

Advanced Linear Algebra
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Lecture: 35
Invariant Direct Sums - 1

So, welcome to lecture series on advanced linear algebra. We have seen in our last lecture if V is having direct sum decomposition by its subspaces W_1, W_2, \dots, W_k then we have seen there exist k linear operators E_1, E_2, \dots, E_k which are basically projection operators such that (i) $E_i^2 = E_i$ (ii) $E_j E_i = 0$ for $i \neq j$ (iii) $I = E_1 + E_2 + \dots + E_k$ (iv) The Range of each E_j is W_j

that is subspace W_i and conversely if we have k linear operators E_1, E_2, \dots, E_k satisfying conditions (i) to (iv), then the vector space V having direct sum decompositions by the subspaces W_1, W_2, \dots, W_k where W_i is the basically range of E_i . Now along this space suppose you want to add the typical characteristics of W_1 and suppose W_1 is invariant under some linear operator T defined on this space V .

I mean if the W_1, W_2, \dots, W_k are invariant subspaces of V under the linear operator T defined on V and if the space has direct sum decompositions then also I will have this scale linear operators E_1, E_2, \dots, E_k satisfying all these four conditions, but apart from this we will also have some extra things.

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Theorem: If $V = W_1 \oplus \dots \oplus W_k$, then \exists k LO E_1, E_2, \dots, E_k s.t.
 (i) each E_i is a projection (i.e. $E_i^2 = E_i$)
 (ii) $E_i E_j = 0$ (iii) $I = E_1 + \dots + E_k$ (iv) Range of E_i is W_i
Conversely If (i)-(iv) holds then $V = W_1 \oplus \dots \oplus W_k$

✓ Let T be a LO on V defined over a field say F .

$V = W_1 \oplus \dots \oplus W_k$
 where each W_i is invariant under T

$\Rightarrow T(W_i) \subseteq W_i$ —

$\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_k$ } $\because V$ is having D.S.D
 $\alpha_i \in W_i$

$T\alpha = T\alpha_1 + T\alpha_2 + \dots + T\alpha_k$
 $= T_1\alpha_1 + T_2\alpha_2 + \dots + T_k\alpha_k$ — T_i is oprt on W_i

Let T be a linear operator on vector space V defined over F and it is given to us say $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ where each W_i is invariant under T that is this implies that, $T(W_i) \subseteq W_i$. So then we have already seen in our last lecture that one can introduce operators like T_1, T_2, \dots, T_k as a restriction of T on W_1, W_2, \dots, W_k and then if I write here, $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_k$.

Since V is having direct sum decompositions, so I can have this type of k -tuple $\alpha_1, \alpha_2, \dots, \alpha_k$ where α will come from W_i and α can be written like this, is coming as a consequence of the criteria I have already stated in above theorem. So implies $I = E_1 + E_2 + \dots + E_k$ so from that also I can have it where $\alpha_j \in W_j$. Then we have, $T(\alpha) = T(\alpha_1) + T(\alpha_2) + \dots + T(\alpha_k) = T_1(\alpha_1) + T_2(\alpha_2) + \dots + T_k(\alpha_k)$, even T is restricted to W_i .

I have given the name like this. See this T_i is operator on the subspace W_i .

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If b_1, b_2, \dots, b_k are ordered basis of W_1, W_2, \dots, W_k
 then $B = \{b_1, b_2, \dots, b_k\}$ is an ordered basis of V
 & $[T]_B = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{bmatrix}$ where A_i is a square matrix
 of $d_i \times d_i$ order
 $\sum_{i=1}^k d_i = \text{Dim of } V$.

Now, question is under what condition each W_i will be T invariant?

Theorem: Let $V = W_1 \oplus \dots \oplus W_k$. Let T be a LO on V .
 Let k LO E_1, E_2, \dots, E_k , which satisfy the cond's
 (i) $E_i^2 = E_i$ (i.e. E_i is a projection)
 (ii) $E_i E_j = 0$ for $i \neq j$ (iii) $I = E_1 + E_2 + \dots + E_k$
 (iv) Range of E_i is W_i
 Then each W_i is invariant under T iff T commutes with

So in this situation, we have seen if B_1, B_2, \dots, B_k are ordered basis of W_1, W_2, \dots, W_k then B

$= \{B_1, B_2, \dots, B_k\}$ is an ordered basis of V and $[T]_B = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_k \end{bmatrix}$ where A_i is a square

matrix of $d_i \times d_i$ which is basically A_i is the matrix representation of T_i on W_i and $\sum_{i=1}^k d_i = \text{Dim of } V$.

