Advanced Linear Algebra
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Lecture – 40
Applications of Primary Decomposition Theorem- II

So welcome to lecture series Advanced Linear Algebra. Today we will discuss about the applications of primary decompression theorems in different domain. Last time we are discussing about the applications in solving initial problems we have defined the concept of generalized eigenvector of specific rank and how it introduces in the independent vectors.

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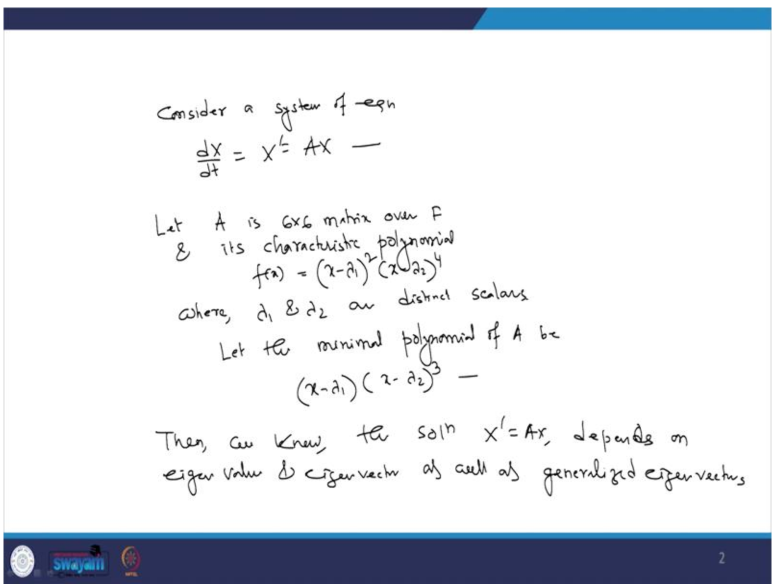

So, in continuation of that so, let me again consider say consider a system of equations autonomous system please $\frac{dX}{dt} = X' = AX$. Let A is a 6x6 over the field F and it is characteristic polynomial f(x) = $(x-\lambda_1)^2\ (x-\lambda_2)^4$ where λ_1 and λ_2 are distinct scalars let the minimal polynomial of A be $(x-\lambda_1)\ (x-\lambda_2)^3$.

Then we know the solution of $X' = AX$, depends on eigenvalue and eigenvectors as well as generalized eigenvector since minimal polynomials the factor $(x-\lambda_1)$ is a linear of first degree. So, certainly if I consider $(A-\lambda_1 I)$ this matrix will have definitely rank as a 4. So, that Null space dimension of the Null space will be 2. So, that I will have 2 linearly eigenvectors associate to $\lambda =$

λ_1. So, in that case one can find two linearly independent solution of the system $X' = AX$ associated to eigenvalue $\lambda = \lambda_1$. But in the case of λ_2 I see that characteristic polynomial is having a factor $(x - \lambda_2)^4$ where as minimal polynomial having factor $(x - \lambda_2)^3$. So, definitely I will have the eigenvectors associated to $\lambda = \lambda_2$ if I consider then $(A - \lambda_2 I)$ certainly I will have rank will be also again 4.

So, that I will have two linearly independent eigenvector since matrix is 6x6 already obtained two linear independent eigenvector $(x - \lambda)$ equivalent of one already obtained two linearly independent eigenvectors associated to $\lambda = \lambda_2$. So, still we need two more. This two more linearly independent vectors will be obtained from the I will have a vector in the null space of $(A - \lambda_2 I)^3$.

There will be vector in the Null space of rank 3 which will give me three linearly independent vectors out of three one will be eigenvectors which is basically already calculated. So, I will have two more vectors which are lead in independent also. So, all together I will have six linearly independent eigenvectors and one can calculate the corresponding complete solution. So, if I have a minimal polynomial like $(x - \lambda_2)^3$ we are 100 percent sure there will be a vector in the null space of $(A - \lambda_2)^3$ say v_3 such that $(A - \lambda_2 I)^3 v_3 = 0$ but $(A - \lambda_2 I)^2 v_3 \neq 0$.

So, I will have a chain of generalized eigenvectors. Now suppose these vectors are known how to find the solutions associated to this generalized eigenvector v_1, v_2, v_3 or if I consider v_1, v_2, v_3, ... v_r. Now let me answer these questions.

