Advanced Linear Algebra
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Lecture – 36
Invariant Direct Sums

Welcome to my lecture series of Advance Linear Algebra. In my last class we have discussed about the direct sum decompositions of subspaces of a vector space V where subspaces are invariant under T. We have seen under what condition the subspaces will be invariant and then we have applied this concept to a diagonalizable operators. So, let me summarize that what we have done last time and also answer the questions what I raised in last terms in the form of this theorem please.

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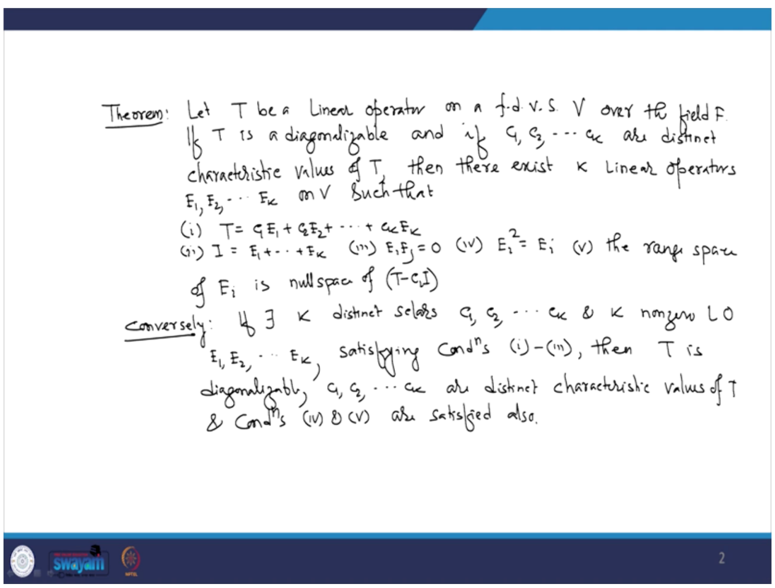

So, what is this let T be a linear operator on a finite dimensional Vector space V over field F. So, let us summarize the result what we did in our last lectures and also answer the question sense which I raised in the end of my lecture. So, I am summarizing in terms of this theorem base it is like this say T be a linear operator on a finite dimensional Vector space V over the field F if T is diagonalizable and if $c_1, c_2, \ldots c_k$ are the distinct eigen values or characteristic values of T.

Then there exists k linear operators $E_1, E_2, \dots E_k$ on V satisfying (i) T = $c_1E_1 + c_2E_2 + \dots + c_kE_k$ (ii) I = $E_1 + E_2 + \dots + E_k$ (iii) $E_j E_i = 0$ for i $\neq j$ (iv) $E_i^{\,2} = E_i$ (v) The range of each E_i is W_i

if I consider T is a diagonalizable. So, certainly this space V will be direct sums of its characteristic spaces and as a consequence of it you will have one can introduce the position of letter $E_1, E_2, \dots E_k$ satisfying all four condition please which we have already seen in our last lectures.

Now the problem is converse. I mean to say if ∃ k distinct scalars $c_1, c_2, \dots c_k$ and k non-zero linear operators $E_1, E_2, \dots E_k$ satisfying condition (i) to (iii) that the operator T = $c_1E_1 + c_2E_2 + \dots + c_kE_k$ and I = $E_1 + E_2 + \dots + E_k$ (iii) $E_j E_i = 0$ for i $\neq j$ then T is diagonalizable and $c_1, c_2, \dots c_k$ are the basically distinct eigen values of this operator T and conditions four and five that is each E_i here will be projection operators and the range of E_i is W_i.

