Advanced Linear Algebra
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Lecture – 43
Inner Products - II

Welcome to the lecture series on Advanced Linear Algebra. In my last lecture, I introduced the concept of Inner Product for different vector spaces. To understand the concept of length of a vector let me introduce one more terminology called norm of a vector.

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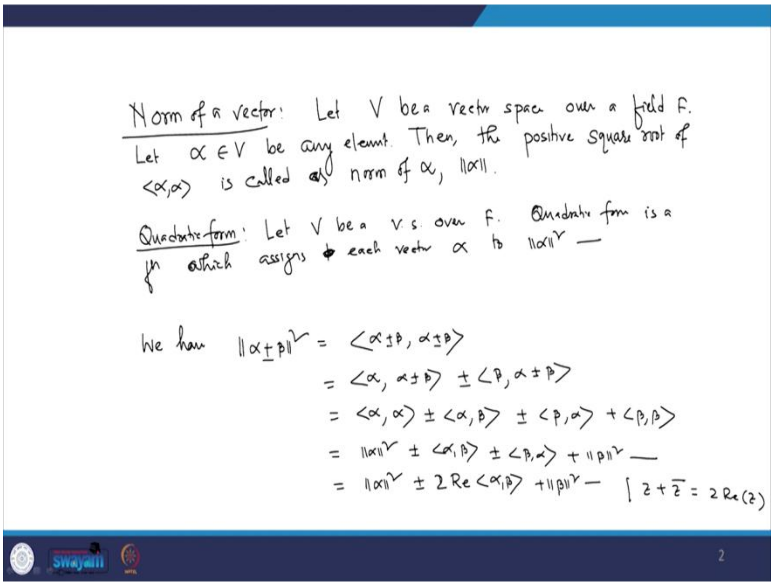

What is that? Let V be a vector space over a field F so, the field here either real number or complex number. Let $\alpha \in V$ be any element then, the positive square root of $< \alpha , \alpha >$ is called as norm of alpha, $\| \alpha \|$ is like this. This $\| \alpha \|$ it can be considered as the length of the vector α.

Let me also define another terminology called quadratic form:- Let V be a vector space over field F. Quadratic form is a function which assign each vector α to $|| \alpha||^2$. Now, using this terminology, norm or quadratic form, we will see the characteristics of the inner product. We have $|| \alpha \pm \beta||^2 = < \alpha \pm \beta, \alpha \pm \beta > = < \alpha, \alpha \pm \beta > \pm < \beta, \alpha \pm \beta > = < \alpha, \alpha > \pm < \alpha, \beta > \pm < \beta, \alpha > + < \beta, \beta > = || \alpha||^2 \pm < \alpha, \beta > \pm < \beta, \alpha > + || \beta||^2 = || \alpha||^2 \pm 2Re < \alpha, \beta > + || \beta||^2$

Where $< \alpha, \beta > + < \beta, \alpha >$ is basically I mean if I consider z is a complex number that $z + \bar{z}$ = 2 Re(z).

Because z + z = 2Re(z) or x + iy + x – iy = 2x.

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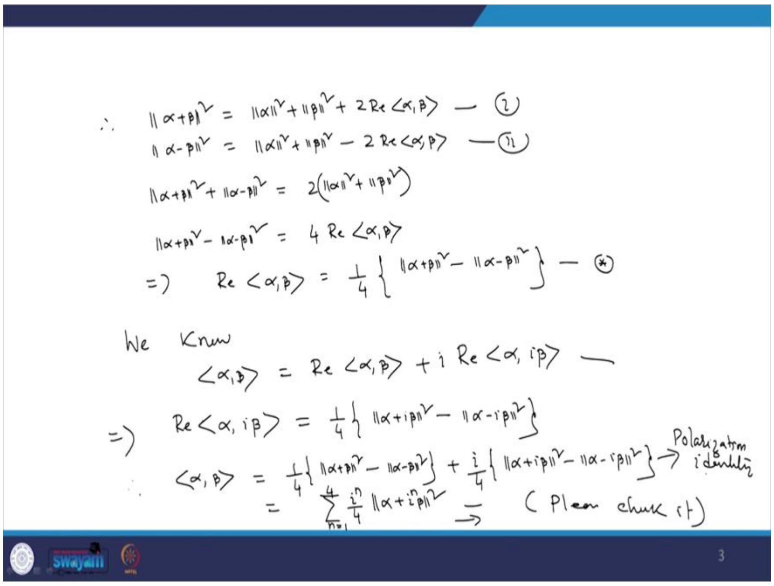

