Advanced Linear Algebra
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Lecture – 42
Inner Products - I

Welcome to lecture series in Advance Linear Algebra. We have studied in physics, even in 11, 12 in **(()) (00:33)** and mathematics also. A vector quantity over the space R^3 as a quantity having magnitude as well as directions.

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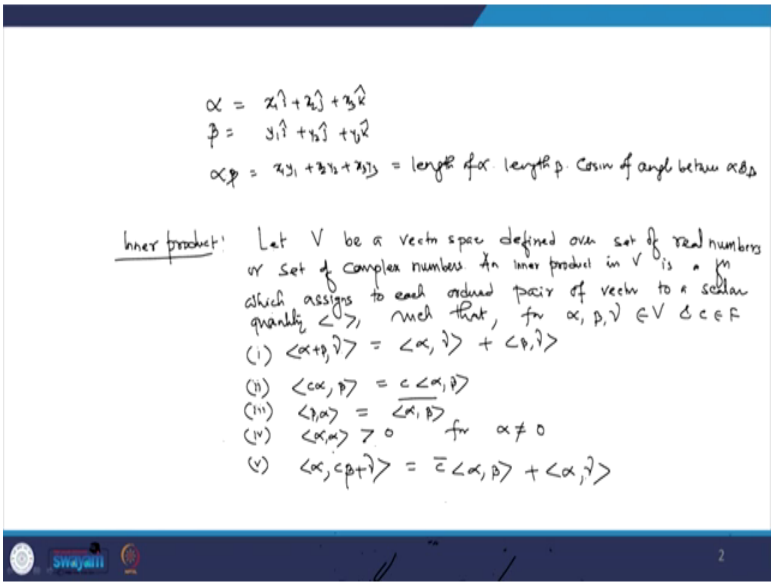

I mean to say, if you consider say $\alpha = x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k}$ and $\beta = y_1\hat{\imath} + y_2\hat{\jmath} + y_3\hat{k}$. Then we have seen that scalar product of the two vectors $\alpha\beta = x_1y_1 + x_2y_2 + x_3y_3 =$ length of α . length of β. cosine of angle between α & β. So that is what we have learned in our in 11th, 12th.

For the case of the I mean dot product of two vectors where we see that vector quantity having magnitude as well as angle. With a similar concept, can be introduced over our generalized vector space or not. Similar concept can be also introduced for the vector space by introducing the concept of inner product. What is inner product? Inner product before talking about the inner product here also.

We have to say, though we will basically restricted to our angle as perpendicular of these two vectors or orthogonality of the two vectors. So, let me define what is inner product on a vector space? Let V be a vector space defined over set of real numbers or set of complex number. I mean to say, the field is basically either real number or complex number. And inner product in V is a function which assign to each order pair of vectors to a scalar quantity $< \ >$.

such that for $\alpha, \beta, \gamma \in$ V and c $\in$ F. I mean F it is basically a field over which vector space is defined. We have to consider, if maybe real number or maybe complex number. So, it satisfied (i) $< \alpha + \beta, \gamma > = < \alpha, \gamma > + < \beta, \gamma >$

This is called linear property of the inner product. (ii) $< c\alpha, \beta > = c < \alpha, \beta >$ (iii) $< \beta, \alpha > = \overline{< \alpha, \beta >}$ (iv) $< \alpha, \alpha > > 0$ for $\alpha \neq 0$. And based on the action (i), (ii), (iii), we have $< \alpha, c\beta + \gamma > = \bar{c} < \alpha, \beta > + < \alpha, \gamma >$

