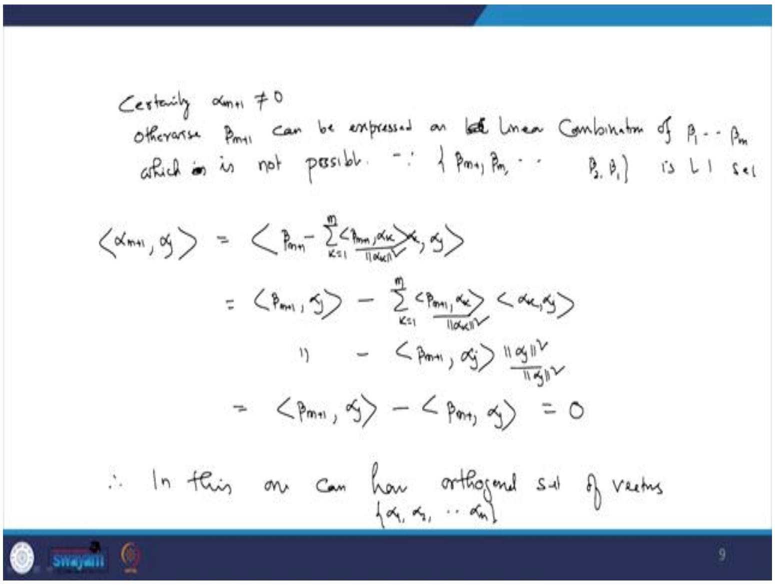

Here, I can see certainly $\alpha_{m+1} \neq 0$. Otherwise, β_{m+1} can be expressed as linear combination of $\beta_1, \beta_2, \ldots \beta_m$ which is not possible because $\{\beta_{m+1}, \beta_m, \ldots \beta_2, \beta_1\}$ is a linearly independent set. So, therefore, $\alpha_{m+1} \neq 0$. Now, $< \alpha_{m+1}, \alpha_j > = < \beta_{m+1} - \sum_{k=1}^{m} \frac{<\beta_{m+1}, \alpha_k> \alpha_k}{||\alpha_k||^2}, \alpha_j >=$ $<\beta_{m+1}, \alpha_j > - \sum_{k=1}^{m} \frac{<\beta_{m+1}, \alpha_k>}{||\alpha_k||^2} < \alpha_k, \alpha_j > = < \beta_{m+1}, \alpha_j > - < \beta_{m+1}, \alpha_j > \frac{||\alpha_j||^2}{||\alpha_j||^2} = < \beta_{m+1}, \alpha_j > - < \beta_{m+1}, \alpha_j > = 0$

We see that if up to 1 to m if this construction is made then 1 can also do for the construction for the m + 1 also. So, in this way, one can go for up to m = n. Also so, in this way, one can have orthogonal set of vectors $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$. So, one can have orthogonal set of vectors like this also by this procedure.

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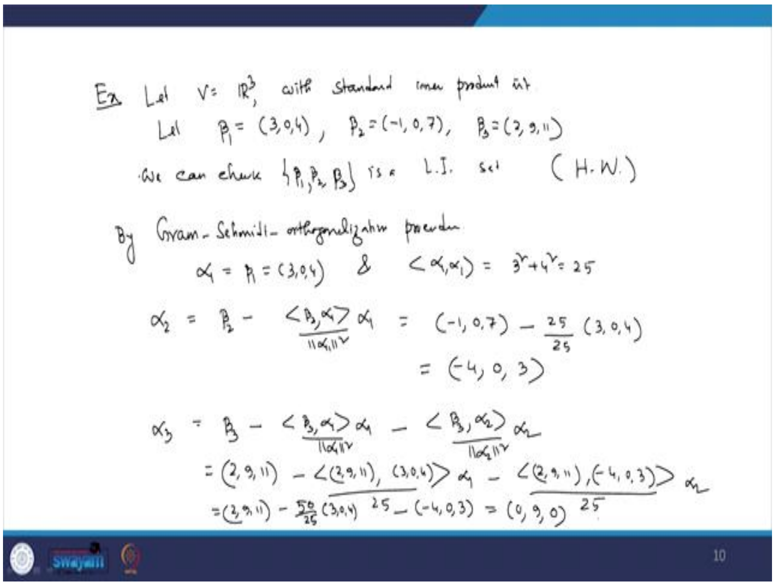

Let me take an Example: - Let V = R^3, **(()) (30:31)** with standard inner product in it. Let $\beta_1 =$ (3,0,4), β_2 = (-1,0,7), $\beta_3 = (2,9,11)$. We can immediately check $\{\beta_1,\ \beta_2, \beta_3\}$ is a linearly independent set. We can check it as a homework. Now, we have to obtain the corresponding orthogonal vectors space. So, by Gram-Schmid orthogonalization procedure $\alpha_1 = \beta_1 = (3,0,4)$

& $< \alpha_1, \alpha_1 > = \ 3^2 + 4^2$ = 25. So, $\alpha_2 = \beta_2$ - $\frac{<\beta_2,\alpha_1>}{||\alpha_1||^2}$ α_1 = (-1,0,7) - $\frac{25}{25}$ (3,0,4) = (-4,0,3) & $\alpha_3 = \beta_3 - \frac{<\beta_3,\alpha_1>}{||\alpha_1||^2}\ \alpha_1 - \frac{<\beta_3,\alpha_2>}{||\alpha_2||^2}\alpha_2 = (2,9,11) - \frac{<(2,9,11),(3,0,6)>}{25}\alpha_1 - \frac{<(2,9,11),(-4,0,3)>}{25}\alpha_2 =$ (2,9,11) - $\frac{50}{25}$ (3,0,4) - (-4,0,3) = (0,9,0)

So, this is basically α_3. So, we obtain $\alpha_1, \alpha_2,\ \alpha_3$ by Gram-Schmid orthogonalization procedure. We can check that all orthogonal set of vectors. So, we will continue this concept in our next class.