So we have seen like this. Now question is under what condition is W_1, W_2, \dots, W_k will be T invariant. Now question is under what condition each W_i will be T invariant? This one. Now to answer this question since most of the places in this course I will consider projection operator, so instead of T_1, T_2, \dots, T_k , I will consider E_1, E_2, \dots, E_k . So to answer this question, I am basically considering this problem.

So, this problem is basically some sort of I am writing theorem. Let $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ and T be a linear operator on V. So let k linear operators E_1, E_2, \dots, E_k which satisfied the condition $\Rightarrow E_i^2 = E_i$ (i.e. each E_i is a projection) (ii) $E_j E_i = 0$ for $i \neq j$ (iii) $I = E_1 + E_2 + \dots + E_k$ (iv) Range of E_j is W_j .

So let $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ and T be a linear operator on V and E_1, E_2, \dots, E_k there are k linear operators which are projection operator satisfying the criteria (i) to (iv). Then each W_i is invariant under T if and only if T commutes with each E_i .

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each E_i
i.e. $TE_i = E_i T$ for $i = 1$ to k

Pf! \Rightarrow It is given $TE_i = E_i T$ for $i = 1$ to k -
claim W_i is invariant under T.
Let $\alpha \in W_i$, $\Rightarrow \alpha = E_i \alpha$
 $\Rightarrow T\alpha = TE_i \alpha = (TE_i)(\alpha) = (E_i T)(\alpha)$ $\because E_i T = TE_i$
 $= E_i(T\alpha) \in W_i$
 $\Rightarrow W_i$ is invariant under T.

(\Leftarrow : only): Given each W_i is invariant under T
claim T commutes with each E_j
 $TE_j = E_j T$ for $j = 1$ to k .

We have $I = E_1 + E_2 + \dots + E_k$
 $\alpha = E_1 \alpha + E_2 \alpha + \dots + E_k \alpha$

i.e. $TE_i = E_i T$ for $i = 1$ to k . So if the operator T commutes with each of the projection operator E_i , then the subspaces W_1, W_2, \dots, W_k will be invariant under T. So let us give a quick proof to this problem. It is given T commutes with each E_i , so $TE_i = E_i T$ for $i = 1$ to k , it is given to us. Claim: W_i is invariant under T. Let $\alpha \in W_i \Rightarrow \alpha = E_i \alpha$ where α is basically range space of E_i .

$\Rightarrow T\alpha = TE_i\alpha = (TE_i)(\alpha) = (E_iT)(\alpha)$ since $TE_i = E_iT$ it is given to us, so $T\alpha = E_i(T(\alpha)) \in W_i$. So this implies W_i is invariant under T . Let us prove the other way that is only if, this first is if, so now I am talking about the only if.

This one basically only if that is given each W_i is invariant under T . Claim that T commutes with each E_j that is $TE_j = E_jT$ for $j = 1$ to k . I have to show that $TE_j = E_jT$. According given statement of the theorem we have $I = E_1 + E_2 + \dots + E_k$. So this implies that $\alpha = E_1\alpha + E_2\alpha + \dots + E_k\alpha$.

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$$\begin{aligned} \Rightarrow T\alpha &= \sum_{i=1}^k TE_i\alpha \\ \text{We have } TE_j\alpha &\in W_j \quad \because E_j\alpha \in W_j \text{ \& } W_j \text{ is invariant under } T \\ \Rightarrow TE_j\alpha &= E_j\beta_j \text{ --- (i) for some } \beta_j \in W_j \\ \text{We have } \alpha &= E_1\alpha + E_2\alpha + \dots + E_k\alpha \\ \Rightarrow T\alpha &= TE_1\alpha + TE_2\alpha + \dots + TE_k\alpha \\ &= \sum_{i=1}^k TE_i\alpha \\ \& \quad E_jT\alpha &= \sum_{i=1}^k E_jTE_i\alpha \\ &= \sum_{i=1}^k E_jE_i\beta_i \quad \text{[from (i)]} \\ \Rightarrow E_jT\alpha &= E_j\beta_j \text{ --- (ii)} \end{aligned}$$

So this implies $T\alpha = \sum_{i=1}^k TE_i\alpha$, We have $TE_j\alpha \in W_j$. Since $E_j\alpha \in W_j$ & W_j is invariant under $T \Rightarrow TE_j\alpha = E_j\beta_j$ for some $\beta_j \in W_j$. We have $\alpha = E_1\alpha + E_2\alpha + \dots + E_k\alpha \Rightarrow T\alpha = TE_1\alpha + TE_2\alpha + \dots + TE_k\alpha = \sum_{i=1}^k TE_i\alpha$.