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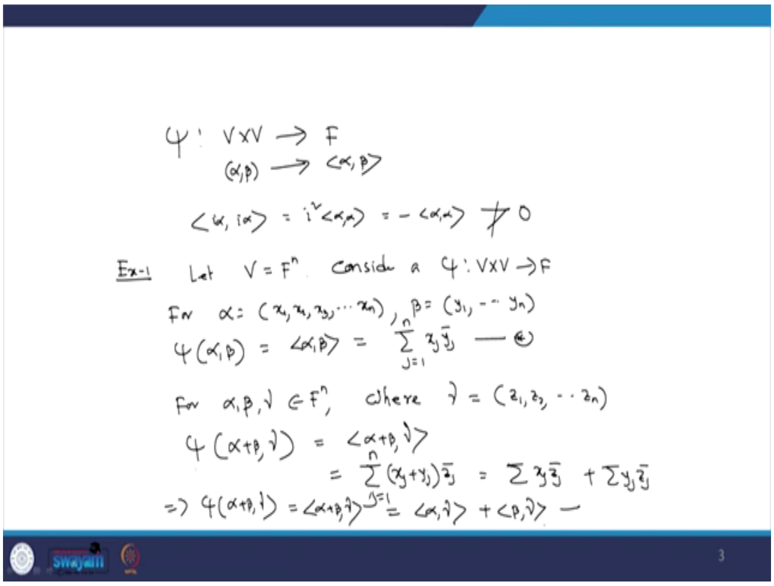

So, here we have seen definition of the inner product like this. Say, suppose a function, φ: VxV $\rightarrow$ F which is mapping $(\alpha, \beta) \rightarrow < \alpha, \beta >$. So, this is the basically inner product I am defining here So, based on this, you can see one thing that if you do not consider (ii) $< \beta, \alpha > = \overline{< \alpha, \beta >}$.

Then this (iv) axiom will not remain valid. When you consider the vector space over the complex number for examples inner product of say $< i\alpha, i\alpha > =$ i$^2 < \alpha, \alpha > = - < \alpha, \alpha > \ngtr 0$ where α is vector, i the imaginary quantity.

So, therefore, to satisfy the fourth axioms, we need the third axiom. So now, let me consider some examples, so, let us consider a very trivial example in the sense of our usual space that is space of n-tuples, let V = F^n. Consider φ: VxV → F for $\alpha = (x_1, x_2, \dots x_n)$, $\beta = (y_1, y_2, \dots y_n)$. Let me define the inner product $\varphi(\alpha, \beta) = < \alpha, \beta > = \sum_{j=1}^{n} x_j \, \overline{y_j}$.

Then claim this function basically introduce an inner product over the space V = F^n. So, to prove that once I have to prove that it is satisfied, the all this first three axioms as well as fourth one also. Because fifth one is coming as a consequence of first three. So, I do not want to do for fifth one.

So, (i), (ii), (iii) and (iv) so, first linear property say we have for $\alpha, \beta, \gamma \in$ F^n $\gamma = (z_1, z_2, \dots z_n)$. Now, if I consider $\varphi(\alpha + \beta, \gamma)$ = $< \alpha + \beta, \gamma > = \sum_{j=1}^{n} (x_j + y_j) \, \overline{z_j} = \sum_{j=1}^{n} x_j \, \overline{z_j} + \sum_{j=1}^{n} y_j \, \overline{z_j}$ $\Longrightarrow \varphi(\alpha + \beta, \gamma) = < \alpha + \beta, \gamma > = < \alpha, \gamma > + < \beta, \gamma >$. So, linear property holds good.

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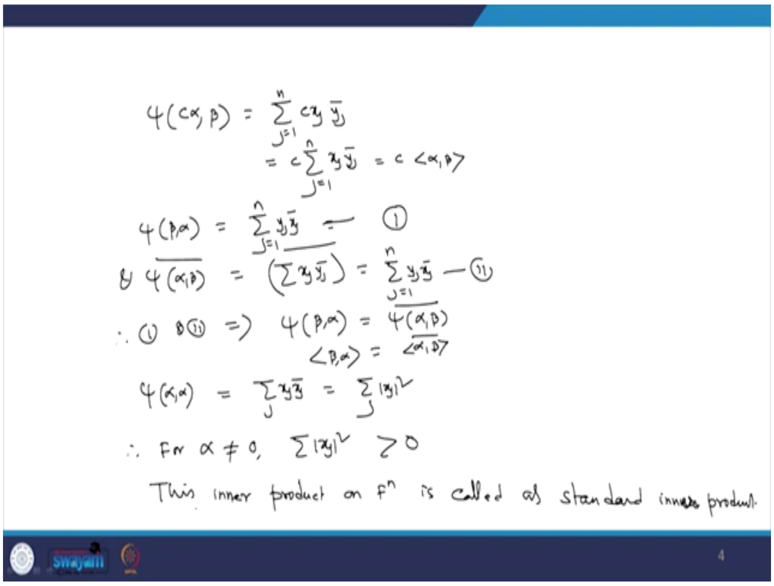

