Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 38
Decomposition of Space and Operator

Welcome to lecture series today we are going to prove the primary decomposition theorem. In last time we have seen using this theorem if a linear operator define over a finite dimensional vector space and the vector space is defined over a an algebraic recluse field F. Then the operator T can be written as sum of a diagnosable matrix and nilpotent matrix stress but primary decommission theorem will talk about generalized way.

So, here the field is not necessary to be even algebraically closed field I mean to say the minimal polynomial may not be product of linear factors. So, let us quickly recall what is the primary decomposition theorem the statement of the theorem is like this say T be a linear operator define over a finite dimensional vector space V over the field F and let p be a minimal polynomial of T such that p = ${p_1}^{r_1}\ {p_2}^{r_2} \ldots {p_k}^{r_k}$.

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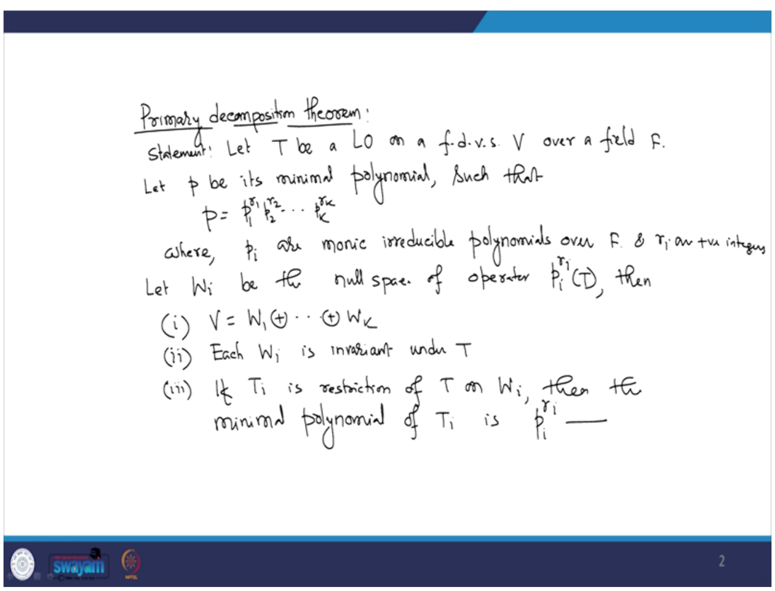

Where ${p_i}^{r_i}$ monic irreducible polynomial over the field F and r_i are positive integers positive integers. Let W_i be the null space of the operator ${p_i}^{r_i}(T)$ then when the polynomial operator

$p_i^{r_i}(T)$, then (I) V = $W_1 \oplus W_2 \oplus \dots \oplus W_k$ where each W_i is invariant under T. So, this two information will say that the identity operator can be written as sum of some projection operator $E_1, E_2, \dots E_k$.

And (iii) if T_i the restriction of T on the W_i where W_i the null space of $p_i^{r_i}(T)$ this operator then the minimal polynomial of T_i is $p_i^{r_i}$.

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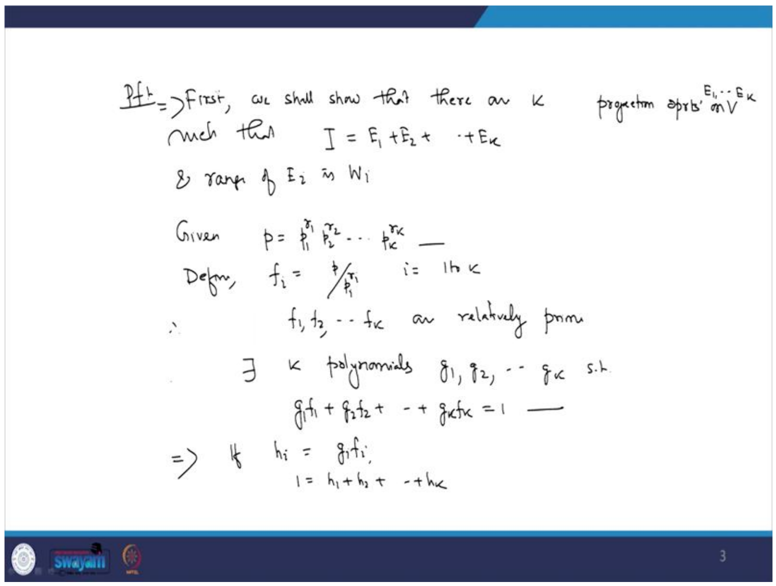

So, let us prove 1 by 1 first, I will show that we shall show that there are k production operator on V such that I = $E_1 + E_2 + \dots + E_k$. So, first let us prove this one please and then I will show that the space having direct sum decomposition piece and the range of E_i is W_i so this is first one, given p = $p_1^{r_1}, p_2^{r_2}, \dots p_k^{r_k}$ define $f_i = \frac{p}{p_i^{r_i}}$ where f_i is polynomial when this factor is taken out that will basically f_i So, it will be i = 1 to k.

