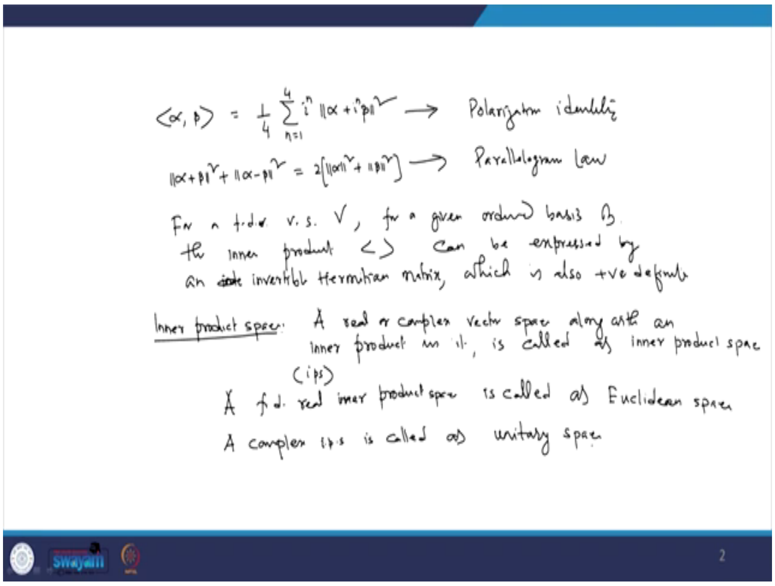
Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 44
Inner Products Spaces - I

Welcome to lecture series an Advanced Linear Algebra. Friends, we have already learned the meaning of inner product in a vector space, and we have seen how inner products give the concept of length of a vector in terms of norm of a vector. We also seen within the norm some important results related to inner products also we have seen.

(Refer Slide Time: 01:01)



Like first is the polarization identity that is inner product of any two vector α, β. If you consider it to in a vector space V then, $< \alpha, \beta > = \frac{1}{4}\sum_{i=1}^{4} i^n || \alpha + i^n \beta||^2$ → polarization identity. And we have also seen that when norm is induced through this inner product.

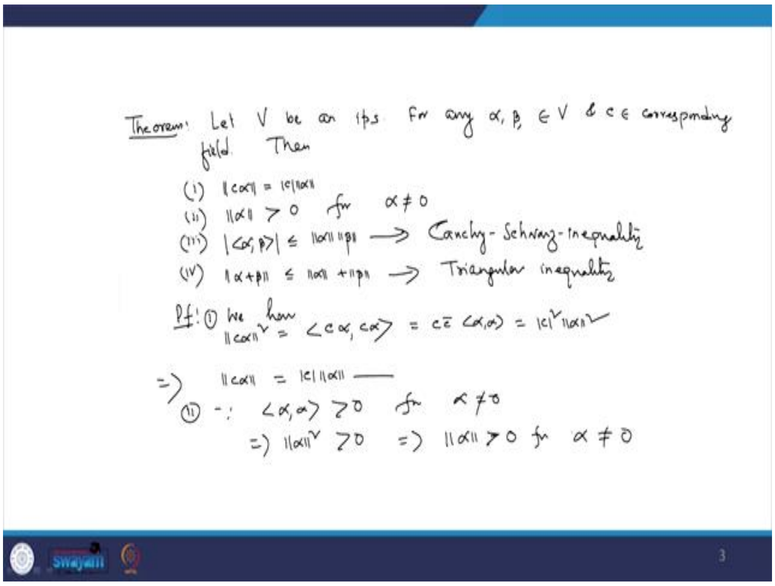
Then, $|| \alpha + \beta||^2 + || \alpha - \beta||^2 = 2(|| \alpha||^2 + || \beta||^2)$ → parallelogram law. See here norm is induced through the inner product and it satisfied parallelogram law and also the inner product of two vectors satisfy such type of polarization identity. This parallelogram law will talk about will tell you later on when you are talking studying in the mathematical analysis.