Second one also, you can see it easily and $\varphi(c\alpha, \beta) = \sum_{j=1}^{n} c x_j \, \overline{y_j}$ = $c \sum_{j=1}^{n} x_j \, \overline{y_j}$ = $c < \alpha, \beta >$ So, second, is also holds good. Now, $\varphi(\beta, \alpha) = \sum_{j=1}^{n} y_j \, \overline{x_j}$ −(i) & $\overline{\varphi(\alpha, \beta)} = (\overline{\sum_{j=1}^{n} x_j \, \overline{y_j}}) = \sum_{j=1}^{n} y_j \, \overline{x_j}$ −(ii). So (i) & (ii) $\Longrightarrow \varphi(\beta, \alpha) = \overline{\varphi(\alpha, \beta)}$ i.e. $< \beta, \alpha > = \overline{< \alpha, \beta >}$ and fourth one $\varphi(\alpha, \alpha) = \sum_j x_j \overline{x_j} = \sum_j |x_j|^2$ so for $\alpha \neq 0$, $\sum_j |x_j|^2 > 0$.

So, it introduce an inner product over the space F^n. So, this inner product on F^n is called as standard inner product.

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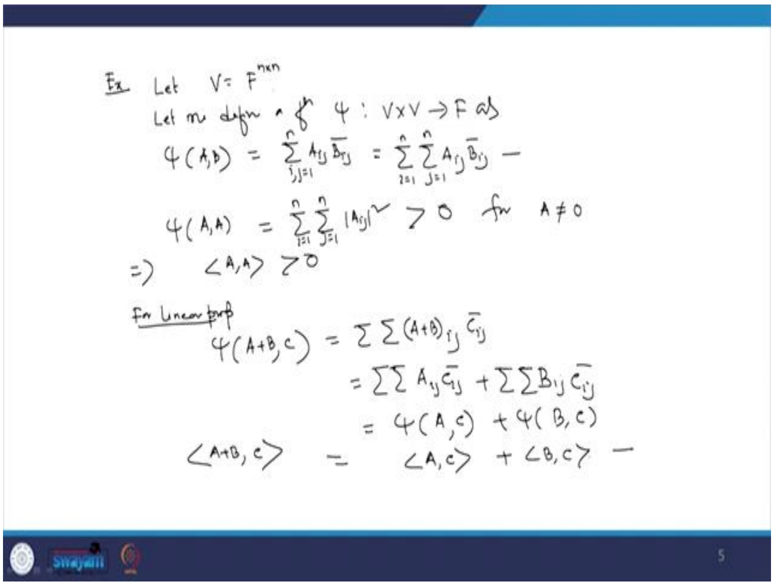

Let me take another examples, so, let me consider space as set of all nxn matrix. Let V = F^{nxn} that is set of all nxn matrix over the field F. Let me define a functions φ: VxV → F as φ (A, B) = $\sum_{i,j=1}^{n} A_{ij}\,\overline{B_{ij}}$ = $\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\,\overline{B_{ij}}$. So you can quickly see that this function φ also introduces an inner product over V.

φ(A,A) = $\sum_{i=1}^{n}\sum_{j=1}^{n} |A_{ij}|^2 > 0$ for A $\neq$ 0 $\Longrightarrow$ $< A, A > \; > 0$. So, it satisfied that fourth condition and we can see linear property also holds good. For linear property we have φ(A+B, C) = $\sum\sum (A+B)_{ij}\overline{C_{ij}}$ = $\sum\sum A_{ij}\overline{C_{ij}} + \sum\sum B_{ij}\overline{C_{ij}}$ = φ(A, C) + φ(B, C)

So, $< A + B, C \; > = < A, C > + < B, C >$. So, linear property also holds good and you can also check $< cA, B >$ = c$< A, B >$ that we can prove it easily from the different vector space. So, it also given inner product over the same space.

