Advanced Linear Algebra
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Lecture – 39
Applications of Primary Decomposition Theorem

So, welcome to the lecture series of Advanced Linear Algebra we have already learned on interesting results about the decomposition of space as well as operators. That is if T be a linear operator define over a finite dimensional vector space V which is also defined over the field F then the space can be decomposed as a direct sum of invariant subspaces. And the operator T will also can be decomposed by finite number of operators which are basically restriction of the operator T to elements of space which is basically primary decomposition theorems.

Today we are going to apply this concept in different places. First, let me consider the applications of these theorems in solving initial value problems.

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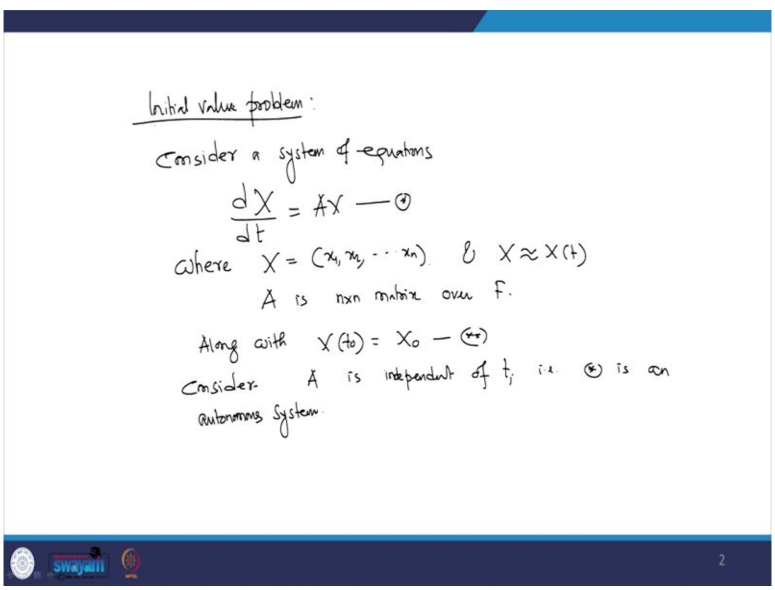

So, let us quickly recall and in this what is initial value problem. So, initial value problem consider a system of equation $\frac{dX}{dt}$ = AX where X = $(x_1, x_2, \ldots x_n)$ and X = f(t), I mean each x_i is function of t, A is nxn matrix over the field F along with X $(t_o) = X_o$. The (*) and (**) is basically gives an initial value problem.

Consider A is independent of t_i i.e. (*) is an autonomous system. So, in this case to find the general solution of the system (*) since A is nxn matrix I mean X belongs to you know n dimension space. So, I will have in linearly independent solutions of (*). So, the general solution must consist of n linearly independent solutions while finding the solution in our theory of OD(ordinary differential) we have seen that we have used the concept of generalized eigenvector.

So, you will see that the primary decomposition theorem at used there to find the general solution of the system star please specifically when system is autonomous.

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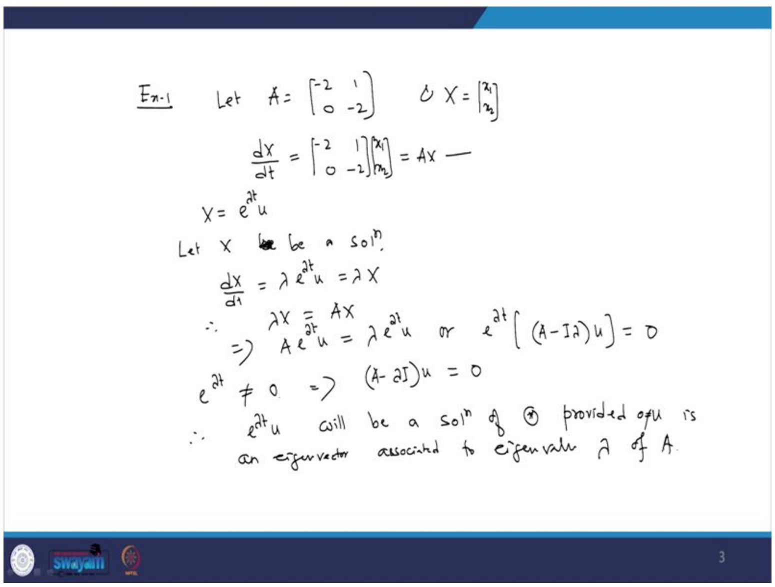