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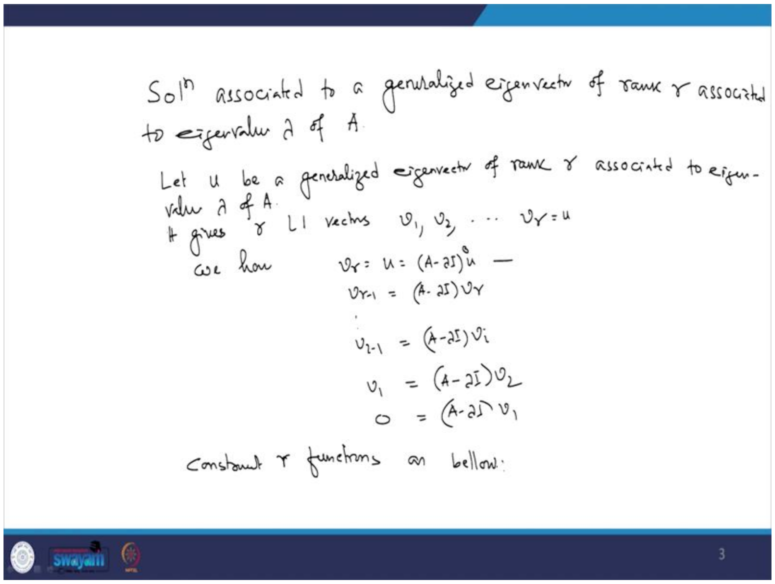

Solution associated to a generalized eigenvector of rank are associated to eigenvalue λ of the matrix A. So, let u be a generalized eigenvector of rank r associated to eigenvalue λ of A so it gives r linearly independent generalized eigenvectors basically $\{v_1, v_2, \dots v_r\}$ = u, I am denoting as u, we know $v_r = u = (\mathrm{A} - \lambda \mathrm{I})^0 u$, $v_{r-1} = (\mathrm{A} - \lambda \mathrm{I}) v_r$, … $v_{i-1} = (\mathrm{A} - \lambda \mathrm{I}) v_i$ $v_1 = (\mathrm{A} - \lambda \mathrm{I}) v_2$, $0 = (\mathrm{A} - \lambda \mathrm{I}) v_1$.

this is already seen it construct r functions as below.

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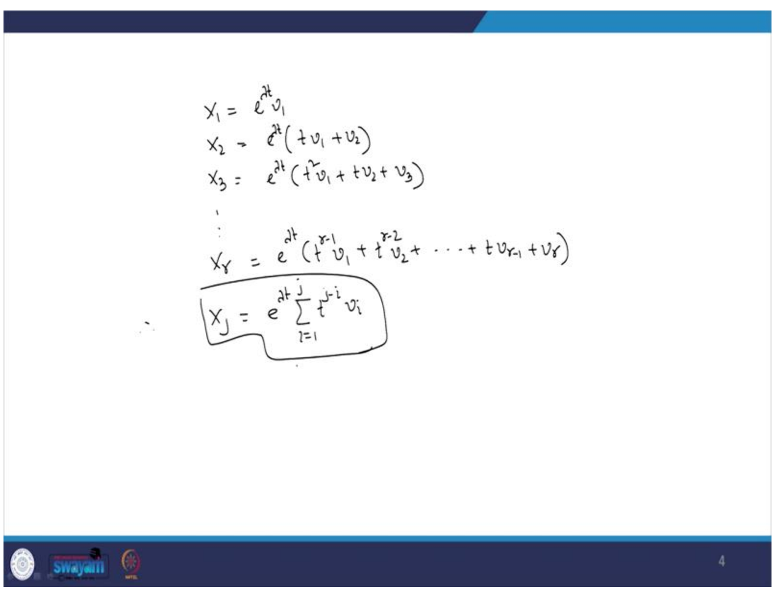

So, let me define our function that, $X_1 = \mathrm{e}^{\lambda t} v_1$, $X_2 = \mathrm{e}^{\lambda t}(t v_1 + v_2)$,

$X_3 = e^{\lambda t}(t^2 v_1 + tv_2 + v_3)$, ... $X_r = e^{\lambda t}(t^{r-1}v_1 + t^{r-2}v_2 + tv_{r-1} + v_r)$, So, I can say in general $X_j = e^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}v_i}{(j-i)!}$. So, I have constructed this is a some r functions like this.