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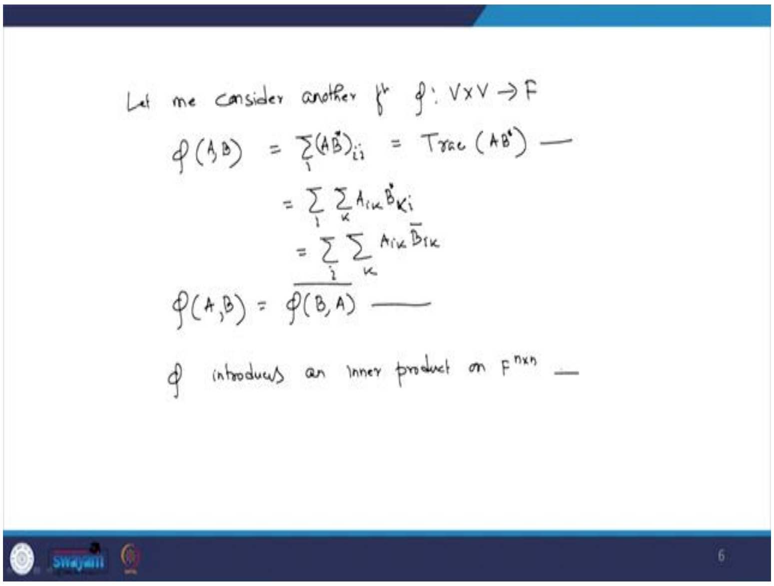

Now over the same space let me consider another function over the same, let me consider another function say φ: VxV $\rightarrow$ F defining by $\varphi(\mathrm{A},\ \mathrm{B})\ =\ \sum_i (A\mathrm{B}^*)_{ij} = \mathrm{Trac}(A\mathrm{B}^*) = \sum_i \sum_k A_{ik}\mathrm{B}^*{}_{ki} = \sum_i \sum_k A_{ik}\,\overline{B_{ik}}$

So, this implies that $\varphi(\mathrm{A},\mathrm{A})$ will be also positive definite quantity. If A $\neq$ 0 and you will also see that it satisfy linear property. It also satisfied that is constant I mean second axioms and it also satisfied the third axioms $\varphi(A, B)\ =\ \overline{\varphi(B, A)}$. One you can check $(A, B)\ =\ \overline{\varphi(B, A)}$. So, as I did for the first example over the same space, this same thing can we do for this? It can be used for this function also.

And here also, you can show that this φ introduces an inner product on your space as a V = F^{nxn}. So, we have taken space as n-tuple space of all nxn matrix and we define the different some in inner product.

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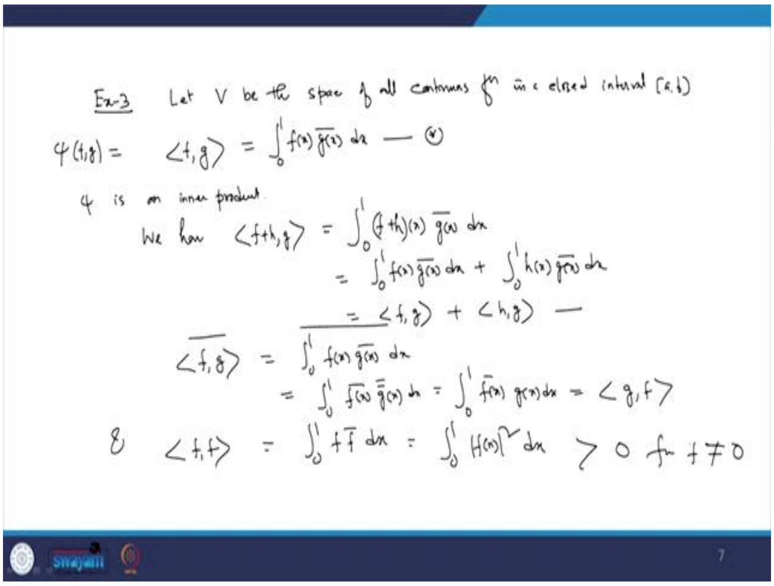