And when operating E_j from the left side, so I have $E_jT\alpha = \sum_{i=1}^k E_jTE_i\alpha = \sum_{i=1}^k E_jE_i\beta_i \Rightarrow E_jT\alpha = E_j\beta_j$ from using (i).

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$$\begin{aligned} \therefore (i) \& (ii) \Rightarrow \\ E_j T \alpha &= T E_j \alpha \\ \Rightarrow E_j T &= T E_j \end{aligned}$$

$\therefore T$ commutes with each E_j

Consider T as a diagonalizable operator on a f.d.v.s V .
Let c_1, c_2, \dots, c_k be its distinct eigen values &
 W_1, W_2, \dots, W_k be the corresponding eigenspaces.

$$V = W_1 \oplus W_2 \oplus \dots \oplus W_k$$

$\therefore \exists k$ l.o E_1, E_2, \dots, E_k s.t.

$$\begin{aligned} (i) E_i^2 &= E_i & (ii) E_i E_j &= 0, (iii) I = E_1 + E_2 + \dots + E_k \\ (iv) \text{ range of } E_i & \text{ is } W_i. \end{aligned}$$

So (i) & (ii) $\Rightarrow E_j T \alpha = T E_j \alpha \Rightarrow E_j T = T E_j$. So T commutes with each E_j . So we have seen the criteria under what condition each subspace W_j will be invariant under T . Now let us apply this concept to a very specific problem when the operator T is having diagonalization structures. I mean when T is diagonalizable, let me see what this theorem says or is it possible to have a simplified structure about this projection operator E_j or not?

So consider T as a diagonalizable operator on a finite dimensional vector space V . See in these theorems here, I have not assumed that whether the vector space has to be finite dimensional or not. So this result what we have considered is valid for finite or infinite both the cases. Now I am applying this concept when T is a diagonalizable operator defined over a finite dimensional vector space V .

Let c_1, c_2, \dots, c_k be its distinct eigenvalues and W_1, W_2, \dots, W_k be the corresponding eigenspaces. Since T is diagonalizable, so eigenvector will form a basis for this space and so certainly V will have a direct sum decomposition by these eigenspaces. So $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$. So since V is the direct sum decomposition of W_1, W_2, \dots, W_k our previous result says that there will be k projection operators.

So E_1, E_2, \dots, E_k where range of E_i is W_i , so that concept will be there. So $\exists k$ linear operators E_1, E_2, \dots, E_k such that that (i) $E_i^2 = E_i$ (ii) $E_j E_i = 0$ for $i \neq j$ (iii) $I = E_1 + E_2 + \dots + E_k$ (iv) The Range of each E_j is W_j . Now the concept that T is a diagonalizable let us see what way this is helping us.

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We have for any $\alpha \in V$
 $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_k$, where $\alpha_i \in W_i$.

$$T(\alpha) = T(\alpha_1) + T(\alpha_2) + \dots + T(\alpha_k)$$

$$= c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_k \alpha_k$$

$$I = E_1 + E_2 + \dots + E_k$$

$$\alpha = E_1 \alpha + E_2 \alpha + \dots + E_k \alpha$$

$$T\alpha = c_1 E_1 \alpha + c_2 E_2 \alpha + \dots + c_k E_k \alpha$$

$$\Rightarrow T = c_1 E_1 + c_2 E_2 + \dots + c_k E_k$$

$$T^2 = (c_1 E_1 + c_2 E_2 + \dots + c_k E_k)(c_1 E_1 + \dots + c_k E_k)$$

$$= c_1^2 E_1 + c_2^2 E_2 + \dots + c_k^2 E_k$$

$$T^r = c_1^r E_1 + \dots + c_k^r E_k$$

So we have for any $\alpha \in V$, $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_k$ where $\alpha_i \in W_i$. I mean to say $\alpha_1, \alpha_2, \dots, \alpha_k$ are basically eigenvectors associated to eigenvalue c_1, c_2, \dots, c_k . And of course we have, $T(\alpha) = T(\alpha_1) + T(\alpha_2) + \dots + T(\alpha_k) = c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_k \alpha_k$ and of course we have seen $I = E_1 + E_2 + \dots + E_k$.