Claim $X_1, X_2, \ldots X_r$ linearly independent solutions of the system $\frac{dX}{dt} = AX$. So, let me first prove that each of them are the solution of this system.

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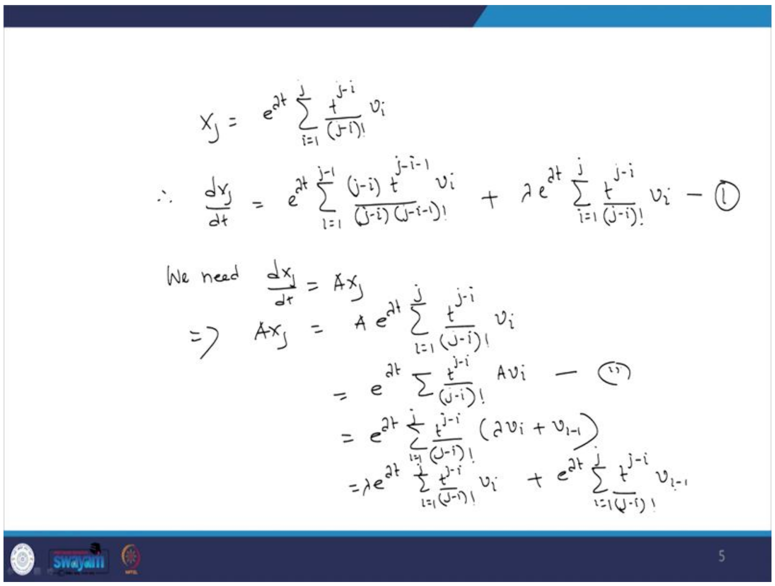

So, we have, $X_j = e^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}v_i}{(j-i)!}$ I did mistake in my construction X_3 is equal to I am defining it here 2! and. So, it is (r-1)! and this is (r-2)! and so on. So, I need a (j-i)!. So, I am defining the are functions, $X_j = e^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}v_i}{(j-i)!}$. So, $\frac{dX_j}{dt} = e^{\lambda t}\sum_{i=1}^{j-i}\frac{(j-i)\,t^{j-i-1}v_i}{(j-i)\,(j-i-1)!} + \lambda e^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}v_i}{(j-i)!}$

We need $\frac{dX_j}{dt} = AX_j \implies AX_j = Ae^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}v_i}{(j-i)!} = e^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}Av_i}{(j-i)!} = e^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}A}{(j-i)!}(\lambda v_i + v_{i-1}) = \lambda e^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}}{(j-i)!}v_i + e^{\lambda t}\sum_{i=1}^{j}\frac{t^{j-i}}{(j-i)!}v_{i-1}$

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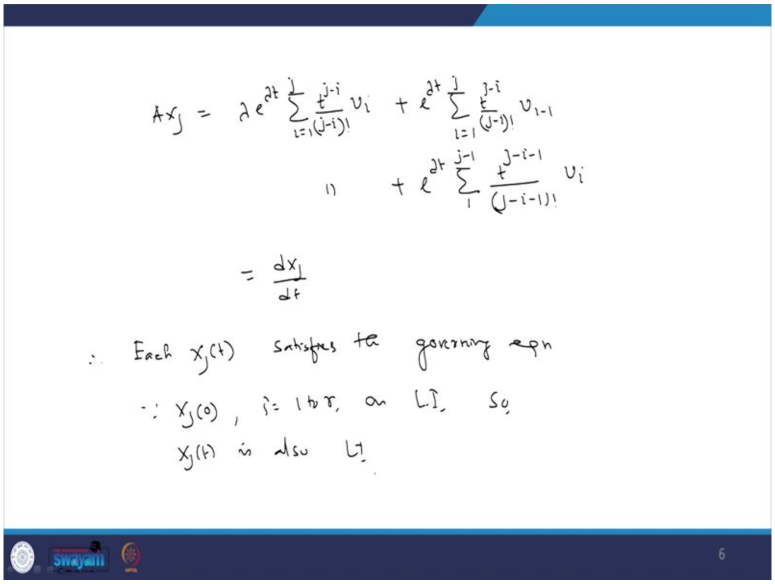