Now, let me take some one example on different spaces So, this will be Example-3:- Let V be the space of all continuous functions in a closed internal (a, b). So, it may be real, valued or complex value depending on the field. So, let me consider in general as a complex, valued function place. So now, I will introduce the inner product as like this notations which is usual functions which mapping from VxV to F. So, I am writing now, for sake of simplicity like this that for any element f and g if

φ(f, g) = $< f,\ g > = \int_0^1 f(x)\,\overline{g(x)}\,dx$. So, this is an interesting example in our mathematics, specifically in analysis which will be used in different cases. Where φ is an inner product space I mean I have to again to check whether it is inner product or not I have to check all the four axioms, linear axioms. we have $< f+h,\ g > = \int_0^1 (f+h)(x)\,\overline{g(x)}\,dx = \int_0^1 f(x)\,\overline{g(x)}\,dx + \int_0^1 h(x)\,\overline{g(x)}\,dx = < f, g > + < h, g >$

So, linear property holds good. Similarly, second property also holds good. You can check it and let me check the third ones. Third one is also very trivials $\overline{< f, g >} = \overline{\int_0^1 f(x)\,\overline{g(x)}\,dx} = \int_0^1 \overline{f(x)}\,\overline{\overline{g(x)}}\,dx + \int_0^1 \overline{f(x)}g(x)\,dx = < g, f >$

So, third 1 also holds good and fourth one $< f, f > = \int_0^1 f\,\bar{f}\,dx = \int_0^1 |f(x)|^2\,dx > 0$ for f $\neq 0$.

Since this is a positive definite functions and you are integrating over the integral 0 to 1. Certainly, it will be greater than 0 for f $\neq$ 0. So, it satisfied all the axioms of the definition of inner product.

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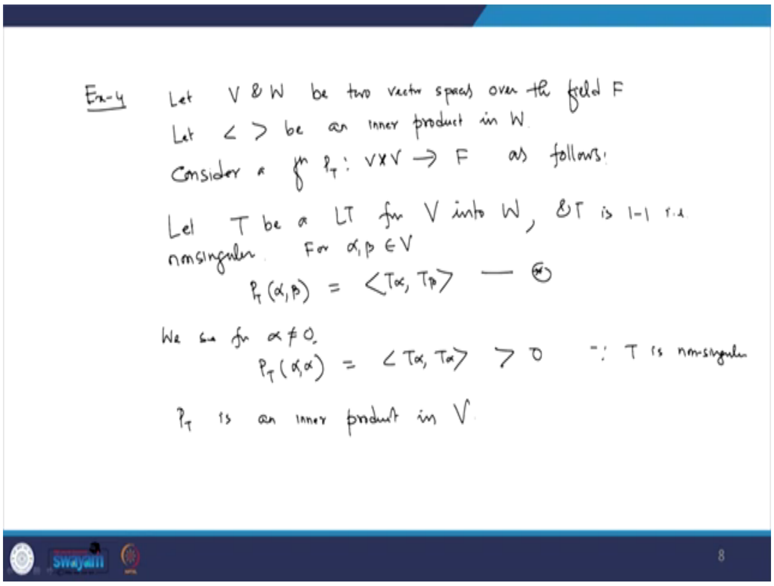

Now, let me introduce another Examples-4. Let V & W be two vector spaces over the field F. Now, let $<\ >$ be an inner product in W but nothing is given about the V. Now, let me define a functions consider function p_T: VxV $\rightarrow$ F as follow. Let T be a linear transformations from V into W and T is 1-1 i.e. that is non-singular.

Now, let me define $p_T(\alpha, \beta)$ where $\alpha, \beta \in$ V such that $p_T(\alpha, \beta) = < T_\alpha, T_\beta >$. I am defining a functions p_T: VxV $\rightarrow$ F which is defining for which is mapping any order pair vector of V to a scalar quantity $< T_\alpha, T_\beta > \in$ F. Because since this inner product given in W so, I can define it like this.

