Advanced Linear Algebra
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Lecture – 41
Applications of Primary Decomposition Theorem-III

Welcome to lecture series an Advanced Linear Algebra. I will continue with the Applications of the Primary Decompression Theorem. We have already seen when it linear operator to define over a finite dimensional vector space. When the minimal polynomial is product of linear factors then the operator is having a block diagonal matrix representations that is Jordan canonical form of the operator we have seen it.

That we have used that we have seen using the one interesting results. That is when you use the primary Decompression theorems. We have seen that any linear operator defined over a final dimensional vector space can be written as a sum of a diagonal matrix and a nilpotent matrix. Then we have use one interesting results related to the nilpotent operators. Like this I mean, if you consider T be a linear operator on a finite dimensional vector space V of index k.

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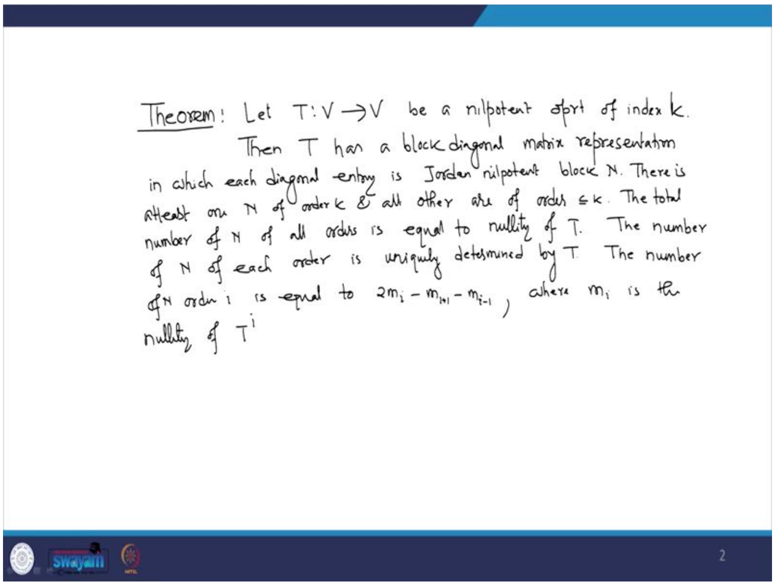

Then T has a block diagonal matrix representations in which each block entry, diagonal entry is Jordan nilpotent block and that will be denoted by say, N. It is like that and each entry will be basically Jordan nilpotent block. I already defined what do you mean by Jordan nilpotent block that is basically some sort of shifting operators we have defined. In that block, diagonal matrix representations, there will be at least one Jordan block nilpotent block N of index of order k.

Since the Jordan nilpotent block, if it is of the index k and is also have to be order k and all other entries will be $\leq$ k. The total number of Jordan nilpotent block N will be exactly equal to the nullity of the operator T. Specifically, the what should be the total number of Jordan nilpotent block of a specific order? Say order i that will be uniquely determined by relation that the number equal to $2m_i - (m_{i+1} + m_{i-1})$, where m_i is the basically nullity of the operator T^i.

So, based on this concept, one can have a block diagonal matrix representation of an nilpotent operator.

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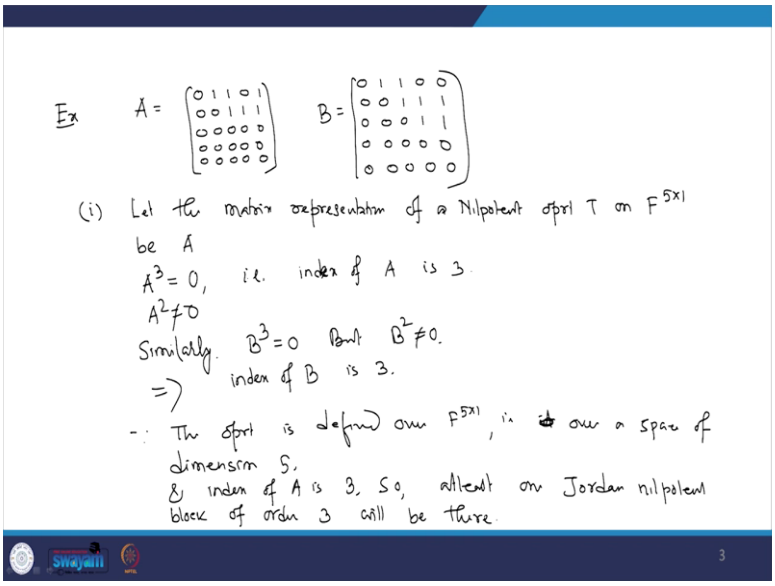

So, for examples if I consider here, I have taken into matrix A = $\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ & B = $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ it is basically you can say the matrix representation of an nilpotent operator say one is a T another is U. We have seen that we can see it here see let me consider first A let the matrix representations of a nilpotent operator T on F^{5x1} be A. Whereas I can consider say another operator whose also matrix representations with respect to standard order basis.

