

Advanced Linear Algebra
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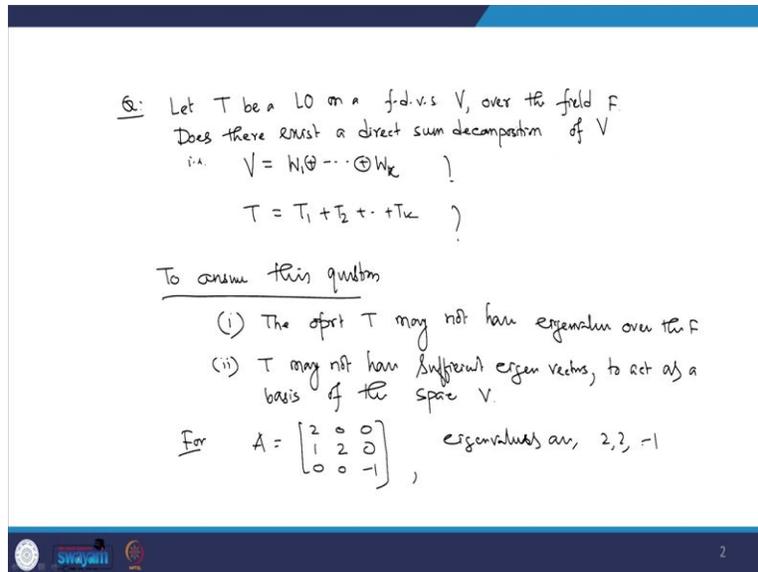
Lecture – 37
Decomposition of space and Operator - I

Welcome to my lecture series of Advance Linear Algebra. We have seen when a linear operator is defined over a finite dimensional vector space where the vector space is having direct sum decomposition of some subspaces which are invariant under T then the operator having direct sun decomposition of some finite operators which are basically restriction of the operator T to corresponding subspaces.

Now question is if I consider a general linear operator on a finite dimensional vector space V then what can we say about this operator whether is it possible to have a direct sum decomposition of the space d under some subspaces. And the operator can be decomposed as a direct sums of some other operators which are basically restriction to the operator the original operator T to the corresponding subspaces.

Let me quickly write down the questions so, that we can see the importance of these questions and how to answer the question.

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So, what I am saying that, Q: - Let T be a linear operator on a finite dimensional vector space V over the field F question is does there exist a direct sum decomposition of vector space V that is does there exist some subspaces like W_1, W_2, \dots, W_k . So, that $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ and the operator $T = T_1 + T_2 + \dots + T_k$. So, this is a basically the question.

So, does a direct sort of decomposition of V like this and operator $T = T_1 + T_2 + \dots + T_k$, i mean does that exist such type of operators in our operators which are basically restriction of T to the corresponding subspaces. As earlier we have seen when the vector space is having direct sum decomposition of some invariant of spaces. So, in that case we have seen that $T = T_1 + T_2 + \dots + T_k$.

Now to answer these questions if I look in terms of the eigenvalue and eigenvector of the operators. So, we see that there are 2 problems, (i) The operator T may not have eigenvalues over the given field. So, that is basically the drawback of the field but this drawback can be overcome if we change this field to a field which is algebraically closed. So, in that case this problem will be resolved.

However, (ii) T may not have sufficient eigenvectors to act as a basis of the space of the space V .

So, for examples if I consider say $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. So, this Matrix this is a triangular Matrix.

So, certainly eigenvalues are 2, 2 and -1. So, $(x - 2)^2(x+1)$ is a characteristic polynomial of this Matrix.

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Eigenvector associated $\lambda = -1$

$$(A-\lambda I)X = 0 \Rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow x_1 = 0, \quad x_1 + 3x_2 = 0, \quad x_3 = 0$$

$$\Rightarrow e_3 = (0, 0, 1)$$

Eigenvector associated $\lambda = 2$

$$(A-2I)X = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = 0, \quad x_3 = 0$$

$$\Rightarrow e_2 = (0, 1, 0)$$

Null space of $(A-2I)^2$

$$(A-2I)^2 X = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\therefore e_1 = (1, 0, 0), e_2 = (0, 1, 0)$

as two L.I. solⁿ of $(A-2I)^2 X = 0$

we see eigenvector associated to eigenvalue $\lambda = -1$ then I will have $(A-\lambda I)X = 0 \Rightarrow$

$$\left\{ \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0 \text{ and } x_1 + 3x_2 = 0, \text{ since}$$

$x_1 = 0$ means $x_2 = 0$. So, this implies $e_3 = (0, 0, 1)$ is the eigenvector, associated to $\lambda = -1$. Now

similarly if I consider eigen vector associated to eigenvalue, $\lambda = 2$, I will have what I will have

$$\text{basically } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0 \text{ and } x_3 = 0 \text{ So, } e_2 = (0, 1, 0) \text{ is the eigenvector}$$

for eigenvalue $\lambda = 2$.