Let me consider some one small examples let A = $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ & X is a 2 dimensional vector. So, let me consider X = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. So, we have $\frac{dX}{dt}$ = $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ = AX we have already seen we have already started on theory OD the solution of this type of system is of the form of say X is solution then it is of the form of, X = $e^{\lambda t} u$

Let X be a solution then we have $\frac{dX}{dt} = \lambda e^{\lambda t} = \lambda X$. So, we have $\lambda X = AX \Longrightarrow$ So, this implies $Ae^{\lambda t} u = \lambda e^{\lambda t} u$ or $e^{\lambda t}[(A - I\lambda)u] = 0$.

$e^{\lambda t} \neq 0 \quad \Longrightarrow (A - I\lambda)u = 0,$ so $e^{\lambda t} u$ will be a solution of (*)equation provided u is an eigenvector provided $u \neq 0$ is an eigenvector associated to eigenvalue λ of A. So, first I have to find the what are the eigenvalues of this matrix first.

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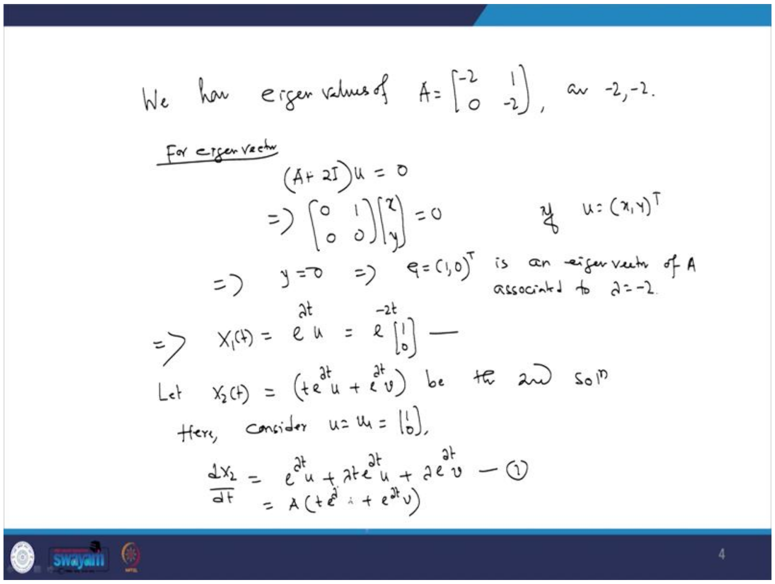

We have eigenvalues of A = $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ is a triangular matrix are -2 -2 because the diagonal it will be the eigenvalues of the triangular matrix. So, what are the eigenvectors. So, for eigenvectors we have (A+2I)u = 0 $\implies \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if u = $[x, y]^T$.

$\implies$ y = 0 $\implies$ $e_1 = [1, 0]^T$ is a an eigenvector of A associated to $\lambda =$ -2. So, this implies I can consider the one solution $X_1(t) = e^{\lambda t}u = e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, this is the one solutions at t = 0, I am getting $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so, which is non-zeo vector.

So, this is one Solutions but we need another solution also another which is also linearly independent and then linear combination of the two solution will give the general solution of the system X just equal to AX, let me consider let $X_2(t) = \left(te^{\lambda t}u + e^{\lambda t}v\right)$ be the second solution. How to know whether this will be the second Solutions how to guess what will the general Solutions all these things so, this I will discuss later on.

For timing you allow me say X(t) equals some function like this since the $\lambda =$ -2 is repeated twice.

So, using the theory of OD for the single variable case based on that spirit what if it $e^{\lambda t}u$ is one then $te^{\lambda t}u$ to the other ones. So, based on that spirit I am writing that $X_2(t) = te^{\lambda t}u + e^{\lambda t}v$.