So, we are getting $||\, \alpha + \beta||^2 = ||\, \alpha||^2 + ||\, \beta||^2 + 2Re < \alpha, \beta > \longrightarrow (i)$ & $||\, \alpha - \beta||^2 = ||\, \alpha||^2 + ||\, \beta||^2 - 2Re < \alpha, \beta > \longrightarrow (ii)$ then from adding (i) & (ii) then $||\, \alpha + \beta||^2 + ||\, \alpha - \beta||^2 = 2(||\, \alpha||^2 + ||\, \beta||^2)$ & from subtract (i) & (ii) then $||\, \alpha + \beta||^2 - ||\, \alpha - \beta||^2 = 4Re < \alpha, \beta > \implies Re < \alpha, \beta > = \frac{||\, \alpha+\beta||^2 + ||\, \alpha-\beta||^2}{4} \longrightarrow (*)$. We also know $< \alpha, \beta >$ = Re$< \alpha, \beta >$ + iRe $< \alpha, i\beta > \implies$ Re $< \alpha, i\beta >$ $= \frac{||\, \alpha+i\beta||^2 - ||\, \alpha-i\beta||^2}{4}$. So finally we see that $< \alpha, \beta >$ = $\frac{||\, \alpha+\beta||^2 - ||\, \alpha-\beta||^2}{4} + + \frac{i\{||\, \alpha+i\beta||^2 - ||\, \alpha-i\beta||^2\}}{4} = \sum_{i=1}^{4} \frac{i^n}{4} ||\, \alpha + i^n \beta||^2$

Please check it. And this is called the polarization identity. So, we have seen the relation between the inner product and norm or quadratic form also. Now, definitely we shall utilize this relation to understand the inner product more deeply. But before that we have a one question suppose in a vector space V of dimension say n the finite dimensional vector space is given to us.

And suppose an order basis is given to us in that vector space is it possible to express the inner product which define that vector space in terms of matrix.

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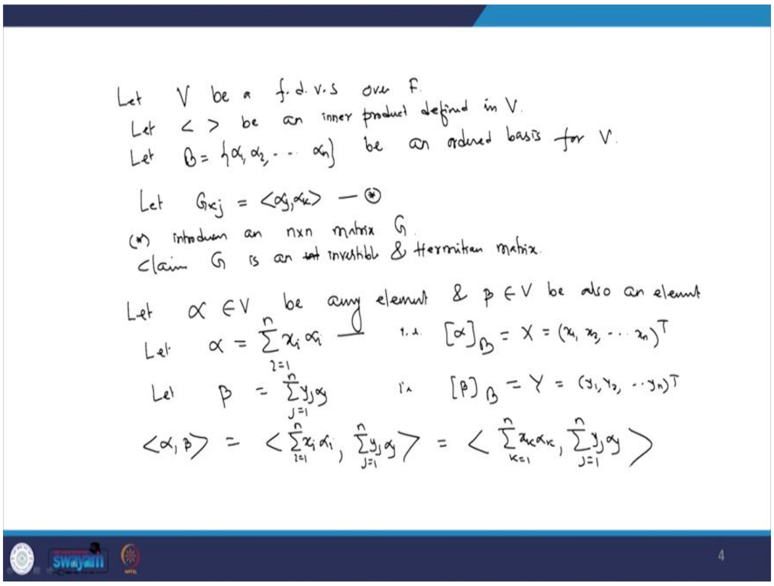

Let me consider like this let V be a finite dimensional vector space over the field F. Let $< \ >$ be an inner product and inner product defined in V. It is given to us let B = $\{\alpha_1, \alpha_2, \ldots \ \alpha_n\}$ be an order basis. Even if I do not give the order basis up to only first two lines that V, be a finite dimensional vector space about the field and an inner product is defined about vector space V.

My question is it possible to express the inner product in terms of matrix concept? So, answer is yes, it is possible, provided an order of basis is given to us. Suppose the an order basis of the B is given to us so, in that case will be able to express the inner product of the vector space V explicitly by matrix. So, let me clarify this claim. So, let B be an order basis for vector space V.