So, $AX_j = \lambda e^{\lambda t} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!} v_i + e^{\lambda t} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!} v_{i-1} = \lambda e^{\lambda t} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!} v_i + e^{\lambda t} \sum_{i=1}^{j-i} \frac{t^{j-i-1}}{(j-i-1)!} v_i = \frac{dX_j}{dt}$

each X_j(t) satisfy is the governing equations and also we will see that at t = 0 it satisfied the also initial conditions. Then I will having one with any independent solution also. So, we already checked that this beyond $v_1, v_2, \ldots\ v_r$ is in an independent and we have also n linen independent solution therefore $x_1, x_2, \ldots\ x_r$ will be also in any independent.

Since $X_j(0)$, i = 1 to r, are linearly independent. So, is $X_j(t)$ is also linearly Independence. So, I have r linearly independent Solutions of the systems $AX = \frac{dX}{dt}$ where we apply generalize eigenvector of rank r. So, we see that the primary decompression theorems is used here to find the solution of the initial value problems $\frac{dX}{dt} = AX$ when system is autonomous case.

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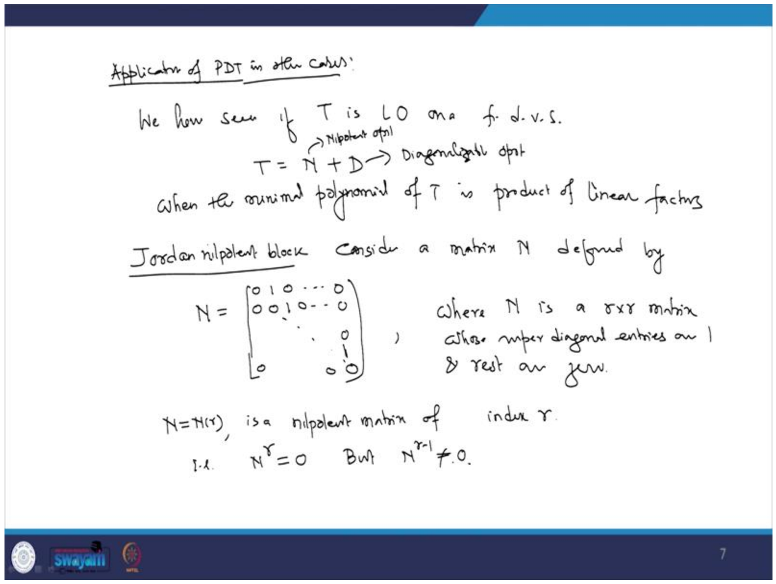

Now let me go the applications of primary decomposition theorems in some other cases. Application of primary decomposition theorems(PDT) in other cases we have seen if T is a linear operator on a finite dimensional vector space then T can be written as sum of a nilpotent operator and a diagonalizable operator D, as T = (N+D) this is we have obtained from the primary decomposition theorems when the minimal polynomial of T is product of linear factors.

I mean to say when the field over which the vector space v is defined is algebraically closed then we are safe. So, you can say that in that case any operator define over the finite dimensional vector space certainly will have decomposition of the form of (N + D) where N is a nilpotent operator and D is a diagonalizable operator. So, that we have already seen.

Let me introduce some specific type of operator or matrix it will show that this is an important operator is nilpotent operator will also have block diagonal forms that element will be some specific structure. So, we are going to talk about that issue. So, before that let me introduce some terminology first is Jordan nilpotent block. Consider a matrix N defined by $N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

So, the diagonal entries are zero super diagonal entries are 1. So, I have defined a matrix like this. So, consider Matrix N of say rxr order defined by like this thing where N is rxr matrix which super diagonal entries are one and rest are zero. So, we will see that this N, I am writing as a basically N = N(r), I mean this is a nilpotent operator or nilpotent matrix of index r.

That is $N^r = 0$ but $N^r \neq 0$, see the way I have written the matrix N, I can immediately construct and linear operator over say F^{rx1} like this.