And I will have only 1 eigenvector associated to eigenvalue $\lambda = 2$ because this Matrix is having rank = 2. So, the nullity = 1. So, I will have like this but interestingly you see if I consider say

$$\text{here } (A-2I) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}. \text{ Now if I consider } (A-2I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} \text{ So, null space of } (A-2I)^2$$

will be what that will basically, $(A-2I)^2 X = 0$ that you have to solve it. So, this means that then

$x_3 = 0$ So, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ as two linearly independent Solutions of $(A-2I)^2 X = 0$.

So, the dimension of the null space of $(A-2I)^2 = 2$, I will have 2 linear independent vector.

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(ii) Each W_i is invariant under T .

(iii) If T_i denote the restriction of T on W_i , then, the minimal polynomial of T_i is $p_i(x)$

✓ $p_1 = (x-a), p_2 = (x-a)^2, \dots, p_k = (x-a_k)$

$p = (x-a_1)^{r_1} (x-a_2)^{r_2} \dots (x-a_k)^{r_k}$

where, a_1, a_2, \dots, a_k are distinct eigen values of T

$D = c_1 E_1 + c_2 E_2 + \dots + c_k E_k$

Then certainly, D will be a diagonalizable operator

we will say D is diagonal part of the operator T

Let $N = T - D$

Now you see the beauty here. So, null space of $(A + I)$ which is span by e_3 so that is let us consider let be W_1 and null space of $(A - 2I)^2$ is spanned by e_1 and e_2 let it be called as say W_2 . then we see that the vector space $V = W_1 \oplus W_2$. So, when I consider the null space of $(A - 2I)^2$ that they are talking about the null space of $(A - 2I)$ then I am getting the vectors linear independent vectors in the null space.

And then we are able to have a direct sum decomposition of the space V in terms of W_1, W_2 . So, this give a hint to think about the whether if I consider any linear operator defined over a finite Dimension vector space whether it is possible to have direct some decomposition of the space as well as operators. So, this is basically we can answer this question in terms of using a more general theorems called primary decomposition theorem please.

So, the primary decomposition theorem will tell us when a linear operator define over a finite dimensional vector space then using that theorem we will be able to say that the operator can be written as a direct sum of finite linear operators and this space also can be written as direct sum decomposition of invariant subspaces. So, let us talk about the what is primary decomposition theorem please. Primary see in my previous example I what I have consider when the characteristic polynomial is factorizable over the space.

But this primary decomposition theorems does not require whether characteristic polynomial will

be factorizable or not. So, it is more General way to prove the existence of direct sum decomposition of the space as well as operator please. So, what is that it is like this let T be a linear operator on a finite dimensional vector space over the field say F let p be the minimal polynomial of T and $p = p_1^{r_1}, p_2^{r_2}, \dots p_k^{r_k}$ where p_i Monic irreducible polynomial over the field F and r_i are positive integers.

Let $W_1, W_2, \dots W_k$ be the null space of $p_1^{r_1}(T), p_2^{r_2}(T), \dots p_k^{r_k}(T)$ then (i) $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$. So, V will have direct sum decomposition of the subspaces $W_1, W_2, \dots W_k$ (ii) each W_i is invariant under T (iii) if T_i denote the restriction of T on W_i then the minimal polynomial of $p_i^{r_i}$. So, this is the theorems he is saying that if a linear operator defined over a finite dimensional vector space V where the space is defined over the field F.

If the minimal polynomial of the operator is product of $p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ where p_i are Monic irreducible polynomial over the field F and if $W_1, W_2, \dots W_k$ are the null space of the operator $p_1^{r_1}(T) p_2^{r_2}(T) \dots p_k^{r_k}(T)$ then $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ and is W_i will be invariant under T. And if T_i denote the restriction of T on W_i then minimal polynomial of T_i is $p_i^{r_i}$.

Suppose this theorem is proof, now what we will get from this theorem. Now this theorem will basically tell us a nice structure of the operators. So, let me consider the situations when Monic irreducible polynomial $p_1, p_2, \dots p_k$ are basically first degree linear factors. So, when $p_1 = (x - c_1), p_2 = (x - c_2) \dots p_k = (x - c_k)$. I mean to say when the field is algebraically closed under that situations I can write down that each monic irreducible polynomial as a product of linear factors.

So, in that situation what I am getting minimal polynomial will be it is of the form of, $p = (x - c_1)^{r_1} (x - c_2)^{r_2} \dots (x - c_k)^{r_k}$ whereas $c_1, c_2, \dots c_k$ are distinct eigenvalues of the operator T then if you define these operator like $D = c_1 E_1 + c_2 E_2 + \dots + c_k E_k$ where $E_1, E_2, \dots E_k$ are projection operators. Actually we will see it here this $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$. So, our previous result says that would be projection operator on $W_1, W_2, \dots W_k$ I mean in that case the identity operator can be written as $I = E_1 + E_2 + \dots + E_k$.