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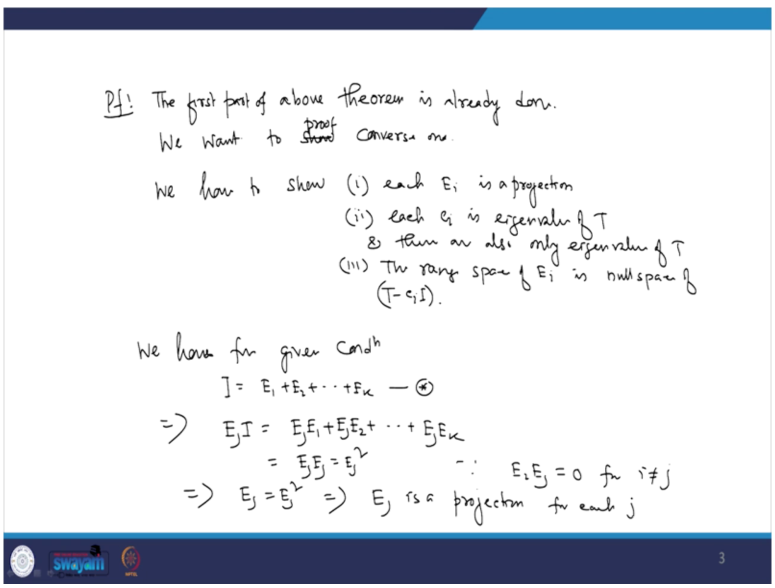

So, now I will prove basically the converse part please. We want to proof converse one. So, what we have to prove. So, we have to prove (i) each E_i is a projection (ii) c_i is eigenvalue of T and these are also only eigenvalue of T (iii) The range space of each E_i is null space of (T-c_i), I if I prove this thing this 3 then certainly T will be diagonalizable. So, that part will be over place.

We have from given condition $I = E_1 + E_2 + \cdots + E_k$ implies if I multiplied E_j from the left side to the both I mean this equations we have $\Longrightarrow E_jI = E_jE_1 + E_jE_2 + \cdots + E_jE_k = E_jE_j =$

$E_j^{\,2}$ since $E_j\, E_i = 0$ for i $\neq j \Longrightarrow E_j = E_j^{\,2}$ $\Longrightarrow$ E_j is a projection for each j = 1 to k. Now we want to show that c_i a constant $c_1, c_2, \dots\ c_k$ are basically eigenvalues of T.

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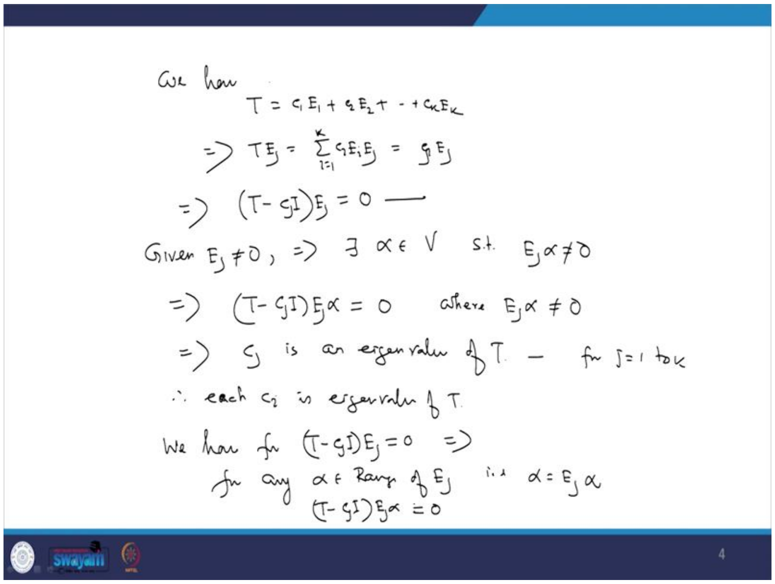

We have T = $c_1E_1 +\ c_2E_2 + \dots\ + c_kE_k$ $\Longrightarrow$ TE_j = $\sum_{i=1}^{k} c_i\, E_iE_j$ = c_jE_j $\Longrightarrow$ (T-c_jI) $E_j = 0$ and given each $E_j \neq 0$, $\Longrightarrow \exists\ \alpha \in$ V s.t. $E_j\alpha \neq 0$ $\Longrightarrow$ (T-c_jI) $E_j\alpha = 0$, where $E_j\alpha \neq 0 \Longrightarrow$ is eigenvalue of T.

because T($E_j\alpha$) = $c_jE_j\alpha$ where $\alpha \neq 0$. So, see there is an eigenvalue of T and for j = 1 to k, because I have multiplied E_j you can do for $E_1, E_2, \dots\ E_k$ and we have that each c_j. So, each c_j is eigenvalue of T.