Let $G_{\alpha_j} = < \alpha_j, \alpha_k > \ \longrightarrow (*)$ because inner product is given to us. So, I shall utilize that given another product to define a scale at G_{α_j} like this. So now, for different values of j and k, j varying from 1 to n, k varies from 1 to n. I will have a matrix G of nxn order. I will show that the inner product, what is given to us in the vectors V can be explicitly defined in terms of this matrix G.

And this matter will have some typical characteristics that is, it is Hermitian and invertible. So, let us see all this thing. So, (*) introduced says and matrix and nxn matrix G, claim G is an invertible and Hermitian matrix. Let $\alpha \in$ V be any element and $\beta \in$ V be also an element. Let $\alpha = \sum_{i=1}^{k} x_i \, \alpha_i$ i.e. $[\alpha]_B$= X = $(x_1, x_2 \ldots x_n)^T$.

And let $\beta = \sum_{j=1}^{k} y_j\, \alpha_j$ i.e. $[\beta]_B$= X = $(y_1, y_2 \dots y_n)^T$. So now, $< \alpha, \beta >$ $= < \sum_{i=1}^{k} x_i\, \alpha_i,$ $\sum_{j=1}^{k} y_j\, \alpha_j > = < \sum_{i=1}^{k} x_k\, \alpha_k,\ \sum_{j=1}^{k} y_j\, \alpha_j >$ where i as a square root of –1.

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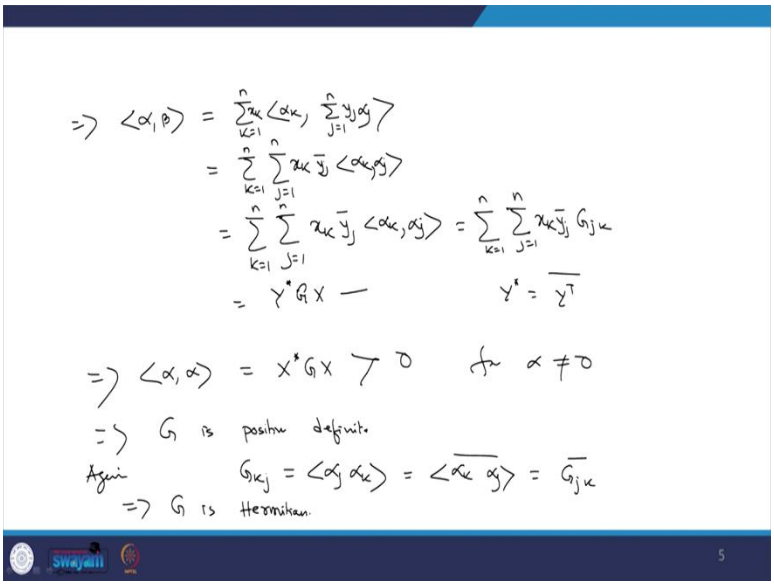

$\Rightarrow \quad < \alpha, \beta > \;=\; \sum_{k=1}^{n} x_k \; < \alpha_k,\ \sum_{j=1}^{n} y_j\, \alpha_j > \;=\; \sum_{k=1}^{n} \sum_{j=1}^{n} x_k\, \overline{y_j} \; < \alpha_k,\ \alpha_j > \;=$ $\sum_{k=1}^{n} \sum_{j=1}^{n} x_k\ \overline{y_j}\ < \alpha_k, \alpha_j > = \sum_{k=1}^{n} \sum_{j=1}^{n} x_k\ \overline{y_j}\, G_{jk} = \mathrm{Y}^* GX,$ where $\mathrm{Y}^* = \overline{\mathrm{Y}^T}$

Because this is scalar quantity and dimension one and we can see by if I consider y equal to what I have considered here y is a column. The transpose mean by signal row then this multiplier G is nxn, matrix and X column. I will have a basically dimension as 1x1 matrix are this is scalar quantity. So, we see $< \alpha, \alpha > = \mathrm{X}^* GX \ > 0$ for $\alpha \neq\ 0$.

So, $\Rightarrow$ G is positive definite. Again, $G_{kj} = < \alpha_j, \alpha_k >$ = $\overline{< \alpha_k, \alpha_j >}$ = $\overline{G_{jk}}$ $\Rightarrow$ G is Hermitian.