Here I will consider u = $u_1 = [1, 0]^T$ that is the eigenvectors associate to $\lambda = -2$ anyhow so, I have written like this now again if this is the solution of the differential equation, $X' = AX$ it must satisfy the equations. So, $\frac{dX_2}{dt} = e^{\lambda t}u + \lambda te^{\lambda t}u + \lambda e^{\lambda t}v = A(te^{\lambda t}u + e^{\lambda t}v)$

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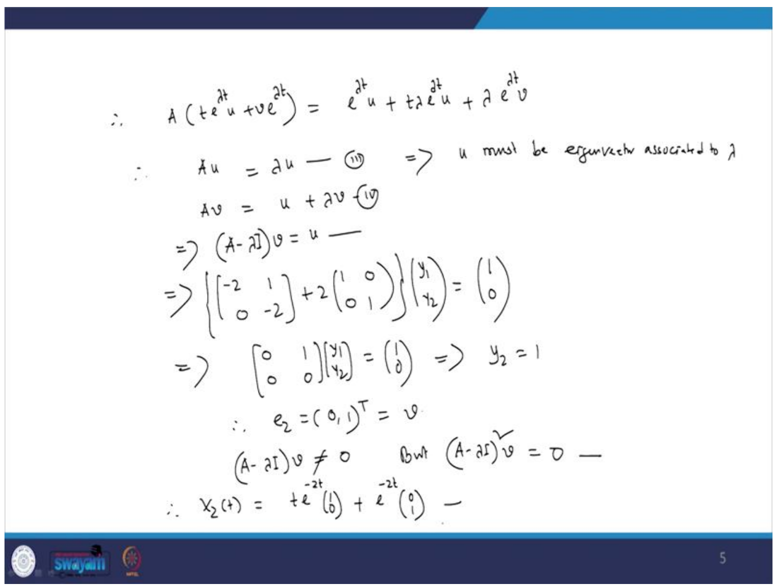

So, I will have A $(te^{\lambda t}u + ve^{\lambda t}) = e^{\lambda t}u + t\lambda e^{\lambda t}u + \lambda e^{\lambda t}v$. So, comparing the coefficient of $te^{\lambda t}$ and $e^{\lambda t}$ I will have from the both sides. So, Au = λu.

And then other one I am getting coefficient to the λt as Av = u + λv. So, this implies Au = λu see I told you that I have to consider u as u_1 eigenvector the justification is coming from this. So, if I consider this is a third equations and this is a fourth equations the third this implies that u must be eigenvector associated to λ. So, which is already calculated and is known.

The second function is v. So, that is satisfying the fourth equations. So, this implies that I am getting (A-λ)v = u. So, the fourth equation can be written like this thing. So, this implies that (A-

λ)v = u $\Rightarrow \{\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow y_2 = 1$. So, I can consider $e_2 = [0, 1]^T$ = v then I am getting v is also linearly independent to u. I am getting two linear independent vectors u and v and we see that (A-λI)v $\neq 0$ because that is equal to u $\neq 0$.

But $(A - \lambda I)^2 v = 0$ why $(A - \lambda I)^2 v = 0$ because for this polynomial the characteristic polynomial is exactly equal to the minimal polynomial which is equal to $(X - \lambda I)^2$. So, this means here $\lambda = -2$. So, therefore any element in the space of f_2 will be basically annihilated by $(A - \lambda I)^2$

So, that is why $(A - \lambda I)^2 v = 0$ and. So, I can write down. So, my second Solutions $X_2(t) = te^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So, this is my second Solution please now you see as t = 0 I am getting $X_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ & $X_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ both are independent to each others. So, therefore X_1 and X_2 are also independent and also independent.