When we see that the norm can be introduced by different ways, not through only inner products then this parallelogram law will see that only satisfied when the norm is induced through inner product only. Another, interesting things we have also seen that when for a finite dimensional vector space V for a given order basis the inner product this can be expressed by an invertible Hermitian matrix which is also positive definite.

Now, we shall utilize this concept of inner product and see the combined impact of vector space and inner product. For that what we will do I will first introduce the concept of inner product space. So, let me define inner product space. So, what is inner product space: - A real or complex vector space along with an inner product in it is called as inner product space(I.P.S.) if finite dimensional real inner product space is called as Euclidean space.

Whereas a complex inner product space is called as unitary space. Vector space equipped with an inner product may be called as Euclidean space or inner product space depending on the if the vector space is **(()) (6:28)** dimensional and real, it is called the Euclidean space. If it is complex, then it is called the unitary space. So, note that this unitary space the dimension of this vector space may be finite, may be infinite also.

(Refer Slide Time: 06:47)



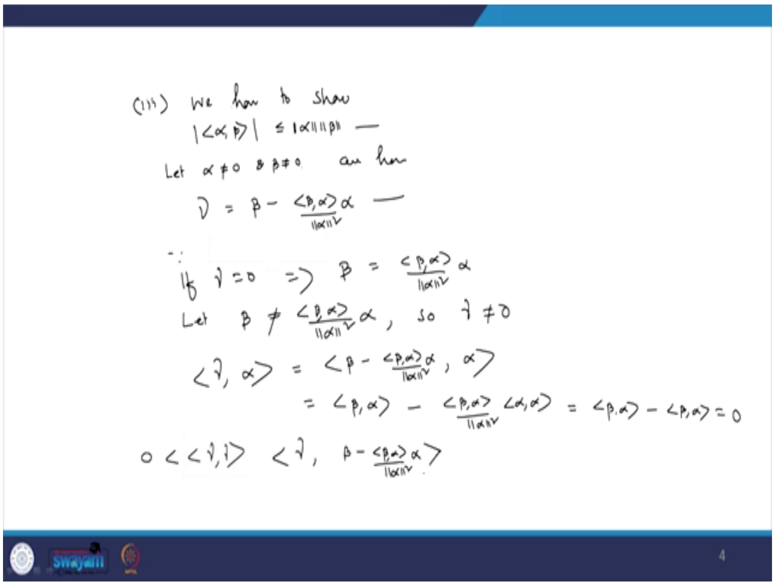
So, when you consider a vector space along with the inner product and you are saying that inner product space definitely, we can have also normal every vectors. And interesting results inner product space can be same. So, these interesting results let me express in terms of theorem. Theorem:- let V be and inner product space, so, it may be real, or it may be complex. For any $\alpha, \ \beta \in$ V and c belongs to the corresponding field. It may be real number, or it may be complex number. Then following properties holds good, (i) $||c\alpha||$ = |c| $||\alpha||$. (ii) $||\alpha|| > 0 \ \ \alpha \ \neq$

0. (iii) $|<\alpha, \beta>| \leq \|\alpha\| \, \|\beta\|$ $\rightarrow$ Cauchy Schwarz inequality. (iv) $||\alpha + \beta|| + \leq \|\alpha\| + \|\beta\|$ $\rightarrow$ triangular inequality.

So, inner product space V, following four axioms of property holds good. See one so, let me prove one by one the. Proof: - (i) we have, $||c\alpha||^2 = <c\alpha, c\alpha> = \bar{c}c <\alpha, \alpha> = |c|^2||\alpha||^2$ $\Longrightarrow$ $\|c\alpha\| = |c| \, \|\alpha\|$, which is a positive quantity.

And (ii) since $<\alpha, \alpha> > 0$ for $\alpha \neq 0 \Longrightarrow$ $||\alpha||^2 > 0$ $\Longrightarrow ||\alpha||$ for $\alpha \neq 0$.