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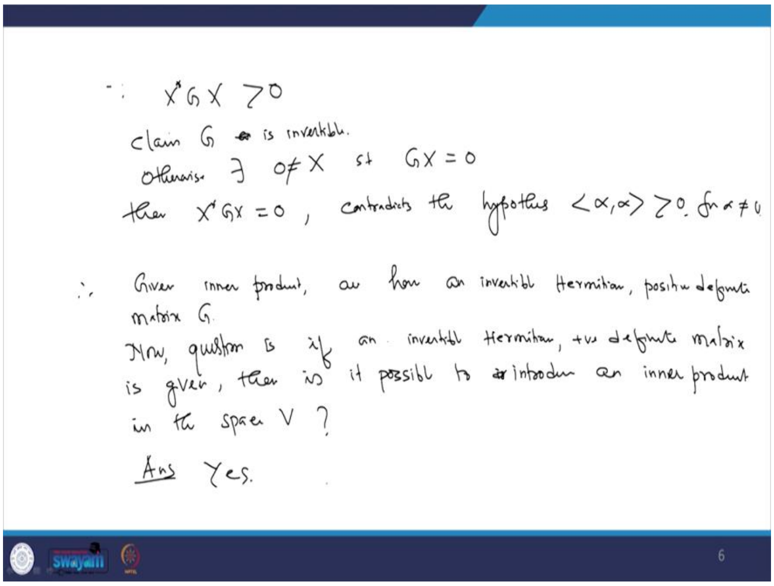

Since, $X^{*}GX > 0$. Claim G will be invertible, otherwise $\exists\ 0 \neq X$ s.t. GX = 0. And then your $X^{*}GX = 0$ contradicts the hypothesis $< \alpha, \alpha > 0$ for $\alpha \neq 0$. So, therefore, G is invertible, so, we have seen G is invertible, G is Hermitian. And even the G_{ii} I mean G_{jj} all the diagonal entry will be also positive quantity.

And this is also positive definite, so, we see that for a given inner product, we are able to produce a matrix nxn matrix, invertible matrix which is also Hermitian and positive, definite matrix. Now, the question is if an invertible Hermitian matrix which also positive definite is given to us, is it possible to introduce an inner product in this space? So, from this I can see so, given inner product, we have an invertible Hermitian, positive definite matrix G.

Now, question is if an invertible Hermitian, positive definite matrix is given to us then is it possible to introduce an inner product in this space V, answer is yes. So, when it order basis is given to us then how to define the inner product will define the inner product like this.

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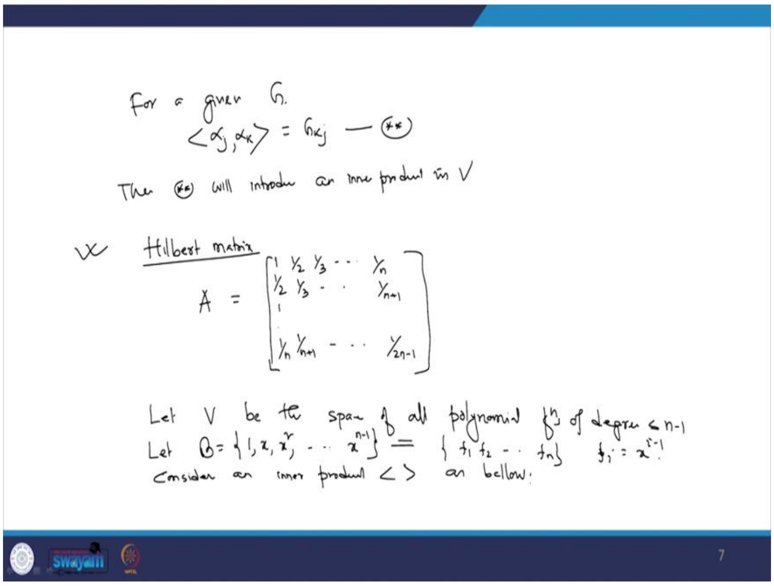

For a given G you can define $<\alpha_j, \alpha_k> G_{kj} \longrightarrow (**)$ So, if I define this is the functions then definitely these functions will introduce then (**) will introduce an inner product in V. So, when this G is Hermitian positive definite and invertible matrix is given to us. So, in that case we can define the inner product in the space as like this.