So, we can check we see for $\alpha \neq 0$, $p_T(\alpha, \alpha) = < T_\alpha, T_\alpha > > 0$. Because T is non-singular so, $T_\alpha \neq 0$. So, therefore, this is non-zero, you can also check that it satisfied all the other axioms because this is definition itself is saying inner product in some W. It is basically introducing an inner product in V.

So, p_T is an inner product in V. So, it introduces an inner product in V. So, from a given another product in W we are able to introduce an inner product in V.

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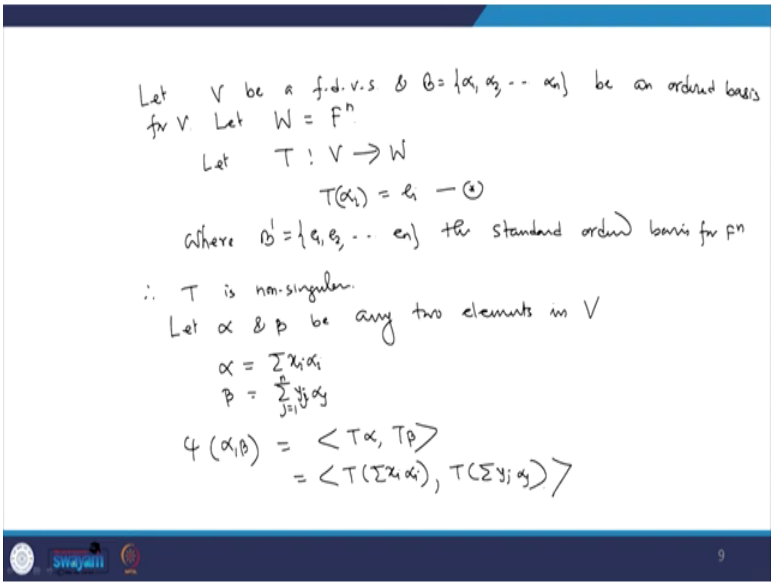

So, if I consider let V be a finite dimensional vector space and B = $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$ be and order basis for this in order basis for V. Let W = F^n. So that one can have 1-1 map **(()) (31:29)** between the V and W. So, let **(()) (31:35)** T: V → W. Also, let me consider that $T(\alpha_i) = e_i$, where B′ = $\{e_1, e_2, \ldots\ e_n\}$.

The standard order basis for F^n, I have chosen the standard order basis in W. So, W = F^n. Now, I am defining a linear transformation T from V to W, $T(\alpha_i) = e_i$. So, certainly so, T is non-singular. Let $\alpha,\ \beta \in$ V. Since I have taken B = $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$ as a basis. So, there is a scalar $x_1, x_2, \ldots x_n$ such that $\alpha = \sum_{i=1}^{n} x_i\, \alpha_i$.

And similarly, for β there exists n scalar, $y_1, y_2, \ldots y_n$ such that $\beta = \sum_{j=1}^{n} y_j\, \beta_j$. Now, if I consider $\varphi(\alpha, \beta) = <\mathrm{T}\alpha, \mathrm{T}\beta> = <\mathrm{T}\sum_{i=1}^{n} x_i\, \alpha_i,\ \mathrm{T}\sum_{j=1}^{n} y_j\, \beta_j>$ where i consider the inner product in F^n as a standard inner product.

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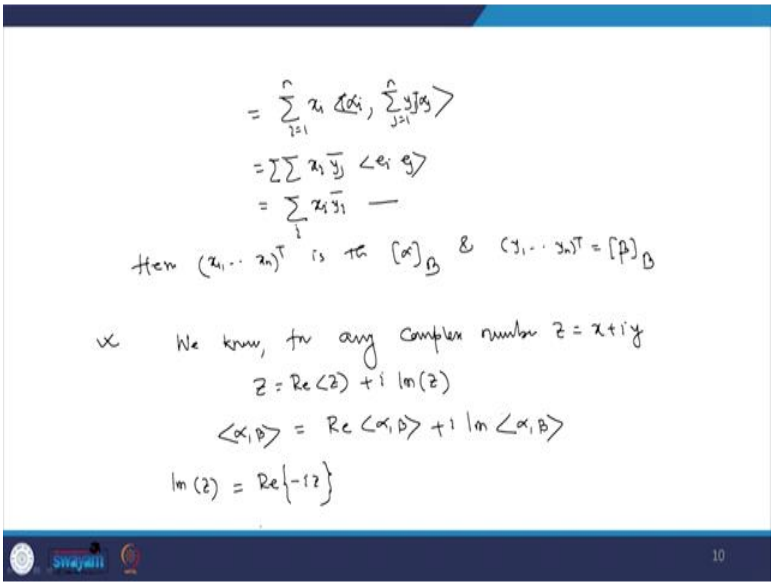