So, then the gcd($f_1, f_2, \dots f_k$) =1. So, I can say that $f_1, f_2, \dots f_k$ are relatively prime they are relatively prime polynomial. So, in number theory there is a results if the gcd($f_1, f_2, \dots f_k$) are relatively prime then there exist $f_1, f_2, \dots f_k$ such that $f_1 g_1 + f_2 g_2 + \dots + f_k g_k$ = 1 when this $f_1, f_2, \dots f_k$ are numbers but this is also true over the you know polynomial field theory also that since $f_1, f_2, \dots f_k$ are relatively prime.

So, there exist k polynomials may also $g_1, g_2, \dots g_k$ such that $g_1 f_1 + g_2 f_2 + \dots + g_k f_k = 1$.

So, I have utilized that results that when the polynomial $f_1, f_2, \ldots\ f_k$ relatively prime or gcd equal to 1 then exist k polynomial $g_1, g_2, \ldots\ g_k$ such that $g_1 f_1 +\ g_2 f_2 +\ \ldots + g_k f_k\ = 1$. So, this implies if $h_i = g_i h_i$ I can write down that 1= $h_1 +\ h_2 +\ \ldots + h_k$.

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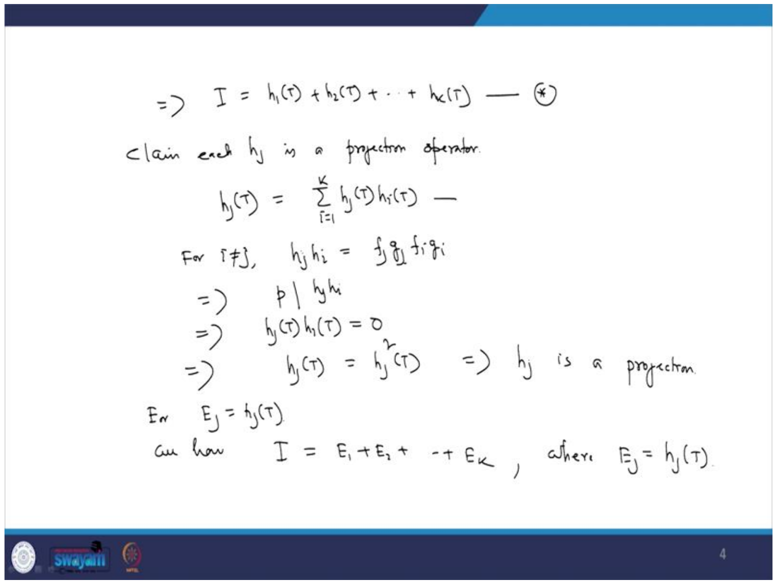

$\Longrightarrow$ I = h_1(T) + h_2(T) + $\ldots + h_k(T)$. Claim is h_j is a projection operator. So, if I multiply if I multiply h_j to the both side of the (*), we have $h_j(T) = \sum_{i=1}^{k} h_j(T)\, h_i(T)$ see note that for i $\neq$ j, $h_j h_i = f_j g_j f_i g_i$.

$\Longrightarrow$ p | $h_j h_i$ because in $f_j {p_j}^{r_j}$ just missing but which is in the f_i $\Longrightarrow$ $h_j(T) h_i(T) = 0$ because p(T) that minimal polynomial operator then p(T) will be zero. So, p(T) is a factor of $h_j(T) h_i(T)$. So, $h_j(T) h_i(T) = 0 \Longrightarrow h_j(T) = {h_j}^2$(T) $\Longrightarrow$ h_j is a projection.

it satisfy the definition of the projection. So, I have i is a sum of projection operators. So, if I consider for $E_j = h_j(T)$ then we have we have I = $E_1 +\ E_2 +\ \ldots + E_k$ where $E_j = h_j(T)$.