Now, we shall utilize this concept to understand one nice problems that is called Hilbert matrix. So, let me consider Hilbert matrix, Hilbert matrix is defined, like this A

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}.$$

Now, this is a standard nxn Hilbert matrix now this matrix immediately we see that it is symmetric matrix.

Say, what I have done it here? If the field complex number is replaced by real number then I will have G, as symmetry matrix instead of Hermitian. So now, question is can I use this concept? That G is in has to be invertible when G is basically coming from as a consequence of for a given inner product. So now, suppose a is like this matrix, so, it is also symmetry matrix.

Is it possible to show that this matrix is also coming as a consequence of a given inner product on a given vector space? If it is then immediately, I can say a will, be an invertible matrix. So now, let us see what is that space? And what is the inner product which will give this matrix? Let V be the space of all polynomial functions of degree ≤ n – 1.

So, let B = $\{1,\ x,\ x^2,\ x^{n-1}\}$ be the standard order basis in V. So, I have taken vector space of dimension n. Accordingly, I have chosen an order basis B = $\{1,\ x,\ x^2,\ x^{n-1}\}$. Now, consider an inner product so, let me define inner product < >.

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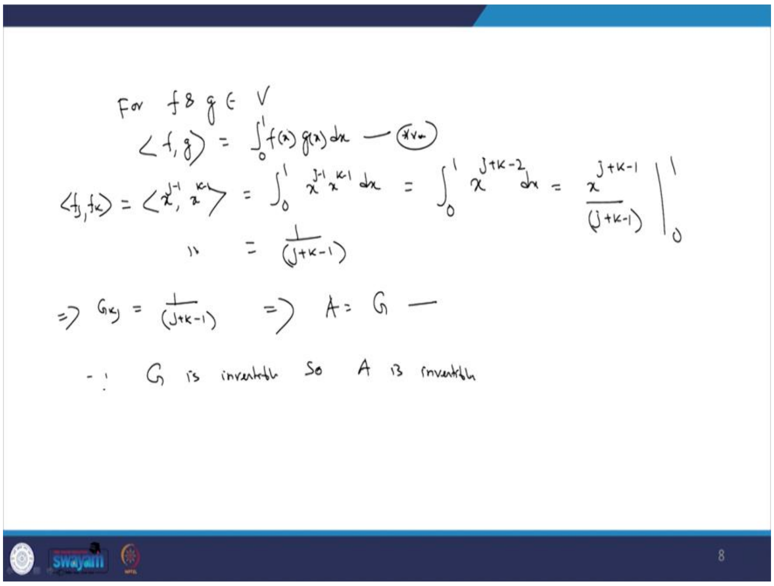

For f & g ∈ V, $< f, g > \ = \int_0^1 f(x)\, g(x)\, dx \ \longrightarrow (***)$ so, let me define this one. Let me consider $< f_j, f_k > \ = \ < x^{j-1}, x^{k-1} > \ = \int_0^1 x^{j-1}\, x^{k-1}\ dx \ = \int_0^1 x^{j+k-2}\ dx \ = \{\frac{x^{j+k-1}}{j+k-1}\}_0^1 \ = \frac{1}{j+k-1}$. Actually I can give the name B = $\{1,\ x,\ x^2,\ x^{n-1}\} = \{f_1, f_2, \ldots f_n\}$, so, $f_n = x^{i-1}$. So, the based on that I can write like this thing.

This implies that if I consider $G_{k_j} = \frac{1}{j+k-1}$. Say, if I compare the Hilbert matrix, see k_j-th entry. So, let me consider the second entry of second row that is $\frac{1}{3}$, G_{22} is equal to how much? $\frac{1}{3}$ here this is G_{22} according this formula, $\frac{1}{2+2-1}$ that again $\frac{1}{3}$. So, you can check that G_{k_j} is equal like this thing. So, this implies that your, A = G, since G is invertible matrix, so, A is invertible.

So, from this we can easily show that the Hilbert matrix is invertible. She will see many applications of this concept to understand more details about inner product. We have to basically solve the many problems which are given the exercise. And will see that many interesting problems can be handled with the help of inner products. I will continue more at this topic in our coming lectures also.