Of course, on F^{5x1} is B. Now, for the case of the A let us see how the operator T, whose matrix representation is A have a block diagonal matrix representations. So that we can this is again some sort of examples we are using to realize this to check the theorems what we have discussed right now? So, we see here first, we have to calculate the what is the nullity I mean index of the A.

We can see $A^3 = 0$ & $A^2 \neq 0$ i.e. index of A is 3. So, it is an nilpotent operator or matrix here. Similarly, you can check that $B^3 = 0$ but $B^2 \neq 0$ i.e. index of B is 3. So, I have taken two matrix A and B which are basically matrix representation up to operator with respect to standard order basis.

And we see here both of them have the nilpotent index is say both are having 3. Now, question is what the block diagonal matrix representation? So, here I have since the operator is defined over space F^{5x1} that is over a space of dimension 5 and index of A = 3. So, one nilpotent block of index of order 3 will be there, an index of A is 3. So, at least, one Jordan nilpotent block of order 3 will be there.

Now, the question is next question is how many Jordan nilpotent block will be there. That depends on the basically nullity of the your operator T or matrix A.

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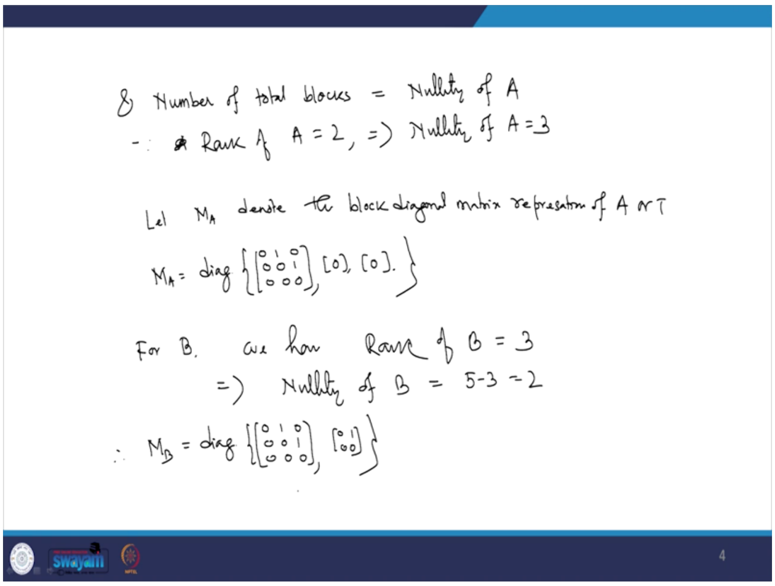

& Number of total blocks = Nullity of A. see A is having 2 non-zero rows which are also linear independent since Rank of A = 2 $\Longrightarrow$ Nullity of A = 3. So, there will be exactly 3 Jordan nilpotent blocks In the matrix representation of the A. So, let M_A denote the block diagonal matrix representations of A or T then $M_A = \{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, [0], [0] \}$ where one block will be of order 3 since the index of A = 3.

there will be two modes so, two modes is basically of index . I will say that I mean of order one and one only because the total number of block has to be exactly equal to 3. So, this is the basically block diagonal matrix representation of the operator T or matrix A. Now, what can you say about the matrix B? For B we have rank of B = 3.

So, $\Longrightarrow$ Nullity of B = 5 – 3 = 2. So, total number of block must be equal to exactly 2. So, this implies, if I denote the M_B denote the corresponding block diagonal representation of the matrix B. Then $M_B = \{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \}$ where diagonal entry has to be block of order 3. So, one will be like this and other one will be of order 2.

So, this is the block diagonal matrix representation of the B. Now, we have seen when an operator minimal polynomial is product of linear factors then we can have Jordan canonical representations. But what can we say when the minimal polynomial is not product of linear factors? How to handle that situation?