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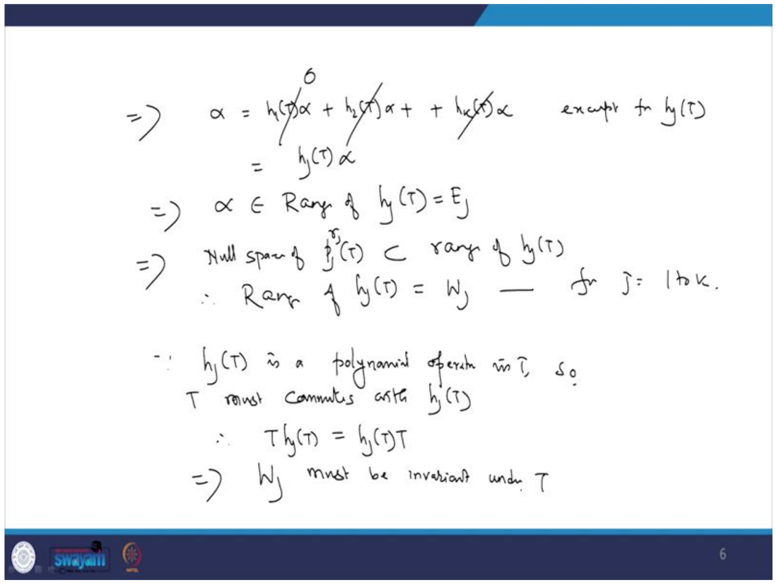

Now I want to show that range of $h_j(T)$ is W_j let $\alpha \in$ range of $h_j(T) \implies \alpha = h_j(T)\alpha \implies p_j{}^{r_j}(T)\alpha = p_j{}^{r_j}(T)h_j(T)\alpha = 0$ since p | $p_j{}^{r_j}h_j$.

So, therefore I will have $p_j{}^{r_j}(T)h_j(T)\alpha = 0 \implies \alpha \in$ Null space of $p_j{}^{r_j}(T) = W_j \implies$ range of $E_j \subset W_j$. Now I want to show it other way let $\alpha \in W_j \implies p_j{}^{r_j}(T)\alpha = 0 \implies h_j(T)\alpha = 0$ for i $\neq$ j.

Since $p_j{}^{r_j} \mid h_j$ for i $\neq$ j So, that is why $h_j(T)\alpha = 0$. So, this implies see $\alpha \in W_j$. So, $\alpha \in$ V. So, $\alpha = h_1(\mathrm{T})\alpha + h_2(\mathrm{T})\alpha + \dots + h_k(T)\alpha = h_j(\mathrm{T})\alpha$ where each of them are zero, except for $h_j(\mathrm{T})$. So, this implies $\alpha \in$ R of $h_j(T)$.

$\implies$ Null space of $p_j{}^{r_j}(T) \subset$ Range of $h_j(T)$. So, Range of $h_j(T) = W_j$ for i= 1 to K. So, we have seen that exists k projection operator $h_1, h_2, \dots h_k$ or $E_1, E_2, \dots E_k$ such that I = $E_1 + E_2 + \dots + E_k$ and W_j is the Range of $h_j(T)$, since $h_j(T)$ is a polynomial is a operator in T. So, T must commutes with $h_j(T)$ because these are all polynomial in T.

So, if I multiply left hand side or right inside of the T it will be remain same. So, T $h_j(T) = h_j(T)$ T $\implies$ since T commutes with the each projection operator. So, by previous results W_j must be invariant under T in previous result we have seen that if the projection operator and each projection operator is commutes with the operator T then only will say that the each W they will be invariant under T.

So, using those results I can immediately say here that W_j must be invariant under T. So, we have seen here the linear operator T whose minimum polynomial is product of like p = $p_1^{r_1}\, p_2^{r_2}\ \ldots\ p_k^{r_k}$ introduces k projection operators $h_1T_1, h_2T_2, \ldots, h_kT_k$ or $E_1 + E_2 + \ldots + E_k$ such that I = $E_1 + E_2 + \ldots + E_k$ and the Range of each E_i is W_i which is invariant under T. So, therefore T can be written as direct sum of its restriction operator $T_1, T_2, \ldots T_k$ on $W_1, W_2, \ldots W_k$.

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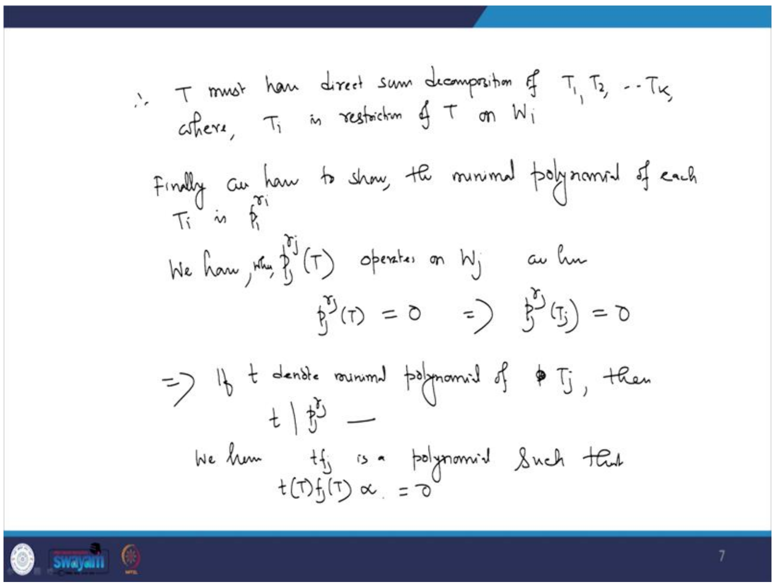