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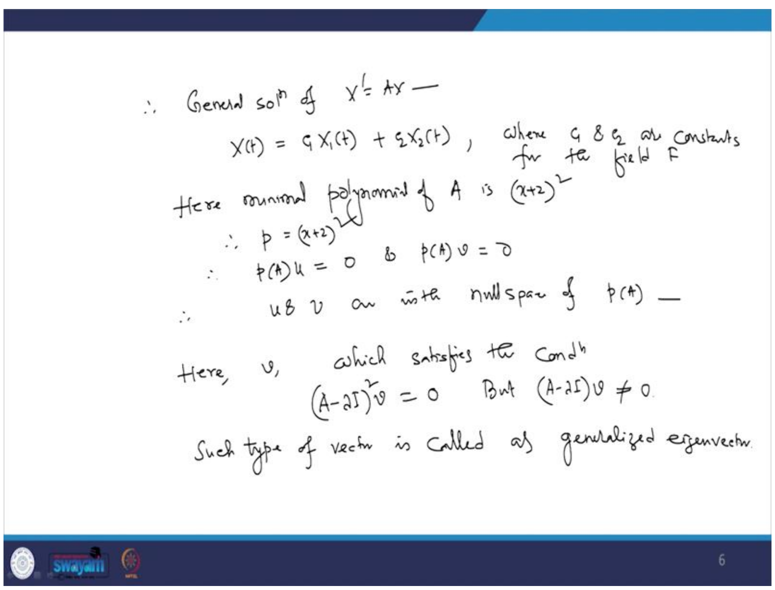

So, General solution General solution of $X' = AX$, I will have $X(t) = c_1X_1(t) + c_2X_2(t)$, where c_1, c_2 are constant over the corresponding field F where c_1 and c_2 are constant standard from the field F overall the system is defined. So, I got this general solution of the system $X' = AX$ but friends I have mentioned that here minimal polynomial of A is $(x + 2)^2$.

So, p = $(x+2)^2$ and I see that which is also characteristic polynomial here also we see that when you consider p(A)u = 0 & p(A)v = 0, u the eigenvectors what I calculate u = $\begin{bmatrix}1\\0\end{bmatrix}$ and v = $\begin{bmatrix}0\\1\end{bmatrix}$. I am not saying that v is eigenvector.

I am saying that only u is eigenvector, v is not so, this implies u and v are in the Null space of p(A). Here v which satisfy the conditions that condition that $(A-\lambda I)^2 v = 0$ but $(A-\lambda I)v \neq 0$ such type of vector is called as generalized eigenvector.

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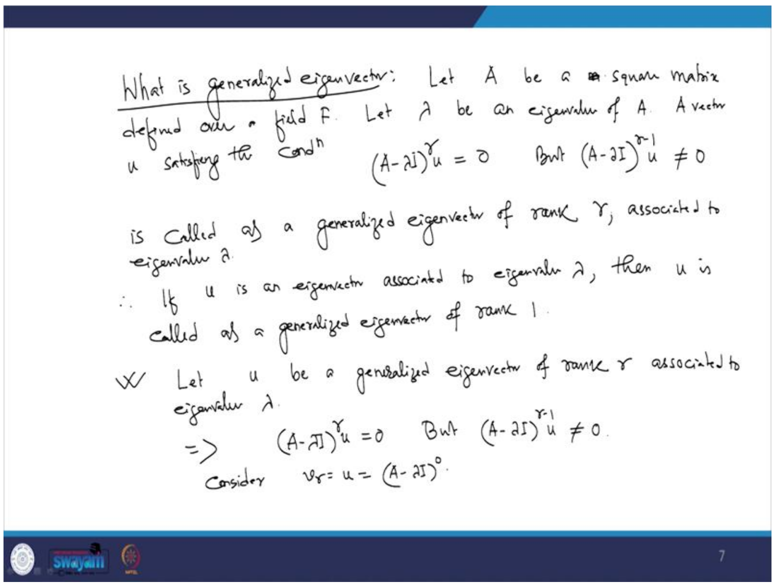

So, let me define what is generalized eigenvectors. Let A be a square matrix define over a field say F or let T be a linear operator defined over a final dimension vector space V over the field F. Let λ be a eigenvalue of A & a vector u satisfying the condition satisfying the condition, $(A-\lambda I)^r u = 0$ but $(A-\lambda I)^{r-1} u \neq 0$ is called as a generalized eigenvector of rank r.

So, actually this I already have seen that in the solution of the initial problem the concept of generalized eigenvector is there. This is called generalized eigenvector rank are associated to eigenvalue λ. So, if you is an eigenvector associated to eigenvalue λ then u is called as a generalized eigenvector of rank 1. Now let us see if I consider a vector generalized eigenvector of rank r also that introduce our vectors which are linearly independent.