= $\sum_{i=1}^{n} x_i < T\alpha_i, \sum_{j=1}^{n} y_j T\beta_j >$ $= \sum\sum x_i \overline{y_j} < e_i, e_j >$ = $\sum x_i \overline{y_j}$ where $T\alpha_i = c$ and $T\alpha_j = e_j$ and $e_i e_j$ = 1 when i =j, $e_i e_j$ = 0 otherwise.

Here $(x_1, x_2, \dots x_n)^T$ is the $[\alpha]_B$ & $(y_1, y_2, \dots y_n)^T$ = $[\beta]_B$. So, we are getting an inner product from the information that an inner product is defined over the W, W = F^n.

From that I am also able to introduce an inner product over space V. So, this is the definitions and some examples of the inner product. We will talk more about this inner product in our coming in next lectures but before con completing this today's lecture, let me define one things. Say we know for any complex number z = x + i y. So, this implies that z = Re(z) + i im(z).

So, similarly, for any $< \alpha, \beta >$ = Re$< \alpha, \beta >$ + i im$< \alpha, \beta >$ over the complex field. But see im(z) = Re{–iz}. Because –iz means see we are basically looking over the imaginary part of z means y. So, y can be written as Re{–iz}.

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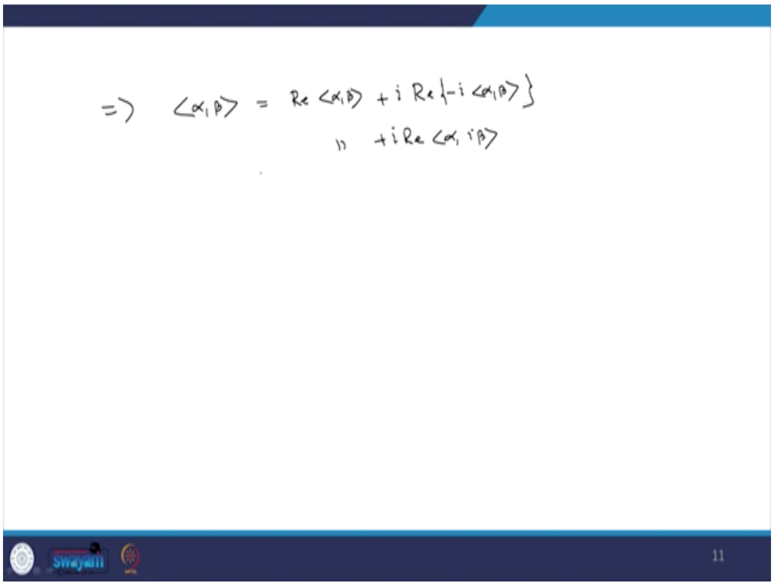

So, $\Rightarrow \; <\alpha,\beta> = Re<\alpha,\beta> + iRe\{-i<\alpha,\beta>\} = Re<\alpha,\beta> + iRe<\alpha,i\beta>$ Because from the property of $<\alpha,i\beta> = -i<\alpha,\beta>$. So, this is a nice algebraic characteristic of the complex number.

So, this I will use for to understand inner product more details which will discuss in the next class. So, we see that any complex quantity can be written as real part of that + i times imaginary part of that one. So, based on this principle for any complex valued $<\alpha,\beta> = Re<\alpha,\beta> + iRe<\alpha,\beta>$

But im$<\alpha,\beta>$ = Re $\{<\alpha,i\beta>\}$. So, this relation will utilize to understand more about the inner product is which will discuss in next class