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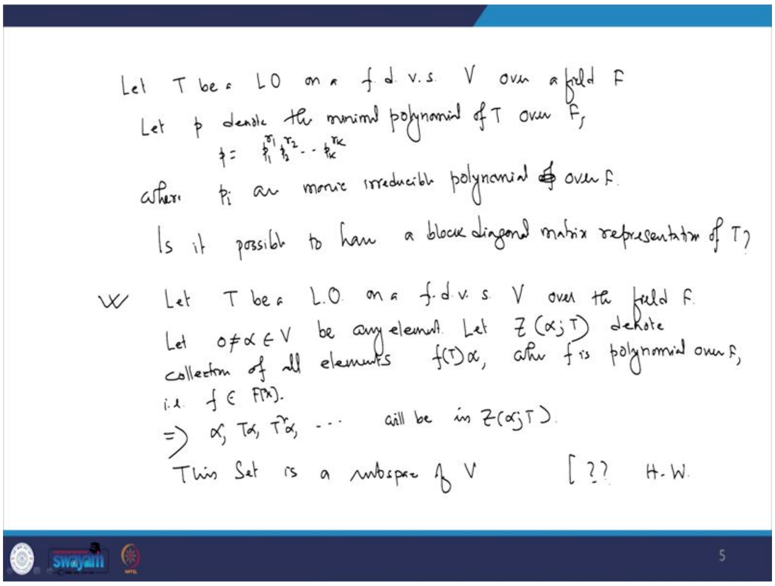

So, it is like this let T be a linear operator on a finite dimensional, vector space V over a field F. Let p denote the minimal polynomial of T over F where p = $p_1{}^{r_1}$ $p_2{}^{r_2}$ ... $p_k{}^{r_k}$ where p_i are monic irreducible polynomial over F. In this case, is it possible to have a block diagonal matrix representation of T? Answer is yes then how to do it?

According to primary decomposition theorems, you will have certainly a direct sum of independent subspaces. But what will be the independent subspaces? For this case we are going to discuss that one. So, for this, let me introduce a concept of cycling subspaces. Let T be a linear operator on a finite dimensional vector space V over the field F. Let $0 \neq \alpha \in$ V be any element.

Let Z(α;T) denote collections of all elements like f (T)α where f is polynomial function over the field F that is I can say, I mean f ∈ F[x], like this polynomial function like this. So, this means that if α ∈ Z(α;T) then Tα will be there T^2 will be there. Because I can consider polynomial as x polynomial as x^2.

So, this implies α, Tα, $T^2\alpha$, ... all will be in Z(α;T). In fact, this set is subspace of vector space. This set is a subspace of V. So, how it is subspace you can easily prove by using a condition that if beta and gamma belongs to Z(α;T). So that cβ+ γ also belong to this once where β is basically some sort of F^1T(α) and γ will be basically some F^1T (α).

So, both will be again, so that is the polynomials that cf_1+ f_2 is a polynomial over the field F so, it will again inside the Z(α;T). So, in that way you can show that this set is the subspace of the vector space.

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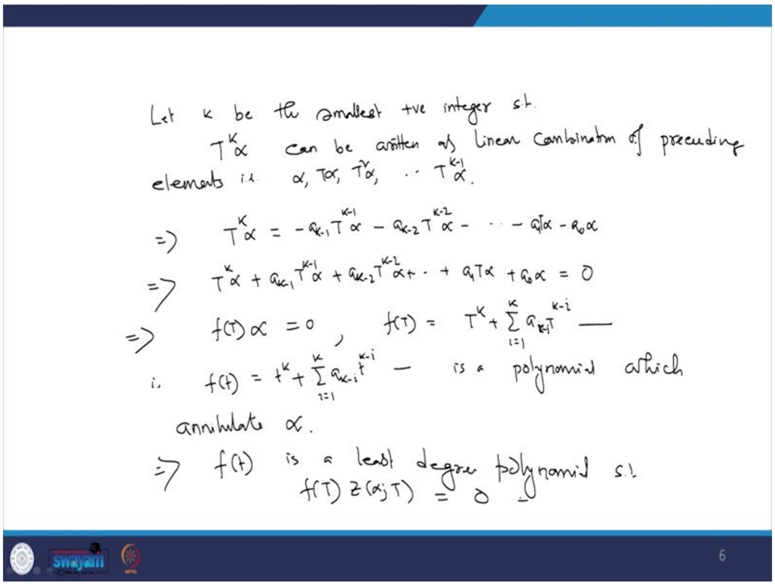

Not only that say, I will say that let k is the smallest positive integer such that $T^k\alpha$ can be written as linear combination of preceding elements i.e. α, Tα, $T^2\alpha$, … $T^{k-1}\alpha$ $\Longrightarrow$ $T^k\alpha$ = -$a_{k-1}T^{k-1}\alpha - a_{k-2}T^{k-2}\alpha - a_1T\alpha - a_0T\alpha$.