(Refer Slide Time: 10:54)



(iii) We have to show, $|<\alpha, \beta>| \leq \|\alpha\| \, \|\beta\|$ $\rightarrow$ Cauchy Schwarz inequality. See if $\alpha = 0$, $\beta = 0$ then this inequality is definitely satisfied. So, Let $\alpha \neq 0$ & $\beta \neq 0$. So, for this case we have, let me consider $\gamma = \beta - \frac{<\beta,\alpha>\alpha}{||\alpha||^2}$

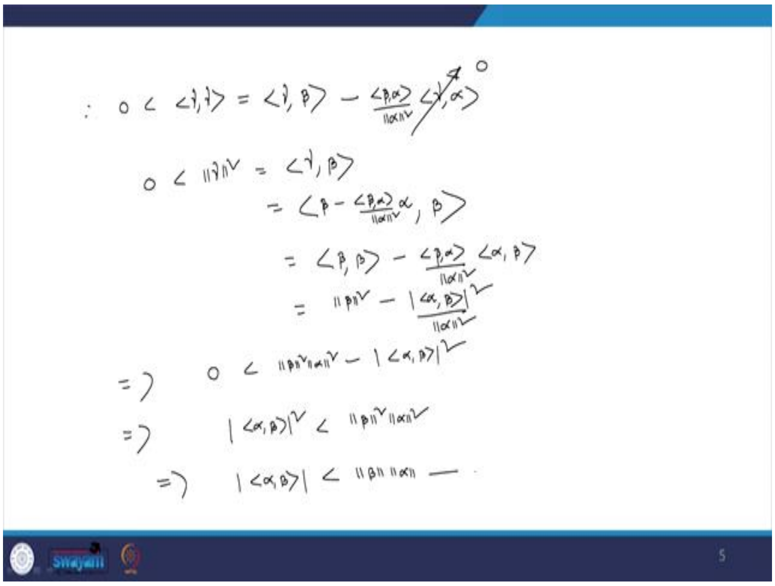
You may raise the question how to think that I have to consider gamma equal to like this and answer this question in my next when coming soon basically. When you are talking about this another terminology that is called orthogonality of the vector. So now, I have taken γ as another vectors. So, it is basically, $\gamma = \beta - \frac{<\beta,\alpha>\alpha}{||\alpha||^2}$

So, we have, if $\gamma = 0 \Longrightarrow \beta = \frac{<\beta,\alpha>\alpha}{||\alpha||^2}$. This is a scalar quantity, so, beta is basically a scalar product of the alpha. Now, let me consider $\beta \neq \frac{<\beta,\alpha>\alpha}{||\alpha||^2}$, I mean beta, not equal to simply a scalar multiple of alpha. So, in that case, so, $\gamma \neq 0$.

We have, $<\gamma, \alpha> = <\beta - \frac{<\beta,\alpha>\alpha}{||\alpha||^2}, \alpha> = <\beta, \alpha> - \frac{<\beta,\alpha>}{||\alpha||^2} <\alpha, \alpha> = = <\beta, \alpha> - <\beta, \alpha> = 0$. So, this implies that we have considering vector γ so, $<\gamma, \alpha> = 0$. We have, $0 < <\gamma, \gamma>$ Because if I have considered $\gamma \neq 0$.

So, this means that $0 < <\gamma, \gamma>$. I mean this will be $<\gamma, \gamma> = <\gamma, \beta - \frac{<\beta,\alpha>\alpha}{||\alpha||^2}, >$

(Refer Slide Time: 14:58)