So, using the same concept I am writing here this e_i basically that position operator. So, if I define a another operator $D = c_1E_1 + c_2E_2 + \dots + c_kE_k$ then that theorem D will be a diagonalizable operator. So, if I can write that $D = c_1E_1 + c_2E_2 + \dots + c_kE_k$ then certainly D will be a diagonal operator. We will say D is diagonal part of the operator T. So, let me Define any N another operator is defined by set $N = (T-D)$. So, if I take out the diagonal part of the operator T that I am saying that say N.

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Then, we have

$$T = TE_1 + TE_2 + \dots + TE_k \quad \because I = E_1 + \dots + E_k$$

$$D = c_1E_1 + \dots + c_kE_k$$

$$\Rightarrow T - D = (T - c_1I)E_1 + \dots + (T - c_kI)E_k$$

$$\Rightarrow N = T - D = \sum_{i=1}^k (T - c_iI)E_i$$

$$\Rightarrow N^2 = \sum_{i=1}^k (T - c_iI)^2 E_i$$

$$\vdots$$

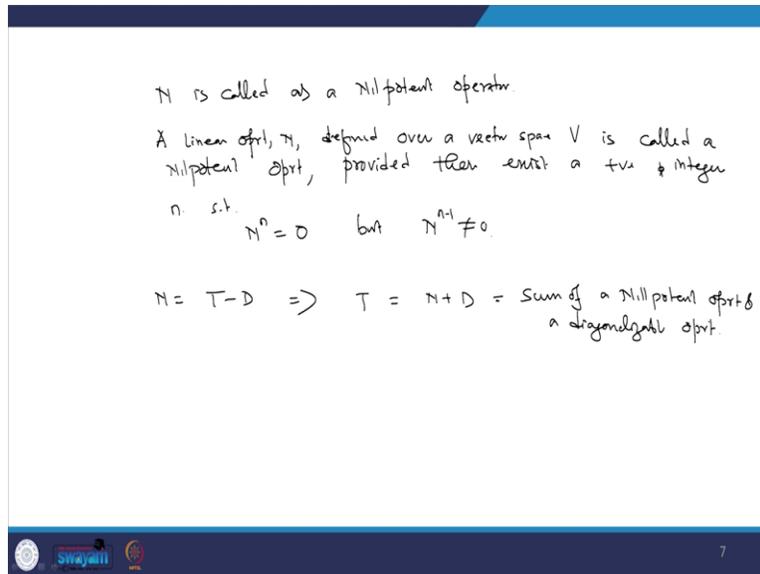
$$N^r = \sum_{i=1}^k (T - c_iI)^r E_i$$

For $r > r_i$ $i = 1$ to k
 $N^r = 0$ \because each $(T - c_iI)^r$ is zero on the range of E_i .

Then we have $T = TE_1 + TE_2 + \dots + TE_k$ & $D = c_1E_1 + c_2E_2 + \dots + c_kE_k$. So, this implies $(T-D) = (T-c_1I)E_1 + (T-c_2I)E_2 + \dots + (T-c_kI)E_k$. So, this implies the new operator $N = T-D = \sum_{i=1}^k (T - c_i) E_i$, $N^2 = \sum_{i=1}^k (T - c_i)^2 E_i$, ... $N^r = \sum_{i=1}^k (T - c_i)^r E_i$

So, see W_i is what W_i is basically according to the primary theorems W_i is the basically range of prediction operator T_i . So, this means that N^r equal to this. So, for $r > r_i$ where $I = 1$ to k , $N^r = 0$ since each $(T - c_i)^r$ is zero on the range of E_i .

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So, this means this N is called as a nilpotent operator what is nilpotent operator a linear operator in defined over a vector space V is called nilpotent operator provided there exist a positive integer n s.t. $N^n = 0$ but $N^{n-1} \neq 0$. So, using primary decomposition theorems when a linear operator T having minimal polynomial as a product of linear factors in that case we have say $N = T - D$

$\Rightarrow T = N + D =$ sum of a Nilpotent operator and a diagonalizable operator. So, this is the beauty of this theorem. So, I can split the operator T into sum of 1 diagonal operators and near power operators. Now this Spectrum this T converted as $N + D$ will certainly help to have a simplified structure of the operator. So, we see that using primary decomposition theorems.

When we apply this theorem to a linear operator whose minimal polynomial or whose characteristic polynomial is product of linear factors in that case the I can have a nilpotent operator and a diagonal level operator D such that the T can be written as $D + N$ and sum of nilpotent operator and diagonal operator. Now this concept will help us to have a simplified structure of the operator please.

So, we will discuss more about all this issue and also the proof of the primary recovery theorems which I have not done it I simply use this results but we will see the proven next class.