Advanced Linear Algebra

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Lecture – 4
Vector Spaces - 1

Welcome to lecture series on advanced linear algebra. In my last class, we have seen to answer the question that whether a system AX=Y has solution or not. The meaning of these questions we have seen in terms of linear combination of vectors for less than or equal to three, I mean to say over the XY plane or over the Euclidean space, we have seen that solution of the linear system of equations is equivalent to the questions whether is there a possible linear combination of the column vectors of your matrix A which will give me right hand side vector space.

This we have seen for n less than or equal to three spaces. The question is; is it possible to generalize the same concept since up to n = 3 we can visualize three-dimensional space, but as you cross n greater than 3 one cannot visualize this picture of vector concept there.

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$$AX = Y \quad - (*) \qquad A = (a_{ij}),\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

where $a_{ij} \in F$ for $1 \le i \le m$, $1 \le j \le n$

& $y_i \in F$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

So, we need to extend this concept more algebraically and for that each Y I have introduced a new concept of vector space which we will talk again space of vectors. As you know in our 11-12 concept vector means it has direction as well as magnitude that is scalar quantity as the magnitude and direction I am talking about the vector part. So, associated with the two numbers, one set of vectors and the set of scalars type of things.

Now, basically we want to generalize this concept for the generalized system AX=Y where A is the mxn matrix over a field say F and Y is basically again mx1 column matrix space or column vector space like what we have written here.

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What is vector space?

Vector space consists of two sets: (i) Set of objects or vector, V
(ii) Field of scalar F, along with two binary operations
(iii) Vector addition, $+$ & (iv) Scalar multiplication, $*$, Such that
w.r.to vector addition V is an abelian group i.e.
(a) closure prop: for any u & $v \in V$ $\quad u+v \in V$
(b) Associative " : For any $u, v, w \in V$, $(u+v)+w = u+(v+w)$
(c) Existence of additive identity i.e. zero element: $\exists\ e \in V$ s.t. for any $u \in V$ $\quad u+e = e+u = u$, $e \rightarrow$ zero element
(d) Existence additive inverse: For any $u \in V$ $\exists\ v \in V$ Such that $u+v = v+u = e$
(e) Commutative prop holds good: For any u & $v \in V$ $\quad u+v = v+u$.

So, let us define vector spaces. The first question will be what is vector spaces? I can write like this a vector space consist of two sets, one set, set of objects or vectors which can be denoted as V and another one is a field of scalar, here I am denoting as standard notation F along with two binary operations. One each call vector addition, I am denoting this as + and other one is scalar multiplication I am denoting as *.

Such that with respect to vector addition V is an abelian group that is satisfied through their property. For any u,v ∈V, u + v ∈V and sum of two vectors is again a vector. Second is associative property, for any u, v, w ∈ V, (u + v) + w = u +(v + w). Third one existence of additive identity that is we can say zero element, I mean ∃ e ∈ V such that for any u ∈ V, u + e = e + u = u, e is called zero element.

Existence of additive inverse. For any u ∈V, ∃ v ∈ V such that u + v = v + u = e. Commutative property, property also holds good that is for any u, v ∈ V, u + v = v + u . So, with these five axioms, V has to be an evident group with respect to vector addition.

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W. r.to scalar multiplication:

(i) For any $c \in F$ & $u \in V$

$c * u \in V$

i.e. closure prop holds good

(ii) For any c_1 & $c_2 \in F$ & $u \in V$

$(c_1 + c_2) * u = c_1 * u + c_2 * u$

(iii) $1 \in F$ & for any $u \in V$

$1 * u = u * 1 = u$

(iv) for any $c \in F$ & $u, v \in V$

$c * (u + v) = c * u + c * v$

Then with respect to scalar multiplication it has to satisfy again five more axioms. See scalar multiplication means in deals with the scalar quantity from the field and vector quantity on the V. This is basically associated with these two sets of elements. So, first is closure property, I mean to say for any c ∈ F and u ∈ V, c * u ∈ V that is closure property.

Property holds good. Second one for any c_1, c_2 ∈ F and u ∈ V, $(c_1 + c_2) * u = c_1 * u + c_2 * u$. So note that this plus is basically binary operation in field. But this plus what I am saying here is vector addition. Three, 1 ∈ F and for any u ∈ V, 1 * u = u * 1 = u. Fourth one for any c ∈ F and u, v ∈ V, c * (u + v) = c * u + c * v that is distributive property over the addition also holds good.