Let u be a generalized eigenvector of rank are associated to eigenvalue λ. So, this implies

$(\mathrm{A} - \lambda\mathrm{I})^{r} u = 0$ but $(\mathrm{A} - \lambda\mathrm{I})^{r-1} u \neq 0$. Consider $v_r = u = (\mathrm{A} - \lambda\mathrm{I})^{0} u$.

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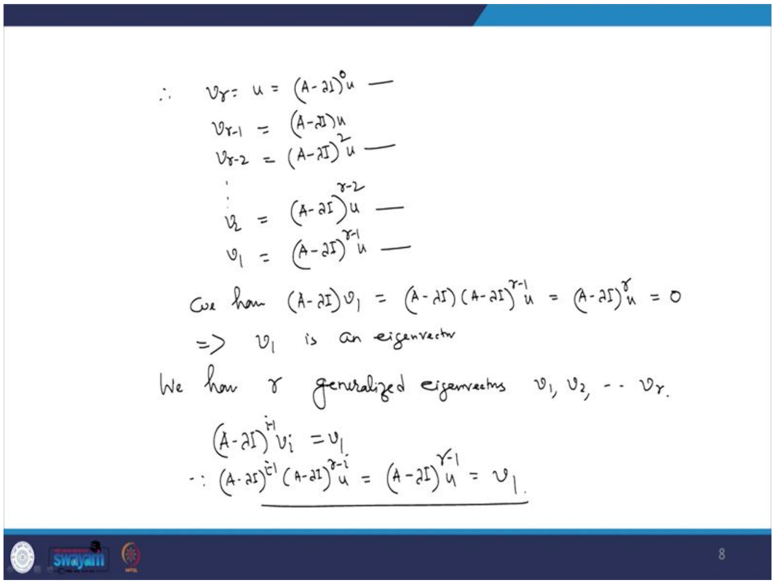

So, we have, $v_r = u = (\mathrm{A} - \lambda\mathrm{I})^0 u$, $v_{r-1} = (\mathrm{A} - \lambda\mathrm{I})u$, $v_{r-2} = (\mathrm{A} - \lambda\mathrm{I})^2 u$, … $v_2 = (\mathrm{A} - \lambda\mathrm{I})^{r-2} u$, $v_1 = (\mathrm{A} - \lambda\mathrm{I})^{r-1} u$. So, I have introduced r vectors, $v_1, v_2, \ldots\ v_r$ we have according to my definitions from the above, $(\mathrm{A} - \lambda\mathrm{I})v_1 = (\mathrm{A} - \lambda\mathrm{I})(\mathrm{A} - \lambda\mathrm{I})^{r-1} u = (\mathrm{A} - \lambda\mathrm{I})^r u = 0$

because we know u is the vector satisfying $(\mathrm{A} - \lambda\mathrm{I})^r u = 0$. So, this implies v_1 is an eigenvector . So, what can you see our v_2 will it be eigenvector, no v_2 cannot be because I will not say that $(\mathrm{A} - \lambda\mathrm{I})v_1 = 0$.

So, you will see that that is equal to some non-zero vector. So, v_2 cannot be an eigenvector however this $v_1, v_2, \ldots\ v_r$ they are basically generalized eigenvectors. So, we have we have r generalized eigenvectors $v_1, v_2, \ldots\ v_r$ if I consider $(\mathrm{A} - \lambda\mathrm{I})^{i-1} v_i = v_1$ So, what can we say about this once $(\mathrm{A} - \lambda\mathrm{I})^{i} v_i$. So, I basically if I consider from 1 to r then this is equal to v_1.

Because v_i will be according to this formula r-i because we have, $(\mathrm{A} - \lambda\mathrm{I})^{i-1}(\mathrm{A} - \lambda\mathrm{I})^{r-i} u = (\mathrm{A} - \lambda\mathrm{I})^{r-1} u = v_1$ and you know so, this is some relation I am getting this.

So, for a generalized eigenvector of rank r we have seen there are r vectors we have from this r vectors.