$\Longrightarrow$ $T^k\alpha$ + $a_{k-1}T^{k-1}\alpha$+$a_{k-2}T^{k-2}\alpha$ + …+$a_1T\alpha + a_0T\alpha$ = 0 $\Longrightarrow f(T)\alpha = 0, f(T)\alpha = T^k + \sum_{i=1}^{k} a_{k-i}\, T^{k-i}$. So, one can write down like this. i.e. f(t) = $t^k + \sum_{i=1}^{k} a_{k-i}\, t^{k-i}$ the polynomial which annihilate α.

Not only αI will say that this implies this f(t) is the least degree polynomial such that f(t) Z(α;T) =0.

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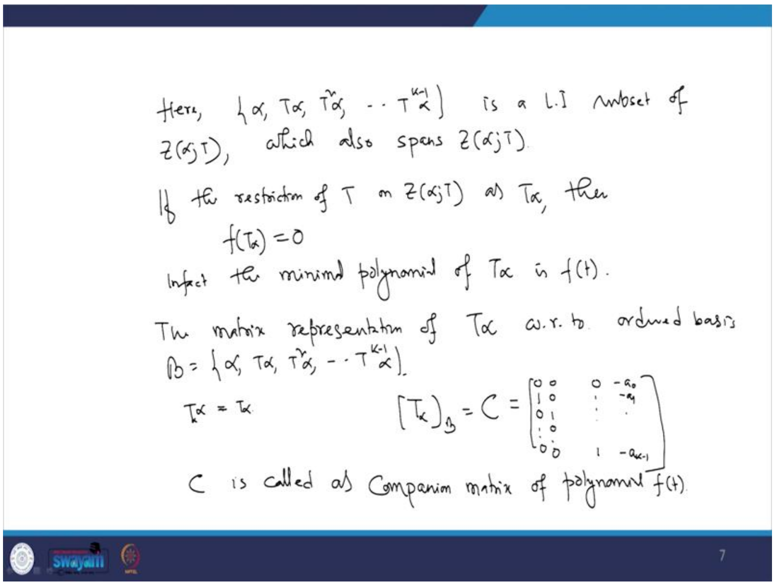

Here also, we can see here { α, Tα, $T^2\alpha$, … $T^{k-1}\alpha$ } is a linearly independent subset of Z(α;T) which also spans Z(α;T). So, this one also, I am giving as a homework you can check it that this is a basis for the subspace Z(α;T). In particular, if the restriction of T to if I denote the Tα, if the restriction of T on Z(α;T) as Tα, then f(Tα) = 0.

In fact, the minimal polynomial of T_α is f(t) the matrix representation of T_α with respect to ordered basis, say, B = { α, Tα, $T^2\alpha$, … $T^{k-1}\alpha$ }. So, I am writing $T_\alpha\alpha = T_\alpha$. So, this denotes the corresponding matrix representation of T.

So, let me denote that matrix representative T alpha with respect to ordered basis B equal to suppose C matrix then the first column will be the image of α and T_α when it expresses a linear combination of this basis element take the coefficient and then the transpose. So, T_α will basically so, I am getting what? And second element will be like this.

So, it will continue up to will have 1 and the last one will be here $T^{k-1}\alpha$ that is equal to: So, if you see this expressions then take the coefficients then I am getting $-a_0$, $-a_1$ and it will be – a_{k-1} $[T_\alpha]_B$ = C = $\begin{bmatrix} 0 & 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & 1 & -a_1 \\ 0 & 1 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & . \\ 0 & 0 & 1 & 0 & -a_{k-1} \end{bmatrix}$ where C is called as companion matrix of polynomial your f(t). So, the matrix representation is here is basically companion matrix of polynomial f(t).

So, we have seen there here if I consider a non-zero element α and if I consider the all the collection of the f(t)α then the subspace.