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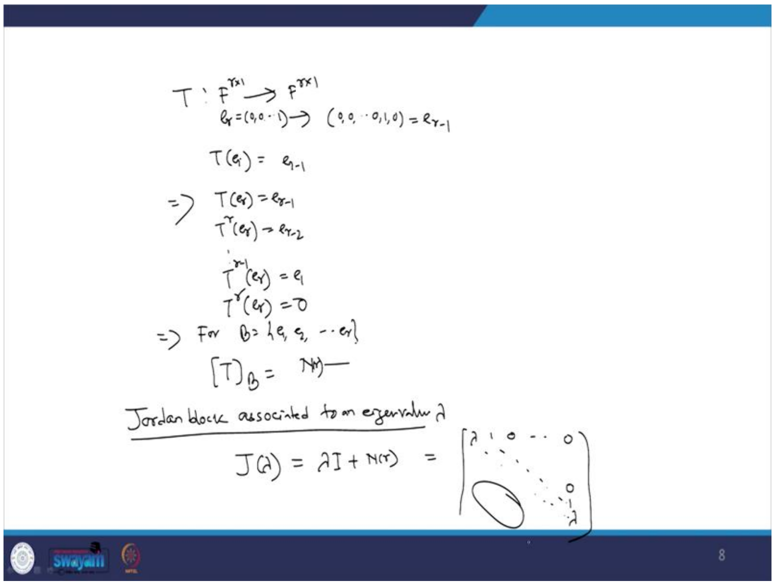

You know we can define a linear transformation say T: $F^{rx1} \rightarrow F^{rx1}$, I am defining like say $e_1 = (0,0, \ldots 1) \rightarrow (0,0,\ldots 0,1,0) = e_{r-1}$. If I defined the map e_r which is mapping e_{r-1}, I mean to say $T(e_r) = e_{r-1} \implies T(e_r) = e_{r-1}$, $T^r(e_r) = e_{r-2}$, $T^{r-1}(e_r) = e_1$. $T^r(e_r) = 0$

If I write the corresponding matrix representations with respect this standard order basis B, I will get the matrix representation is basically your N which you can say it because the $e_1 \rightarrow 0$, $e_2 \rightarrow e_1$ So, like this.

So, this implies if this implies for B = $\{e_1, e_2, \ldots e_n\}$ order basis, $[T]_B = N_r$, for the same place. So, we see that this Jordan nilpotent block define of index r define like this is basically coming as a consequence of the sifting operator T: $F^{rx1} \rightarrow F^{rx1}$, defined by $e_1 \rightarrow e_{r-1}$. So, that way please.

And let me also define another matrix that is called Jordan block associated to an eigenvalue λ. Let me define this once Jordan block associated to an eigenable λ, I am defining like this $J(\lambda)$ it is basically again it is a square matrix say of order rxr then I am defining here it is basically you know, $J(\lambda) = \lambda I + N(r)$, where N is Jordan nilpotent block matrix of index r.

So, $N^r = 0$ but $N^r \neq 0$ and the N having structure like what I have already done that except super diagonal terms where all is super diagonal term is equal to one rest of the terms are equal to zero I am defining another matrix called Jordan block of say order rxr as

$$J(\lambda) = \lambda I + N(r) = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$. So, this is my Jordan block associated to eigenvalue λ.

So, based on this definitions of the Jordan block assisted eigenvalue λ and Jordan nilpotent block N let me also use a one theorems related to the nilpotent operator or nilpotent matrix of index r if you consider.

Then there is a classical result that if N is a nilpotent operator of index r then this nilpotent operator have a block diagonal matrix representations where the diagonal entries will be Jordan nilpotent Matrix of different index. So, let me State about the theorem please.

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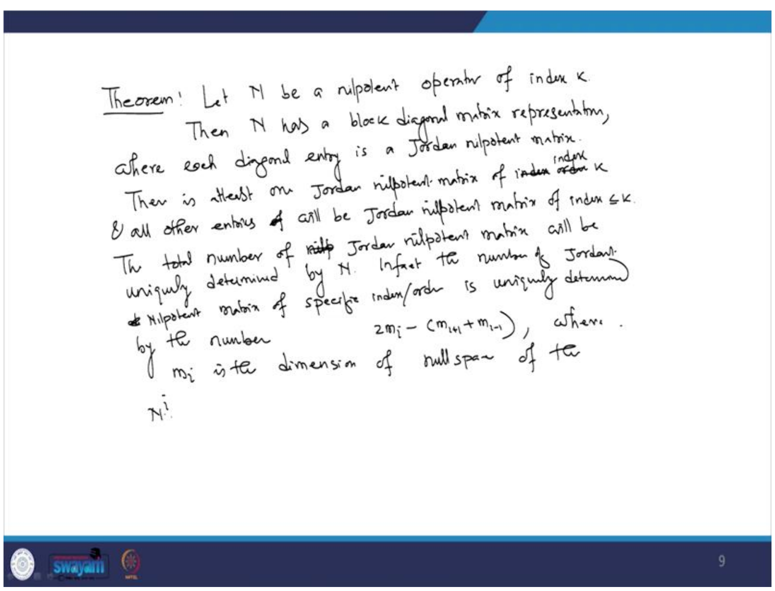