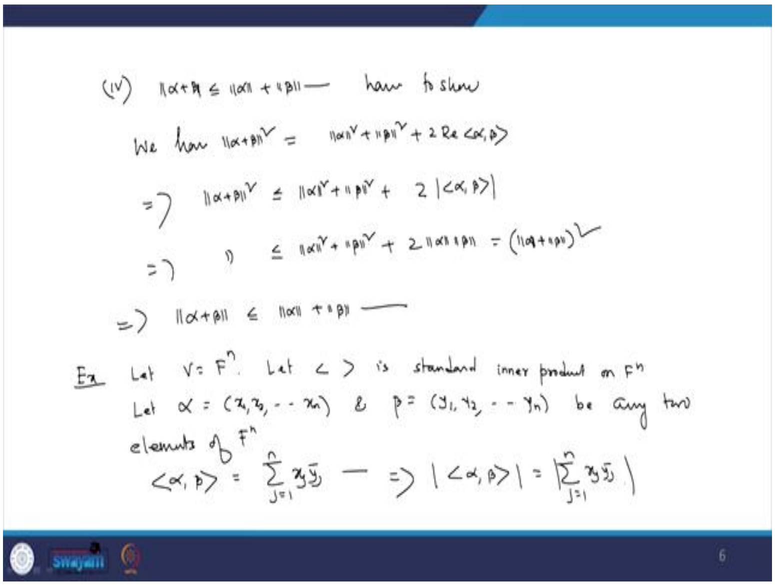


So, $0 < <\gamma, \gamma> = <\gamma, \beta> - \frac{<\beta,\alpha>\alpha}{||\alpha||^2} <\gamma, \alpha>$, where $<\gamma, \alpha> = 0$ already we have seen it $\Rightarrow 0 < ||\gamma||^2 = <\gamma, \beta> = <\beta - \frac{<\beta,\alpha>\alpha}{||\alpha||^2}, \beta> = <\beta, \beta> - \frac{<\beta,\alpha>}{||\alpha||^2} <\alpha, \beta> = ||\beta||^2$ $\frac{|<\alpha,\beta>|^2}{||\alpha||^2}$

$\Rightarrow 0 < ||\beta||^2 ||\alpha||^2 - |<\alpha, \beta>|^2$. So, here I have taken the strictly less than zero is strictly less than this quantity based on $\gamma \neq 0$. $\beta \neq 0$, beta not equal to again linear scalar multiple of α. So, if we do not consider that one then it will be less than equal to also.

$\Rightarrow |<\alpha, \beta>|^2 < ||\beta||^2 ||\alpha||^2 \Rightarrow |<\alpha, \beta>| < ||\beta|| \, ||\alpha||$ Anyhow, so, equality will come into the picture if you consider that $\beta = \beta = \frac{<\beta,\alpha>\alpha}{||\alpha||^2}$, I mean scalar multiple of α. So, in that case equality will come to the picture.

(Refer Slide Time: 18:08)



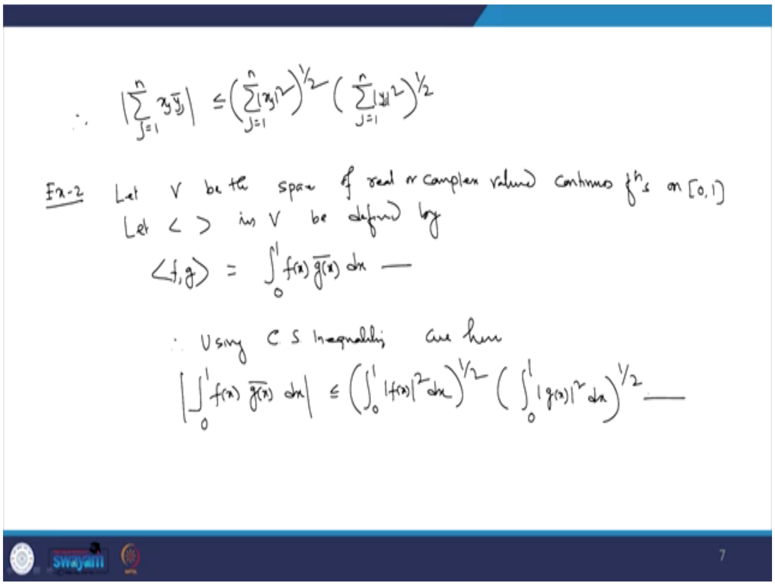
The next result was that triangular inequality. We have to show that (iv) $||\alpha + \beta|| + \leq \|\alpha\| + \|\beta\|$ $\rightarrow$ triangular inequality. So, for this we have seen $||\alpha + \beta||^2 = ||\alpha||^2 + ||\beta||^2 + 2Re < \alpha, \beta >$. This already have established this result. $\Longrightarrow ||\alpha + \beta||^2 < ||\alpha||^2 + ||\beta||^2 + 2| < \alpha, \beta > |$, So, I can say real will be vanished because we have taken the absolute one.