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(v) For any c_1 & $c_2 \in F$ & $u \in V$

$(c_1 \cdot c_2) * u = c_1 * (c_2 * u)$ —

equipped with two binary operation vector addition & scalar multiplication

Definition: A non empty set V is said to be vector space over a field F, provided V is an abelian group w.r.to vector addition & w.r.to scalar multiplication it satisfies following five axioms

(i) closure prop: for any $c \in F$, $u \in V$ $c * u \in V$

(ii) associative " For any $c_1, c_2 \in F$ & $u \in V$ $(c_1 \cdot c_2) * u = c_1 * (c_2 * u)$

(iii) $\exists\, 1 \in F$ s.t. for any $u \in V$, $1 * u = u = u * 1$

(iv) For $c_1, c_2 \in F$ & $u \in V$ $(c_1 + c_2) * u = c_1 * u + c_2 * u$

(v) for any $c \in F$ & $u, v \in V$, $c * (u + v) = c * u + c * v$

And the last one is the fifth one for any c_1 and $c_2 \in$ F and u $\in$ V, $(c_1.c_2)$ * u = c_1* ($c_2 *$ u). So, with respect to scalar multiplication is satisfied all these five axioms and with respect to vector addition it is an abelian group. So, then only we can say it is a vector space. Definition of a vector space I can say like this. A nonempty set V is said to be vector space over a field F, I would have to say that V equipped with two binary operations vector addition and scalar multiplication.

Equipped with binary operations is said to be vector space over the field F provided V is an abelian group with respect to vector addition and with respect to scalar multiplication satisfies following five axioms that already I have discussed. One is closure property, for c $\in$ F, u $\in$ V, c *u $\in$ V. Associated property type for any c_1, $c_2 \in$ F and u $\in$ V, $(c_1.c_2)$ * u = c_1* ($c_2 *$ u).

It is not exactly associative, but it is type of associative. Third there exist 1 $\in$ F such that any u belongs to V, 1 star u equal to u and equal to also u star 1. Fourth for c_1, $c_2 \in$ F and u $\in$ V, (c_1+ c_2 * u = $c_1 * u$+ $c_2 *$ u and for distributive the property over the addition also holds good for any c $\in$ F and u, v $\in$ V, c* (u + v) = c * u + c * v. So, this is the definition of the vector space. Now, let me consider some examples.

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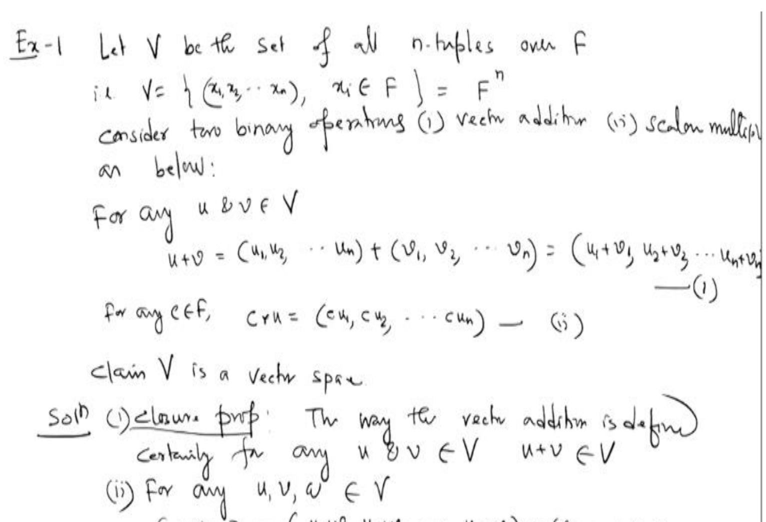

Let V be the set of all n-tuple over F that i.e. V ={ $(x_1, x_2, \ldots x_n), x_i \in F$ }= F^n, this can be written also it is like in a standard notation that is F^n Okay. So, V be the collection of all the n-tuples like this over the F, F is any field, it may be real number, it may be complex number. Consider two binary operations, one is vector addition and two is say scalar multiplication at below.

For any u &v ∈ V, u + v = $(u_1, u_2, \dots\ u_n) +\ (v_1, v_2, \dots\ v_n)$ = $(u_1 + v_1, u_2 + v_2, \dots\ u_n + v_n)$ and this is my vector addition 1. And scalar multiplication is like this, for any c ∈ F, c * u = (cu_1, c$u_2, \dots$ cu_n) , so this is my scalar multiplication over this given set V. Claim, V is the vector space.

Now to prove that one I have to show that V satisfies all the axioms in the definition of vector space, I mean to say V is an abelian group with respect to binary operation one and with respect to scalar multiplication two satisfy all the five axioms. So, first let us quickly check that whether V satisfied all the axioms of the Abelian group or not. First closure property. Closure property means what I have to check?

I have to check suppose u and v are the two n-tuples I have considered over the F, I have to show that the sum of two n-tuple is again an n-tuple over the same. So, I can say that the way the vector addition is defined certainly for any u,v ∈ V, u + v ∈ V, so closure property holds good. What can you say about the associative property, let me quickly check that once.