Now we also have from (T-c_jI) $E_j = 0$ $\Longrightarrow$ for any $\alpha \in$ Range of E_j, i.e. $\alpha = E_j\alpha$, (T-c_jI) $E_j\alpha = 0$.

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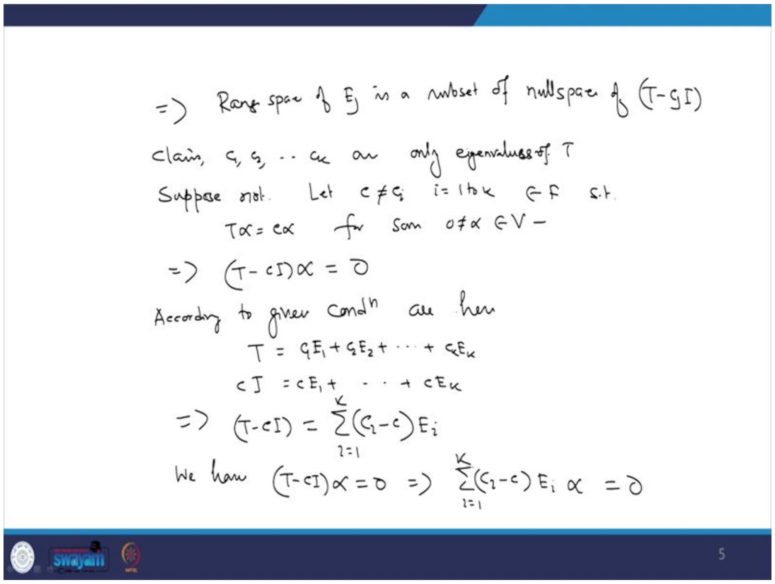

$\Longrightarrow$ of E_j is a subset the subset of null space of (T-c_jI). So, you see that the range space of the each E_j operator which of I have proved the projection operator is basically a subset or Subspace of the null space of (T-c_jI), I want to show that the range space of E_j is exactly equal to null space of (T-c_jI) but before proving that once let me first prove that $c_1, c_2, \dots\ c_k$ are the only eigenvalues of the operator T, claim $c_1, c_2, \dots\ c_k$ are only eigenvalue of T.

Suppose not let c $\neq$ c_i, i = 1 to k $\in$ F s.t. Tα = cα for some 0 $\neq$ α $\in$ V. So, this implies (T-cI) α = 0 according to given condition we have, T = $c_1E_1 + c_2E_2 + \dots + c_kE_k$ and I = $E_1 + E_2 + \dots + E_k$. $\Longrightarrow$ (T-cI) $= \sum_{i=1}^{k}(c_i - c)\,E_i$. So, just if I multiply c upon the sides then I am getting c.

So, if I multiply c to both the sides of this expression I = $E_1 + E_2 + \dots + E_k$, then I will get Ic= $cE_1 + cE_2 + \dots + cE_k$. I will have T and subtract that (T-cI) if I consider then I am having basically $\sum_{i=1}^{k}(c_i - c)\,E_i$. So, I have like this thing. Now we have according to our hypothesis (T-cI) α = 0 $\Longrightarrow$ $\sum_{i=1}^{k}(c_i - c)\,E_i\,\alpha = 0$.