Let N be a nilpotent operator of index k. Then N has a block diagonal matrix representation where each diagonal entry is a Jordan nilpotent matrix, there is at least one Jordan nilpotent matrix of either index or order we can same here of index because is basically index k means definitely it has to be of order also I mean kxk of in order k matrix of index k or order k.

All others entries will be Jordan nilpotent matrix of I can say index not to confuse or index $\leq$ k, the total number of Jordan nilpotent matrix will be uniquely determined by the operator N or matrix both are same if you consider any operators fine if you consider matrix that is also fine. So, this is basically a classical theorem which talk about the block diagonal matrix representation of any nilpotent operators and nilpotent Matrix of specific index.

In fact the number of Jordan nilpotent matrix of specific index or order is uniquely determined by the number, $2m_i - (m_{i+1} + m_{i-1})$ where m_i is the dimension of N^i. So, this is the classical results the proof is bit lengthy that is true right now I shall use this result to give a Jordan canonical representation of a operators when the operator is defined over the vector space which is also defined over the field and field is algebraically closed.

So, using this results and within the previous result that when you consider linear operator define over a final dimensional vector space that operator can be retained as a sum of a diagonal matrix and Jordon nilpotent matrix and using the previous result that every nilpotent operator will have a diagonal representation the block diagonal representations where each entry will be basically nilpotent matrix.

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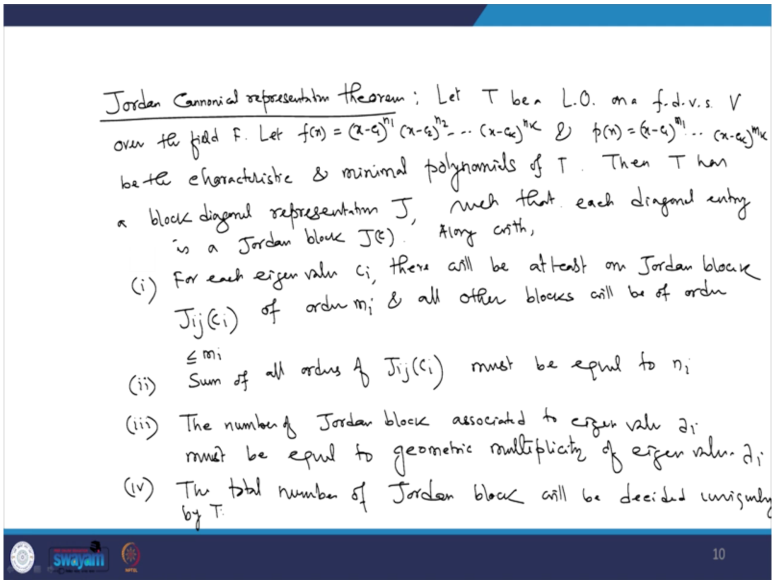

So, based on that let me give a talk about theorems called Jordan canonical representation theorem says that let T be a linear operator on a finite dimensional vector space V over the field say F. Let f(x) $= (x - c_1)^{n_1}$ $(x - c_2)^{n_2}$ ... $(x - c_k)^{n_k}$ & p(x) $= (x - c_1)^{m_1}$ $(x - c_2)^{m_2}$ ... $(x - c_k)^{m_k}$ be the characteristic and minimal polynomials of T.

So, I have considered f(x) as a characteristic polynomial and p(x) is the minimal polynomial of the operator T. So, that polynomials are factorizable over the field that is given then T has a block diagonal representations say J, such that each diagonal entry is a Jordan block something like J(c) along with, (i) For each eigenvalue c_i, there will be at least one Jordan block $J_{ij}(c_i)$ of order m_i that is the basically the $(x - c_i)^{m_i}$ in the minimal polynomials and all others blocks will be of order $\leq m_i$.