So, T must have direct sum decomposition of $T_1, T_2, \ldots T_k$ where T_i each restriction of T on W_i. So, you see that the space is also having direct sum decomposition of inherent subspaces $W_1, W_2, \ldots W_k$ and T is also having direct sum decomposition of operator $T_1, T_2, \ldots T_k$ finally we have to show to show the minimal polynomial of is T_i is $p_i^{r_i}$ we have, when $p_i^{r_i}$(T) this polynomial operator on W_j we have $p_i^{r_i}$(T) = 0.

$\Longrightarrow$ $p_j^{r_j}(T_j)$ = 0, when $p_j^{r_j}(T_j)W_j$ means restricted toW_j. So, it is basically T_j. So, $p_j^{r_j}(T_j)$ = 0.
$\Longrightarrow$ if t denote minimal polynomial of T_j then t | $p_j^{r_j}$ we have tf_j is a polynomial such that you know t$(T)f_j(T)\alpha = 0$ where t$(T)f_j(T)\alpha \in$ V.

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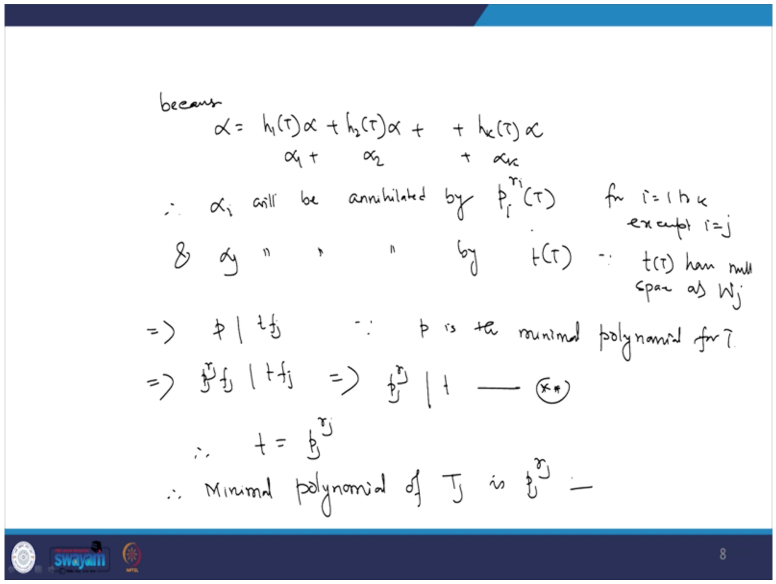

Why because $\alpha = h_1(T)\alpha + h_2(T)\alpha + \ldots h_k(T)\alpha$. So, I will have a component over the you know like $\alpha_1 + \alpha_2 + \ldots + \alpha_n$. So, this α_i will be I mean α_i will be annihilated by $p_i{}^{r_i}$(T) because α is coming from W_i and W_i is the Null space of $p_i{}^{r_i}$(T) for I = 1to k except i ≠ j and α_j will be annihilated by t(T) because t(T) having Null space as W_j.

So, this implies t(T) because α_j by t(T) because α_j element in W_j and t(T_j) we have already mentioned that t(T_j) = 0 because t is the minimum polynomial of T_j see t is the minimal polynomial T_j. So, therefore α_j will be annulated by t(T) this implies p | tf_j since p is the minimal polynomial for T and we see that for any element $\alpha \in$ V is animated by t(T_j)(T).

So, this implies p | tf_j. $p_j{}^{r_j} f_j \mid tf_j$. So, this implies $p_j{}^{r_j} \mid t$. So, in one place we have seen the $p_j{}^{r_j} \mid t$ and another case $p_j{}^{r_j} f_j$ | p. So, these two informations together say that so, t = $p_j{}^{r_j}$. So, minimal polynomial of T_j is $p_j{}^{r_j}$.

So, this is the proof of this theorem and you will see this theorem can be used in variety of places which will discuss in our next classes.