So then this means that $\alpha = E_1 \alpha + E_2 \alpha + \dots + E_k \alpha$, $T\alpha = c_1 E_1 \alpha + c_2 E_2 \alpha + \dots + c_k E_k \alpha \Rightarrow T = c_1 E_1 + c_2 E_2 + \dots + c_k E_k$. So when T is diagonalizable, then also I am getting there $T\alpha$ where $\alpha = E_1 \alpha + E_2 \alpha + \dots + E_k \alpha$. So $T\alpha = TE_1 \alpha + TE_2 \alpha + \dots + TE_k \alpha$, but $TE_1 \alpha = c_1 E_1 \alpha$ because $E_1 \alpha$ is a eigenvector associated to the eigenvalue c_1 .

So similarly, I will have $T = c_1 E_1 + c_2 E_2 + \dots + c_k E_k$. Previously when T is not diagonalizable, suppose this diagonalizable is not there, so I can only write down the identity operator as a sum of projection operator E_1, E_2, \dots, E_k but when T is diagonalizable, I mean in that case your eigenspaces are also invariant, then I am getting what?

I am getting the operator T as a sum of $c_1 E_1 + c_2 E_2 + \dots + c_k E_k$. Now this information will certainly help to talk about the structure of the projection operator as a function of T because projection operator E_1, E_2, \dots, E_k all belongs to your $L(V, V)$. So certainly E_1, E_2, \dots, E_k one will be able to express as a linear combination of $T^0, T^1, T^2, \dots, T^r$, I mean basis of the $L(V, V)$.

But for this diagonalizable operator case we can show that this projection operator is having typical simple structures, what is that, that I am showing it here. Suppose this T is diagonalizable

then I say that T is of this structure. So this implies T^2 will be equal to how much? It will be $T^2 = (c_1E_1 + c_2E_2 + \dots + c_kE_k)(c_1E_1 + c_2E_2 + \dots + c_kE_k) = (c_1^2E_1 + c_2^2E_2 + \dots + c_k^2E_k)$
 So using the property that $E_iE_j = 0, E_i^2 = E_i$.

So in general, $T^r = (c_1^rE_1 + c_2^rE_2 + \dots + c_k^rE_k)$.

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\Rightarrow For any polynomial $g(x)$ over F

$$g(T) = g(c_1)E_1 + g(c_2)E_2 + \dots + g(c_k)E_k$$

In particular, when $g(x) = p_j(x) = \prod_{i \neq j} \frac{(x - c_i)}{(c_j - c_i)}$

we have $p_j(c_i) = \delta_{ij}$ (Kronecker delta fn)

$$= \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$$

we have

$$p_j(T) = p_j(c_1)E_1 + \dots + p_j(c_{j-1})E_{j-1} + p_j(c_j)E_j + \dots + p_j(c_k)E_k$$

$$\neq p_j(c_j)E_j = E_j$$

$\Rightarrow E_j = p_j(T)$

So this implies that for any polynomial $g(x)$ over the field F on which the vector space is defined, $g(T) = g(c_1)E_1 + g(c_2)E_2 + \dots + g(c_k)E_k$. So the operator $g(T)$ was defined over the vector space V . So in particular, when the polynomial $g(x)$ equal to having Lagrangian interpolation polynomial say, $g(x) = p_j(x) = \prod_{i \neq j} \frac{(x - c_i)}{(c_j - c_i)}$.

I mean $((x - c_1)(x - c_2) \dots (x - c_k)$ excluding the $(x - c_j)$ and the denominator it will be, this is $(c_j - c_1)(c_j - c_2) \dots (c_j - c_k)(c_j - c_k)$ if I consider like this then we have $p_j(c_i) = \delta_{ij} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$ So if I consider this type of polynomial function, then we have $p_j(T) = p_j(c_1)E_1 + p_j(c_2)E_2 + p_j(c_{j-1})E_{j-1} + p_j(c_j)E_j + \dots + p_j(c_k)E_k = p_j(c_j)E_j = E_j \Rightarrow E_j = p_j(T)$

where p_j is a Lagrangian interpolation polynomial function. So this is the beauty of this result. So we see that when T is diagonalizable, the projection operator having structure of the form $p_j(T)$ and that p_j is basically Lagrangian interpolation polynomial. We will continue with the applications of this invariant direct sums.

And more specifically for first I will talk about the diagonal operators because I imagined here that if T is diagonalizable and V is having direct sum decompositions, then the operator T having a typical structure like $c_1E_1 + c_2E_2 + \dots + c_kE_k$. But if suppose the T is operator defined over the vector space V and V is also having direct sum decomposition of suppose W_1, W_2, \dots, W_k where c_1, c_2, \dots, c_k are distinct constants, then whether the operator T will be diagonalizable or not that have to be proved, so this thing I will prove in my next class.