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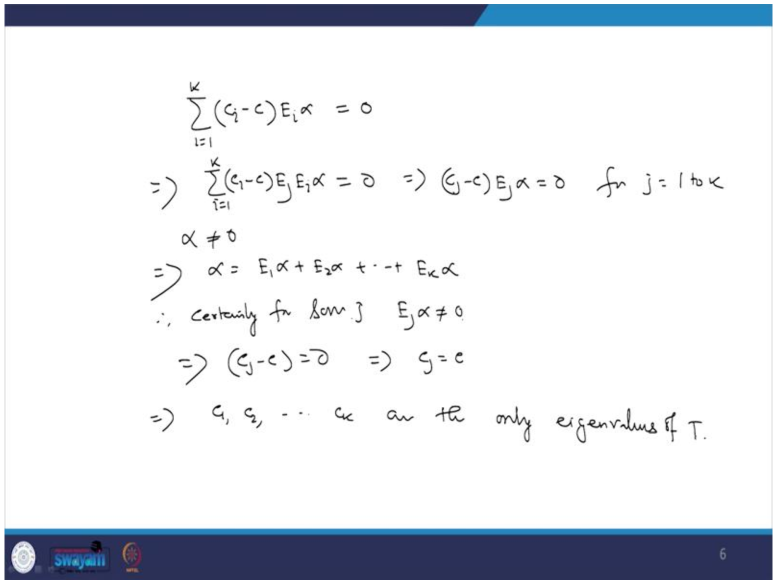

$\sum_{i=1}^{k}(c_i - c)\, E_i\, \alpha = 0 \implies \sum_{i=1}^{k}(c_i - c)\, E_j E_i\, \alpha = 0 \implies (c_j - c)E_j\alpha = 0$ for j = 1 to k because I have multiplied E_jE_j and I am getting this 1. $\alpha \neq 0 \implies$ you know according to this tracks are given condition, $\alpha = E_1\alpha + E_2\alpha + \ldots + E_k\alpha$.

So, certainly for some j, $E_j\alpha \neq 0, \implies (c_j - c) = 0 \implies c_j = c$ because product of two non-zero quantity number cannot be zero over F $\implies c_j = c \implies c_1, c_2, \ldots c_k$ are the only eigenvalues lose of the operator T so if I somehow can show that that range space of each E_j is exactly null space of (T-c_jI) then I can say that the operator T is diagonalizable.

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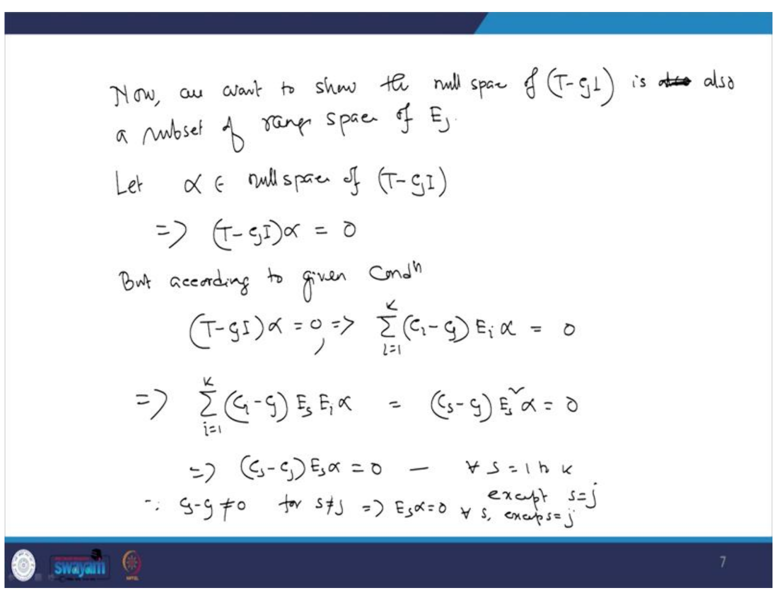

So, now we want to show the null space of (T-c_jI) is also a subset of range space of E_j. So, you want to show this one. So, let $\alpha \in$ null space of (T-c_jI). So $\Longrightarrow$ (T-c_jI) α = 0 but according to given condition (T-c_jI) α = 0, $\Longrightarrow \sum_{i=1}^{k}(c_i - c_j)\, E_i\, \alpha = 0$.