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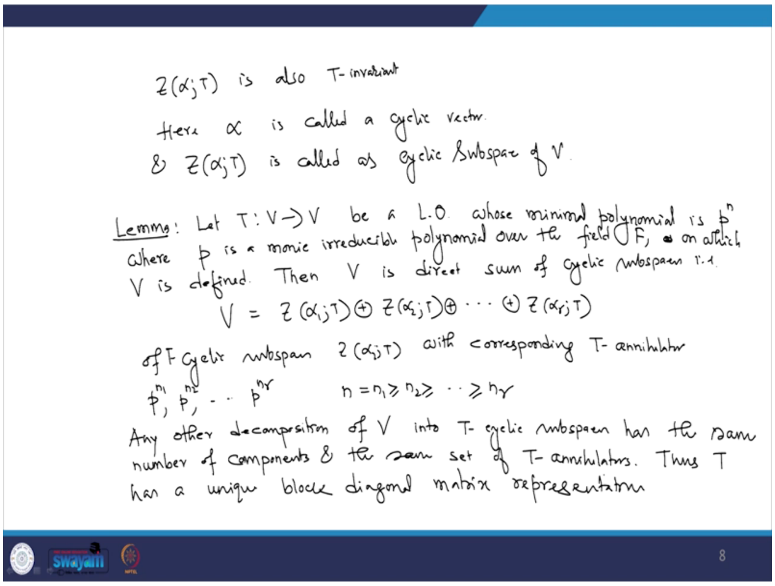

And it is of course another thing is that this subspace here, Z(α;T) is also T-invariant. Because element are of Z(α;T) is equal to something some sort of g(t)α. So, certainly T(g(t))α will be also inside the Z(α;T). So that is, I can say, Z(α;T) is also T-invariant. Apart from that here, alpha it is called a cyclic vector, and Z(α;T) is called as cyclic subspace of vector space V.

Why α is called cyclic vector? Because all the elements of Z(α;T) is basically linear combination of the element of α, Tα, $T^2\alpha$, $T^3\alpha$ … $T^{k-1}\alpha$ like that. So, this is called the cyclic that is why it is α cyclic vector. Now, this concept will help us to basically split the space V as a direct sum of cyclic subspaces. And this proof is very lengthy proof. So, this proof I will leave it on you as a self-study or homework.

You can try if you have any difficulty, you will definitely discuss this proof over the tutorial class. So now, let me write down the one nice results where this concept of cyclist of space has been used. And how the space is decomposed as a sum of cyclic subspaces? So, we see that when you consider non-zero vector α then we can write down the a we can introduce a cyclic recycling subspace is Z(α;T).

Now, based on this spirit there is a one result which is very classical results. And of course, following this result, there is also final another interesting result which is basically talk about the block diagonal representation of the linear operator, defined over a **(()) (39:06)**. This result in terms Lemma I am saying like this say, T: V → V where the minimal polynomial of the operator T is p^n.

And p is a monic irreducible polynomial over the field F on which V is defined. Then we will have a direct sum decompositions of T cyclic subspaces I mean there will be some vector $\alpha_1, \alpha_2, ...\ \alpha_n$ such that V = Z(α_1;T) ⊕Z(α_2;T) ⊕ … ⊕ Z(α_r; T) where Z(α_1;T) is p^{n_1} for Z(α_2;T) is p^{n_2}.

And Z(α_r;T) is p^{n_r} where n = $n_1 \geq n_2 \geq \cdots \geq n_r$. So, one can have a direct some decomposition of the space V where subspaces are T cyclic subspaces which are, of course invariant T. And the T annihilator I mean the least degree monic irreducible polynomials of the corresponding cyclic subspaces are basically p^{n_1} p^{n_2} ... p^{n_r}.

So, here in this theorem, it is also saying like if you consider see this is this decomposition is not unique, is saying that there will be another decomposition also. But even if you consider the any other decomposition of V in T cyclic subspaces, the number of components will be exactly same. The set of T-annihilators also will be same. Thus, T has a unique block diagonal matrix representations.

This result says basically, if you consider linear operator on **(()) (41:30)** vector space V with minimal polynomial is p^{n_r}. Then space will have a direct sum of decomposition of T cyclic subspaces by introducing cyclic vectors $\alpha_1, \alpha_2, ...\ \alpha_n$. And the corresponding T annihilators the p^{n_1} p^{n_2} ... p^{n_r}. Now, using these results this Lemma basically when the minimal polynomial is not only one irreducible, polynomial p.