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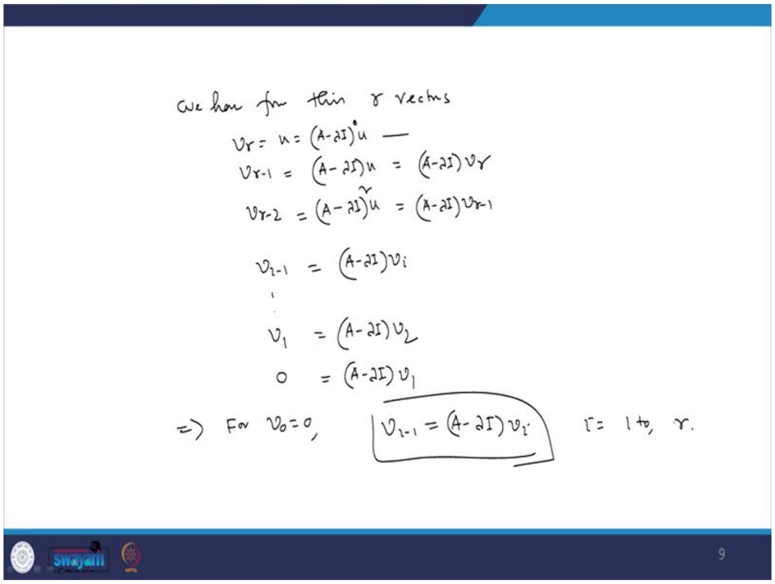

We have, $v_r = u = (\mathrm{A} - \lambda \mathrm{I})^0 u$, $v_{r-1} = (\mathrm{A} - \lambda \mathrm{I})u = (\mathrm{A} - \lambda \mathrm{I})v_r$, $v_{r-2} = (\mathrm{A} - \lambda \mathrm{I})^2 u = (\mathrm{A} - \lambda \mathrm{I})v_{r-1}$, … $v_{i-1} = (\mathrm{A} - \lambda \mathrm{I})v_i$ $v_1 = (\mathrm{A} - \lambda \mathrm{I})v_2$, $0 = (\mathrm{A} - \lambda \mathrm{I})v_1$

So, if I consider $v_0 = 0$ then I have this type relations. So, this implies that this implies for $v_0 = 0$, I can write a relation like you know $v_{i-1} = (\mathrm{A} - \lambda \mathrm{I})v_i$ for i = 1 to r.

So, I have r vectors generalize eigenvectors from the generalized eigenvector u of rank r this is called the I mean this gives r vector which is called a chain of generalized eigenvectors of length r. So, the set $v_1, v_2, \ldots\ v_r$ is called a chain of generalized eigenvector of length 1.

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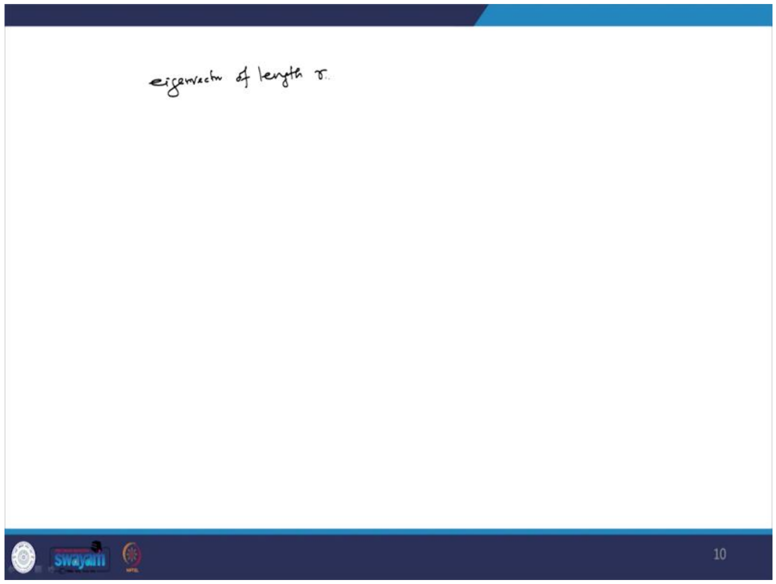

So, we see that if we consider a generalized eigenvector of rank r it introduces a chain of r generalized eigenvectors $v_1, v_2, \ldots\ v_r$ where v_i is eigenvectors of rank 1. I am giving to you that prove that this chain of r vectors are linearly independent consider it as a homework. We will continue this in next class also.