For any u, v, w ∈ V, (u + v) + w = $(u_1 + v_1, u_2 + v_2, \dots\ u_n + v_n)$ + $(w_1, w_2, \dots\ w_n)$. So, this is again finally going to be again (u + v) + w = $(u_1 + v_1 +\ w_1, \dots\ u_n + v_n +\ w_n)$.

So, I am getting like this. Now, this number is over the field F which is also satisfied the associative property.

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$$(u+v)+w = \left(u_1+(v_1+w_1),\ u_2+(v_2+w_2),\ \cdots\ ,u_n+(v_n+w_n)\right)$$

$$= u + (v+w) \text{ —}$$

(iii) Consid $e = (e_1, e_2, \cdots e_n)$ be a zero element of V

For any $u \in V$, $u + e = u$

$$\Rightarrow \left(u_1+e_1, u_2+e_2, \cdots u_n+e_n\right) = \left(u_1, u_2, \cdots u_n\right)$$

$$\Rightarrow u_1+e_1 = u_1 \Rightarrow e_1 = 0$$

$$\Rightarrow e = (0, 0, 0 \cdots 0)$$

(iv) for any $u \in V$, $u = (u_1, u_2, \cdots u_n)$, then Let $v = (v_1, v_2, \cdots v_n)$

s.t. $u + v = e$

$$\left.\begin{array}{l} u_1+v_1 = 0, \\ u_2+v_2 = 0 \\ \vdots \\ u_n+v_n = 0 \end{array}\right\} \Rightarrow v = (-u_1, -u_2, \cdots -u_n)$$

So therefore, I can write down this one as (u + v) + w = $(u_1 + (v_1 + w_1),\ u_2 + (v_2 + w_2), \dots\ u_n + (v_n + w_n))$ = u + (v + w), so associative property holds good. What will be our existence of zero element? Third one, see for zero element.

I will say that consider e = $(e_1, e_2, \dots\ e_n)$ be a zero element of V and according to definition of the vector space we have for any u ∈ V, u + e = u. So, this implies I will have $(u_1 + e_1, u_2 + e_2, \dots\ u_n + e)$ = $(u_1, u_2, \dots\ u_n)$. Since, these two n-tuples are equal, therefore comparing each component they have to also equal and considering this, this implies that $u_1 + e_1$= u_1. This implies that e_1 = 0 is a number from the field F, so e_1 = 0. Similarly, e_2= 0, e_3 = 0 and e_n = 0. So, this implies that equal e = (0, 0, 0, …0). So, this is the zero element of F^n space, on V equal to F^n space. What can you say about the additive inverse? For any u ∈ V, u = $(u_1, u_2, \dots\ u_n)$, Then let v = $(v_1, v_2, \dots\ v_n)$ which is again an n-tuple which belongs to V.

Suppose, if it is an inverse, additive inverse of u such that u + v = e. So, this implies that we will have $u_1 + v_1 = 0, u_2 + v_2 = 0, \dots\ u_n + v_n = 0$. This implies that my v = $(-u_1, -u_2, \dots - u_n)$ This is the inverse of this element u.

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Commutative prop: For any u & $v \in F^n$

$$u+v = (u_1+v_1, u_2+v_2, \dots u_n+v_n) = (v_1+u_1, \dots v_n+u_n)$$
$$= v+u$$

Similarly, one can check F^n satisfies all othe axioms w.r.to scalar multiplication

$\therefore$ $V = F^n$ is a vector space.

Ex: Let S be a subset of $\mathbb{R}$

Let V be the collection of all fns from S into $\mathbb{R}$

Consider vector add^n & scalar multiplication as below:

Vector add: For any f_1 & $f_2 \in V$

$$(f_1+f_2)(s) = f_1(s) + f_2(s) \quad \text{— (i)}$$

Scalar mult: $$(c \cdot f)(s) = c\, f(s) \quad \text{— (ii)}$$

Next is commutative property. We have to check whether commutative property holds good or not. For any u, v ∈ F^n, we have u + v = $(u_1 + v_1, u_2 + v_2, \dots u_n + v_n)$ = $(v_1 + u_1, \dots v_n + u_n)$ = v + u. Similarly one can check F^n satisfies all other axioms with respect to scalar multiplication also. So, V = F^n is a vector space. Now, if I consider n = 1 then we have V = F. So here set of a vector is F and set of scalar V also F. So, we can prove in the same way that F over F will be also a vector space. Let me consider the F equal to say a complex number that is my set of vector is a complex number and the set of scalar that is field as a real number.