(ii) Sum of all order of $J_{ij}(c_i)$ $= n_i$ that is algebraic multiplicity of the eigenvalue c_i. So, it will be I have written c_i. So, it will be sorry then c_i can I should write c_i. So, sum of all order of $J_{ij}(c_i)$ $= n_i$ that is algebraic multiplicity of the eigenvalue. (iii) The number of Jordan block associated to eigenvalue c_i must be equal to geometric multiplicity of eigenvalue c_i.

(iv) The total number of Jordan block will be decided uniquely by the operator T. So, this is called Jordan canonical decomposition theorems he is saying that if T be a linear operator on a finite-dimensional vector space when the characteristic polynomial or minimal polynomial are factorizable over the field then the operator T will have a block diagonal matrix representation J where the diagonal entries will satisfy these four hypothesis I mean four condition please.

(i) for each eigenvalue there will be at least one Jordan block $J_{ij}(c_i)$ of order c_i and all other will be of $\leq m_i$. (ii) Sum of all order of $J_{ij}(c_i)$ must be equal to geometric multiplicity of the eigenvalue c_i that is must be equal to n_i. (iii) The number of Jordan block associated to eigenvalue c_i must be equal to geometric multiplicity of c_i. And the total number of Jordan block will be decided uniquely by the operator T.

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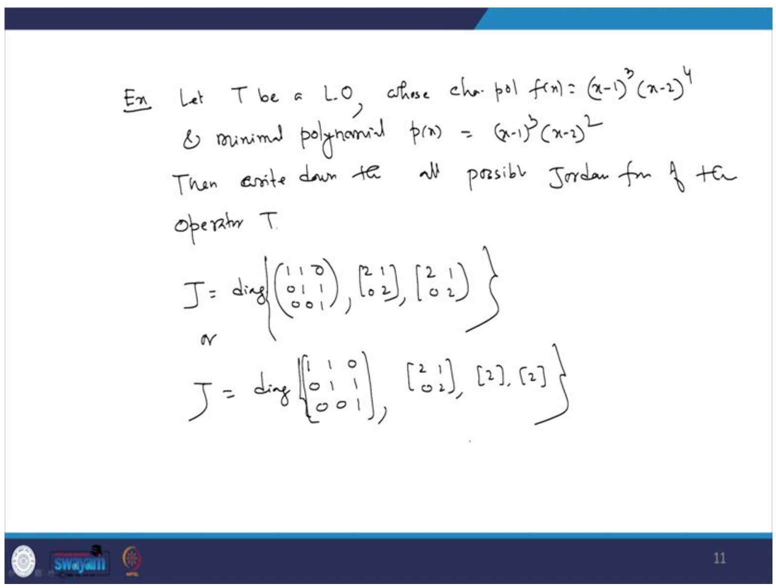

So, let me take an examples let T be a linear operator which characteristic polynomial f(x) $=(x-1)^3\,(x-2)^4$ and minimal polynomial we are, p(x) $=(x-1)^3\,(x-2)^2$. Then write down the all possible Jordan form of the operator T. So, Jordan form would be like this one is like this, J equal to I can say diagonals first will be since the minimal polynomial is p(x) $=(x-1)^3\,(x-2)^2$ and in characteristic polynomial also I mean they are $(x-1)^3$ that is there.

So, one is will be 1 then Jordan block to $\lambda = 1$. Then second entry Associated to the eigenvalue λ = 2 the Jordan block will be at least one will be of order 2.. So, other ones it may be of order 2 other order one also because 2 + 2 = 4 that is the basically the algebraic multiplicity of diagonal two.

So, it may be one is like this. So, this is one possible Jordan block diagonal representation, J = { $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ } or J = { $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $[2], [2]$ }. So, I have only 2 possibilities here but friends if it is given that the geometric multiplicity of eigenvalue $\lambda = 2$, is exactly equal to say 2 then I will have only first option.

If it is 3 then I will go for the third option is it possible to have geometric multiplicity of eigenvalue $\lambda = 2$ as 4 no otherwise minimal polynomial will be $(x-1)^3\,(x-2)$. So, this is the simplified form of the corresponding operator T which is defined over a finite dimension vector space here I have taken says F^7 then I have the Jordan form is like this. So, we will continue these applications for other situation also.