It is a product of suppose many irreducible polynomials. So, in that case, the operator will also have a block diagonal matrix representations and of course the space will also have direct sum of decomposition which is like this. So, in the past first case, when only the minimal polynomial is p^{n_r}.

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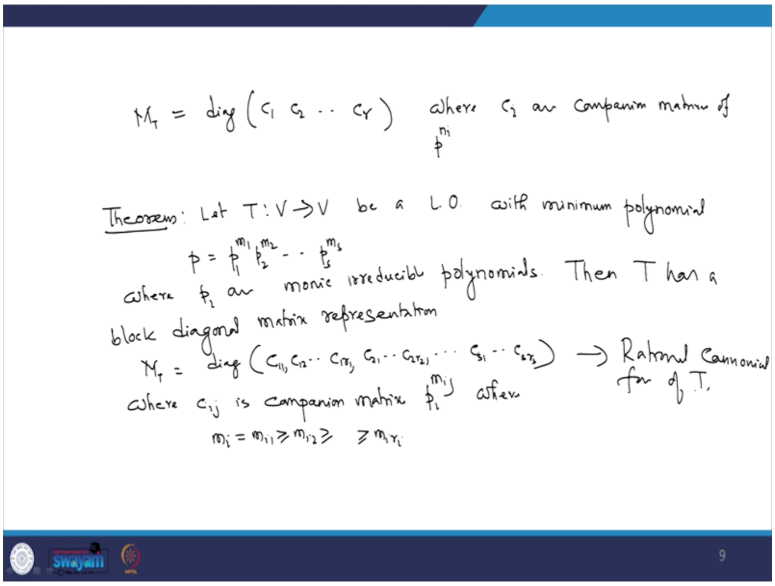

The block diagonal matrix representation will be M_T = diag$(c_1, c_2, \ldots\ c_r)$ which is basically block diagonal matrix whose diagonal entries are $c_1, c_2, \ldots\ c_n$ which are basically companion matrix of p^{n_i}. Already we have defined what the meaning of the companion matrix of a polynomials. So, if the polynomial p is known and p^{n_i} is also will be known. So, based on that only you can calculate the corresponding c_i.

Now, the generalize of the above Lemma which is coming in terms of this theorems, is saying that if T be a linear operator on V to V with minimum polynomial is say, p = ${p_1}^{m_1}\ {p_2}^{m_2} \ldots\ {p_s}^{m_s}$ where p_i in the monic irreducible polynomial then T has a block diagonal matrix representations M_T = diag$(C_{11}, C_{12}, \ldots\ C_{1r_1}, C_{21}, \ldots C_{2r_2}, \ldots C_{3r_3})$ → Rational canonical form of T.

Where C_{ij} is the companion matrix which is polynomial ${p_i}^{m_{ij}}$ where again same relation, holds good that is $m_i = m_{11} \geq m_{ir_1} \geq \cdots \geq m_{ir_i}$. So, this both the result I mean the Lemma analysis theorems the proof of this both the problems are basically I am living on you because it is very lengthy. So, you can try if you have any difficulties, definitely will discuss over the tutorial class or you can ask by email.

If we have any difficulties to understand, the proof will definitely clarify it. Now, let me apply this results to find the block diagonal relatives representation of different problem.

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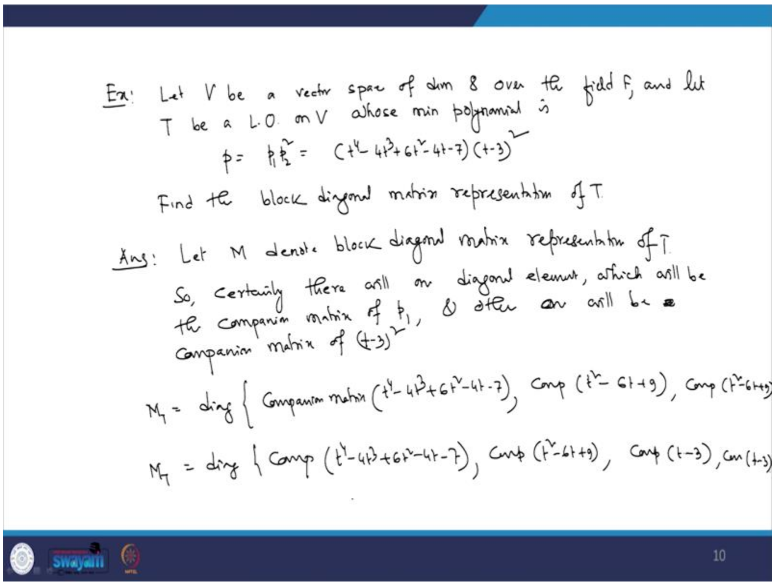