Now question is whether set of complex number over set of real number will be a vector space? Answer is yes. You can quickly check, it satisfies all the axioms of the definition of the vector space. But if we consider a set of vectors as a real number and set of scalars as a complex number and operation is vector addition and scalar multiplication, you can see that it will be not a vector space because it will not close with respect to scalar multiplication.

You can think about it, you can take it as home assignment also. So, now let me take another example. Let S be a subset of the real number R. Let V be the collection of all functions from S into R. Consider vector addition and scalar multiplication as below. Vector addition like this for any f_1 & f_2 ∈ V, $(f_1 + f_2)(s) = f_1(s) + f_2(s)$. So, this is my vector addition. And scalar multiplications I can define like now any scalar c ∈ R here, (c * f) = c . f(s), this is normal product because f(s) also belongs to R and c also belongs to R, so it will be normal multiplication this one. So, this is my scalar multiplication.

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One can easily show V is a vector space.

closure

Assoc: $((f_1+f_2)+f_3)(s) = (f_1+f_2)(s) + f_3(s)$

$= f_1(s) + f_2(s) + f_3(s)$

$= f_1(s) + (f_2+f_3)(s)$

$\Rightarrow (f_1+f_2)+f_3 = f_1+(f_2+f_3)$

W. r. to. scalar multiplication

It also satisfies all other axioms of vector space

$\therefore$ It is a vector space.

Ex Let V be the collection of all polynomial fns of degree $\leq n$ over $\mathbb{R}$.

Consider Vec. add'n, for p & $q \in V$

Now, vector addition and scalar multiplication you can check. The set of all this function is also vector space, I mean one can easily show V is a vector space. I mean first we have to show the satisfied that V which is an abelian group with respect to vector addition and with respect to scalar multiplication it satisfied again five more axioms. See if I take f_1, f_2 to any functions from V, $(f_1 + f_2)(s) = f_1(s) + f_2(s)$, this also belongs to V because each of them belongs to real number.

The $f_1(s)$belong to real number, $f_1(s)$ belongs to real number. So, $f_1(s) + f_2(s)$ also belongs to real number. So this implies $f_1 + f_2$ which is defined from S to R is again a function, therefore it is closed. Similarly, you can check it also satisfies associative property. So closure property we can immediately check like this. Associative property how to check? Associative property you can check like this. Say for $f_1, f_2, f_3 \in$ V one can say that $((f_1 + f_2) + f_3)(s) = (f_1 + f_2)(s) + f_3(s) = f_1(s) + f_2(s) + f_3(s) = f_1(s) + (f_2 + f_3)(s) = f_1 + (f_2 + f_3)$. So, associative property also holds good. What over the zero element?

The zero function itself is zero element. What about the inverse? If f_1 is there, I can say $-f_1$ also there. So, f_1 + $-f_1$ is 0 function and f_1 since it is defined over the basically real line which is basically abelian, so therefore a commutative property also holds good for the set V. With respect to scalar multiplication, see for any c_1 given definition, it is again clear that c x f (s) is basically c x f is again a function from S to R.

So closure property holds good with respect to multiplication and you can also check for four more axioms with respect to scalar multiplications which also holds good for the given set of vectors over the given set of scalar planes. So, we see that this is also vector space. It also satisfied by all other axioms of vector space, so it is a vector space. The next example we can consider let V be the collection of all polynomial functions over R.

Consider vector addition as for p & q $\in$ V. For sake of simplicity let me consider collection of all polynomial function of degree less $\leq$ n over R.

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$$p = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n$$

$$\& \quad q = q_0 + q_1x + \cdots + q_nx^n$$

$$p+q = p_0+q_0 + (p_1+q_1)x + (p_2+q_2)x^2 + \cdots + (p_n+q_n)x^n \quad \text{— (i)}$$

$$\text{For } c \in R, \quad c*p = cp_0 + cp_1x + \cdots + cp_nx^n \quad \text{— (ii)}$$

Now, $p = p_o + p_1x + p_2x^2 + \cdots + p_nx^n$ and $q = q_o + q_1x + q_2x^2 + \cdots + q_nx^n$. So let me define vector addition p+q=$(p_o + q_o)$+$(p_1 + q_1)$x+$(p_2 + q_2)$ x^2+…+$(p_n + q_n)$ x^n. So, this is my vector addition. And scalar multiples as for c $\in$ R, c*p=$cp_o + cp_1x + cp_2x^2 + \cdots + cp_nx^n$. So, this is my scalar multiplication.

Again, one can check with respect to this vector addition and scalar multiplication this set of collection of all the polynomial of degree less than or equal to n is again a vector space. So, I hope you have understood the definitions of what is vector space by these couple of examples, I will definitely continue with that. So, we have defined what is our vector space. We have seen different vector spaces through these different examples. So, we will continue in our next class the further development in these directions. Thank you.