$\Longrightarrow < ||\alpha||^2 + ||\beta||^2 + 2\big||\alpha|\big| \, \big||\beta|\big| = (||\alpha|| + ||\beta||)^2$ by Cauchy Schwarz inequality that is from (iii) already proved. So, $\Longrightarrow ||\alpha + \beta|| + \leq \|\alpha\| + \|\beta\|$. This is triangular inequality.

So, you will see the application of (iii) & (iv) is rigorous. In many places we use Cauchy Schwarz inequality. We also use the concept of triangular regulatory. Now, let me consider some examples. Let V = F^n. Here this field is real number or complex number. Let inner product is standard inner product on F^n.

So, let $\alpha = (x_1, x_2, \dots x_n)$ & $\beta = (y_1, y_2, \dots y_n)$ be any two elements of V = F^n. Then we have, $< \alpha, \beta > = \sum_{j=1}^{n} x_j \, \bar{y_j} \Longrightarrow | < \alpha, \beta > | = | \sum_{j=1}^{n} x_j \, \bar{y_j} |$ **(()) (22:27)**.

(Refer Slide Time: 22:31)

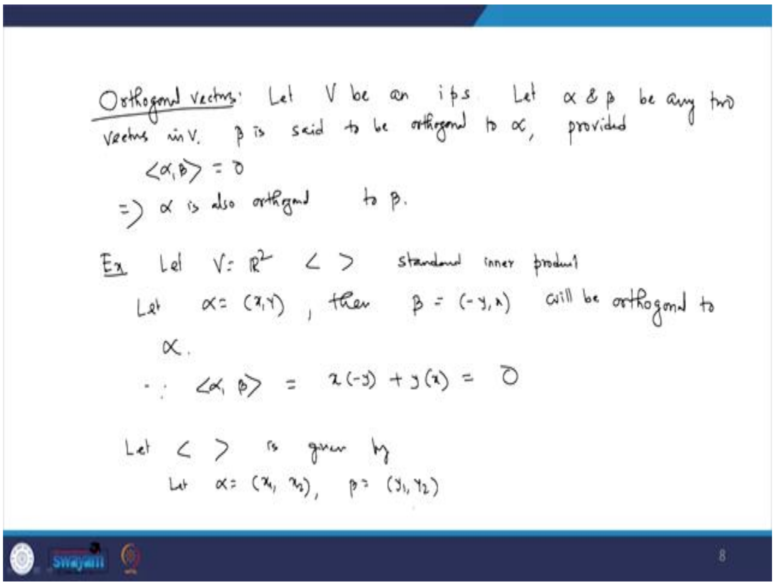


So, according to the Cauchy-Schwarz inequality, $|\sum_{j=1}^{n} x_j \, \overline{y_j}| \leq (\sum_{j=1}^{n} |x_j|^2)^{1/2} \; (\sum_{j=1}^{n} |y_j|^2)^{1/2}$. Now, this is over F^n. Now, you are, let me consider another, Example-2: - That is a space of say continuous function say. Let V be the space of real or complex valued continuous function on the close interval [0, 1]. And let the < > is V be defined by for any f & g ∈ V. In < f, g > = $\int_0^1 f(x) \, \overline{g(x)} \, dx$.

Now, according to this Cauchy-Schwarz inequality so, using Cauchy Schwarz inequality, we have, $| \int_0^1 f(x) \, \overline{g(x)} \, dx \,| \leq (\int_0^1 |f(x)|^2 \, dx)^{1/2} \; (\int_0^1 |g(x)|^2 \, dx)^{1/2}$

So, we will discuss more problems in my next classes. So, I hope the concept of this inner product space and some properties in inner product space is clear.