Because, T = $c_1E_1 + c_2E_2 + \ldots + c_kE_k$ and I = $E_1 + E_2 + \ldots + E_k$. So, c_jI = $c_jE_1 + c_jE_2 + \ldots + c_jE_k$. So, when I consider (T-c_jI)$\alpha = \sum_{i=1}^{k}(c_i - c_j)\, E_i\alpha$. So, this implies that if I multiplied E_s from the both the sides I will have $\sum_{i=1}^{k}(c_i - c_j)\, E_sE_i\alpha = (c_s - c_j){E_s}^2\alpha = 0$.

because $E_sE_i = 0$, when i ≠ s So, that this implies that I will have this type of relation $\Longrightarrow (c_s - c_j)E_s\,\alpha = 0$ for all s = 1 to k except s = j, because I am getting when s = j then $(c_j - c_j) = 0$. So, it is not coming to the picture.

Now since $(c_s - c_j) \neq 0$ for s ≠ j, this implies $E_s\alpha = 0$ for all s, except s = j.

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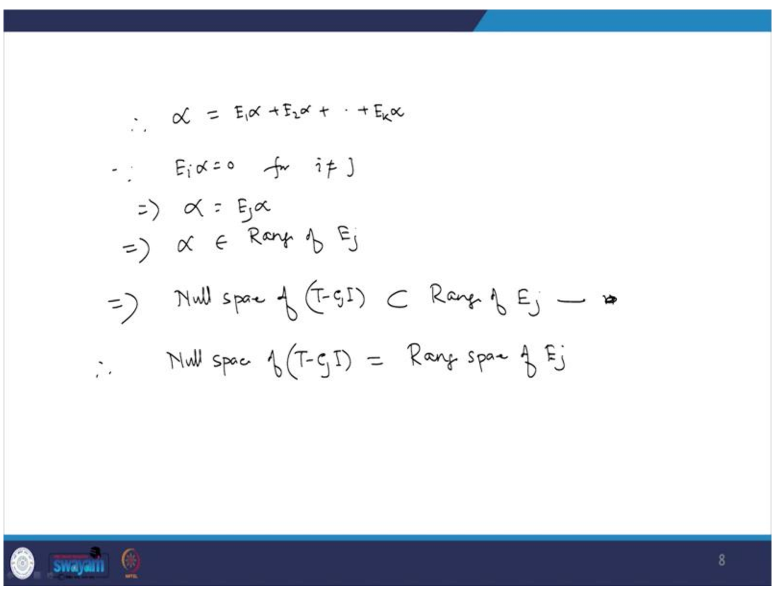

So, we know $\alpha = E_1\alpha + E_2\alpha + \ldots + E_k\alpha$. So, $E_i\alpha = 0$ for i ≠ $j \Longrightarrow \alpha = E_j\alpha \Longrightarrow \alpha \in$ range of E_j. So $\Longrightarrow$ the null space of (T-c_jI) is also a subset of range of E_j. So, already we have seen the range of E_j is a subspace of null space of (T-c_jI) and. Now I am showing that now space of (T-c_jI) = Range space of E_j.

So, from these 2 results I can say that null space of (T-c_jI) is exactly equal to ring space of easier. So, what we have seen it here when we have a linear operator T Define on the vector space V and when there are K distance scalar $c_1, c_2, \dots c_k$ and k distinct and non-zero linear operator $E_1, E_2, \dots E_k$ and the operators $E_1, E_2, \dots E_k$ satisfied the condition that the they operate T = c_1E_1 + c_2E_2 + … + c_kE_k.

And I = $E_1 + E_2 + \dots + E_k$ and the operator E_i having characteristic that $E_iE_j = 0$ for i $\neq j$ then we have seen that each E_i here is basically projection operator and projection of the space V and $c_1, c_2, \dots c_k$ are distinct eigenvalues and they are also only eigenvalues of the operator T and ring space of its linear operator E_i is basically noun space of (T-c_iI).

So, the operator T is then direct sums of subspaces $w_1, w_2, \dots w_k$ and. So, it is a diagonalizable operator. So, T is a diagonal operator on this vector space V. So, this results basically encourage will encourage us to think about the when the operator T is not diagonalizable in that case what will be the structure of the position operator when the space is decomposed by some direct sum of subspaces of corresponding vector space. So, this will certainly give a clue which will discuss in next class please.