So, first let me consider so, I have taken examples like here V is a vector space of dimension 8 over the field F and T be a linear operator defined over 8 whose minimal polynomial p = $p_1\, {p_2}^2$ = $(t^4 - 4t^3 + 6t^2 - 4t - 7)(t-3)^2$. So, we have to find the possible block diagonal, matrix representations of T.

Let M denote block diagonal matrix representation of T. So, certainly there will be one diagonal element at least one diagonal element which will be the companion matrix of p_1. And other one will be companion matrix of $(t-3)^2$ because the dimension is 8, p_1 having degree is 4. So, if I consider companion matrix of $(t-3)^2$ then if I consider 2 then 2 + 2 = 4 + 4.

So, this is on possibilities, other another 2 other 1 will be at least will be $(t-3)^2$. So, the possible block diagonal matrix will be like this. I can say that one is like this. I can say, M_T = diag(companion matrix $(t^4 - 4t^3 + 6t^2 - 4t - 7)$, comp $(t^2 - 6t + 9)$,), comp $(t^2 - 6t + 9)$.

because 2 + 2, 4 + 4 = 8 because t is defined over this one. So, this is one possibilities and other than maybe M_T = diag(comp $(t^4 - 4t^3 + 6t^2 - 4t - 7)$, comp $(t^2 - 6t + 9)$,), comp $(t-3)$, comp $(t-3)$) so that 1 + 1 = 2 + 2, 4 + 4 = 8.

So, we see there are two possibilities when possible form of the block diagonal matrix representation of the your operator T here. So, we know how to calculate the companion matrix of this ones.

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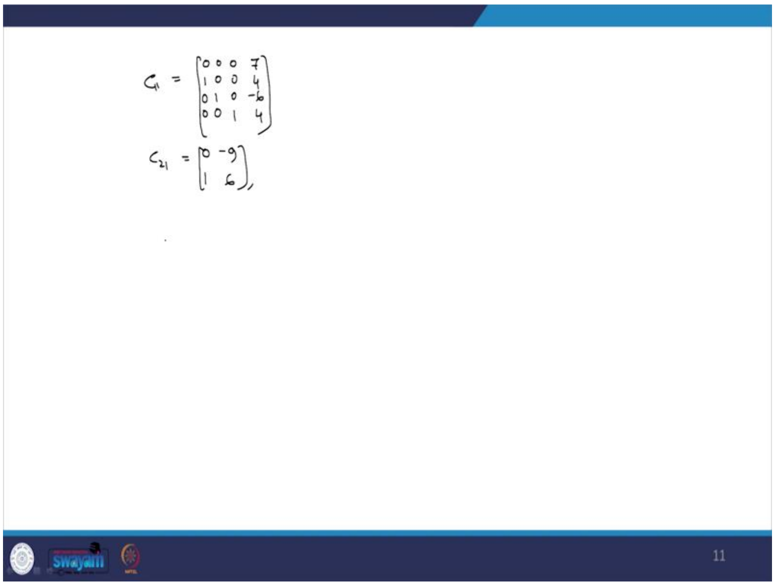

So, if I consider, $C_{11} = \begin{bmatrix} 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$. So, see the companion matrix are related to the p_1.

And your $C_{21} = \begin{bmatrix} 0 & -9 \\ 1 & 6 \end{bmatrix}$ and this is also equal to same as C_{22}. And so, in this way you know how to calculate the companion matrix. So, you can have a block diagonal, matrix representation of the operator T.

So that is all basically the applications of the primary decomposition theorems because this theorems what we have used I mean we have written mention here. This is basically based on above Lemma, as well as primary decomposition theorems. So, I hope you are getting some idea about how to decompose the space server less operators using time and equivalent theorems.

And how to get a Jordan canonical form? And actually this representation what I have done it here? This is called the rational canonical form. I know this is called the rational canonical form of the operator T.