(Refer Slide Time: 26:24)

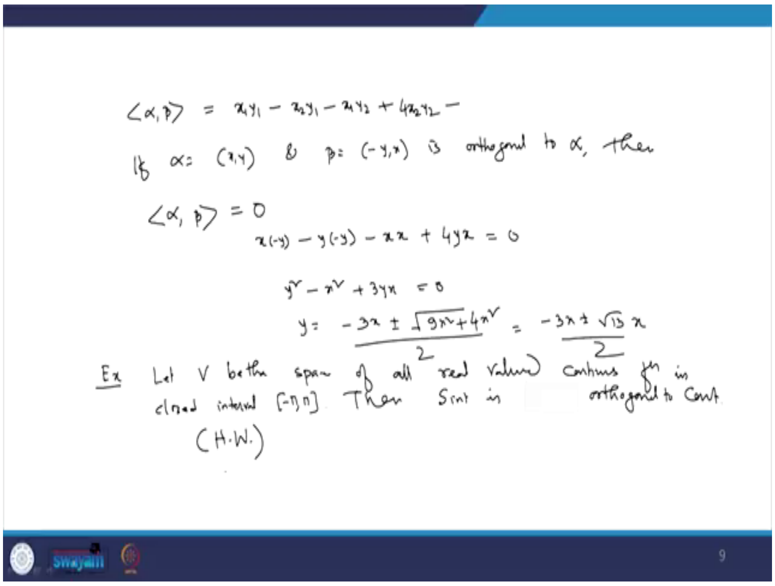


Now, let me introduce one more terminology interest in terminology called Orthogonal vectors: - Let V be an inner product space(i.p.s.). Let α & β we any two vectors in V, β is said to be to be orthogonal to α provide it $<\alpha,\beta> = 0 \implies \alpha$ is also orthogonal to β.

For Example: - Let me consider, V = R^2 and let me consider, the standard inner product one as < > is a standard inner product. So, let α = (x, y), then β = (-y, x) will be orthogonal to α.

Since, $<\alpha,\beta>$ = x(-y) + y(x) = 0. If I consider you are the same space, let me change the inner product. Let < > is given by this that let me consider say let $\alpha = (x_1, x_2)$ & $\beta = (y_1, y_2)$

(Refer Slide Time: 29:33)



And let me define $< \alpha, \beta >$ = x_1y_1- $x_2y_1 - x_1y_2 + 4x_2y_2$. Already we have seen this introduce an inner product over the R^2. Now, if $\alpha = (x, y)$ & $\beta = (-y, x)$ is orthogonal to α. Then will have, $< \alpha, \beta >$ = 0 but in this case, what I am getting here, if I consider that as in the case of the previous one standard inner product space.

That are for $= (x, y)$ & $\beta = (-y, x)$. Now, here $\beta = (-y, x)$ will be orthogonal, provided let me see what the conditions are. See this, $< \alpha, \beta >$ = 0 $\Rightarrow -xy + xy - xx + 4yx = 0$ $\Rightarrow$ $y^2 - x^2$ +3xy = 0 $\Rightarrow y = \frac{-3x \pm \sqrt{9x^2+4x^2}}{2} = \frac{-3x \pm \sqrt{13}x}{2}$

(()) (32:19) So, y will definitely related by x in this way again we can see over the space of say continuous function. Another let me take another Example: - Let V be the space of all real valued continuous function in closed interval say [$-\pi$, π] then sint is orthogonal to cost. So, this you can check it as a homework. So, we will discuss more about this orthogonal sets and orthogonal vectors in our next class.

So, we see that the two vectors if they are orthogonal, they have to satisfy certain criteria and then again we have seen that over the space of real continuous function. In the closed interval and [$-\pi$, π] the function sint is orthogonal to cost or cost is orthogonal to sint. So, we will discuss more on this orthogonal sets and some properties of this in our next class.