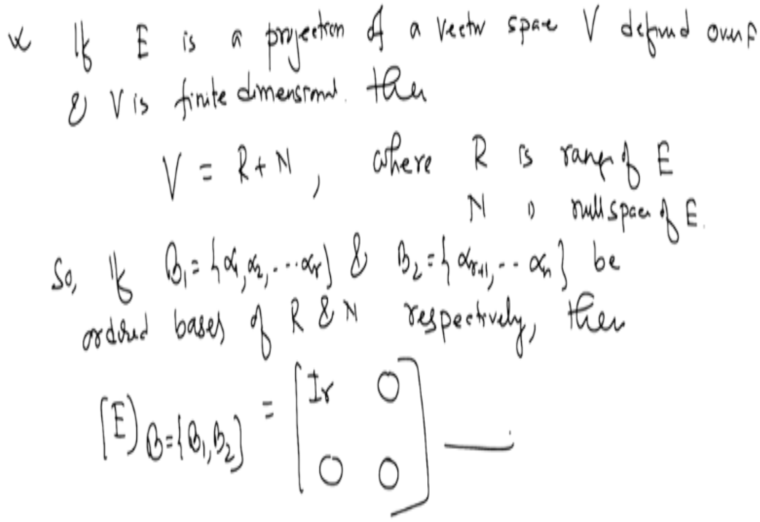
Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture: 34
Direct Sum Decompositions - 2

So welcome to lecture series. In my last lecture, we have introduced the concept of direct sum decomposition of the space, I mean direct sum of subspaces and we have seen if I consider the projection operator on a vector space, then projection operator also split the space into direct sum of two subspaces.

(Refer Slide Time: 00:54)



We have seen if E is a projection operator the projection of a vector space V which defined over F and V is finite dimensional, then V = R +N, where R is range of E and N is null space of the operator E. So if $B_1 = (\alpha_1, \alpha_2, \dots \alpha_r)$ and $B_2 = (\alpha_{r+1}, \alpha_{r+2}, \dots \alpha_n)$ be ordered bases of R and N respectively, then $[E]_{B=\{B_1,B_2\}} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Anyhow we shall utilize this simplified structure to simplify our original operator. So to move in that directions, I will see that how this projection operator helps to decompose the spaces.

(Refer Slide Time: 03:51)

W Consider V be a f.d.v.s. over F. Let

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k \quad \text{—— } (*)$$

Defn a fn $E_j : V \to V$ —

as bellow:

From (*) we have for any $\alpha \in V$

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k \text{ —, where } \alpha_i \in W_i$$

This expression is unique ??

If not, let

$$\alpha = \beta_1 + \beta_2 + \cdots + \beta_k, \quad \beta_i \in W_i$$

then $(\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) + \cdots + (\alpha_k - \beta_k) = 0$

$\Rightarrow \alpha_1 - \beta_1 = 0$ for $i = 1$ to k $\because$ V is direct sum of subspace $W_1 \cdots W_k$

So to talk in the directions let me first consider V be a finite dimensional vector space over the field F. Let V $= W_1 \oplus W_2 \oplus \ldots \oplus W_k$. Suppose this information is given to us. Is it possible to introduce projection operator of the vector space V where the range space of the different projection operator will be $W_1, W_2, \ldots W_k$ etc. So let us see whether we can within this direction decomposition of the space V can I obtain a linear operator? So let me define a function E_j: V $\to$ V.

We know from star(*), we have for any $\alpha \in$ V, I will have $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k$ where $\alpha_i \in W_i$ and this expression is unique. Why? Because if not, let $\alpha = \beta_1 + \beta_2 + \ldots + \beta_k$, then we have, $(\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) + \ldots + (\alpha_k - \beta_k) = 0 \Rightarrow \alpha_i - \beta_i = 0$ for i = 1 to k, since V is direct sum of subspaces $W_1, W_2, \ldots W_k$. So, this implies that for any $\alpha \in$ V, I will have basically a unique representation like this thing.

(Refer Slide Time: 07:17)

Consider $E_j : V \to V$

$E_j(\alpha) = \alpha_j$

$\Rightarrow$ Range of E_j is W_j

& Nullspace of E_j is $(W_1 + W_2 + \cdots + W_{j-1} + W_{j+1} \cdots + W_k)$

$\therefore$ for $i \neq j$,

$E_i(E_j(\alpha)) = E_i(\alpha_j) = 0 \quad \because \alpha_j \in W_j$

& Null space of E_i is $(W_1 + \cdots + W_j + \cdots + W_{i-1} + W_{i+1} + \cdots + W_k)$

$\Rightarrow E_iE_j = 0$ for $i \neq j$

& $E_j(E_j(\alpha)) = E_j(\alpha_j) = E_j(\alpha)$

$\Rightarrow E_j^2 = E_j \quad \therefore$ E_j is a projection operator of V

Consider, E_j: V $\to$ V, $E_j(\alpha) = \alpha_j$ $\Rightarrow$ Range of E_j is W_j and Null space of E_j is sum of ($W_1 + W_2 + \ldots + W_{j-1} + W_j + W_{j+1} \ldots + W_k$). So this is the Null space of E_j. So for i $\neq j$, $E_i(E_j(\alpha))$ = $E_i(\alpha_j) = 0$, since $\alpha_j \in W_j$ and W_j is basically the part of the Null space of E_i and Null space of E_i is $(W_1 + W_2 + \ldots + W_{j-1} + W_j + W_{j+1} \ldots + W_k)$.

So this one, this implies $E_i\, E_j = 0$ for i $\neq j$ & $E_j\left(E_j(\alpha)\right) = E_j(\alpha_j)$ = $E_j(\alpha)$. So this implies that $E_j{}^2 = E_j$, since E_j the projection operator of the vector space V.

(Refer Slide Time: 10:23)

& range of E_j is W_j

$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$

$= E_1\alpha + E_2\alpha + \cdots + E_k\alpha$

$\Rightarrow I = E_1 + E_2 + \cdots + E_k$ —

Theorem: Let V be a f.d.v.s. over the field F. Let V is direct sum of subspaces $W_1, W_2, \cdots W_k$ i.e.

$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ —

Then, there exist k-operators $E_1, E_2, \cdots, E_k$ such that

(i) $E_j^2 = E_j$ (ii) $E_iE_j = 0$ for $i \neq j$

(iii) $I = E_1 + E_2 + \cdots + E_k$, (iv) Range of E_j is W_j

Conversely, if there are k-operators $E_1, E_2, \cdots E_k$ on V, where V is a f.d. v.s over F. The operators

Range of E_j is W_j and we see that $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k = E_1\alpha + E_2\alpha + \ldots + E_k\alpha \implies I = E_1 + E_2 + \ldots + E_k$. Now we see that if V be a finite dimensional vector space and if V = $W_1 \oplus W_2 \oplus \ldots \oplus W_k$.

Then one can have projection operator $E_1, E_2, \ldots E_k$ where E_i satisfy the conditions like $E_iE_j = 0$ for i $\neq j$ and I = $E_1 + E_2 + \ldots + E_k$ and the range of $E_j = W_j$. So now let me see if I suppose a projection operator $E_1, E_2, \ldots E_k$ are given and this operators $E_1, E_2, \ldots E_k$ satisfied the conditions like I = $E_1 + E_2 + \ldots + E_k$, second $E_iE_j = 0$ for i $\neq j$ and the range of $E_j = W_j$.

Then is it possible to say that V = $W_1 \oplus W_2 \oplus \ldots \oplus W_k$, I mean V will have direct sum decomposition of $W_1, W_2, \ldots W_k$ is the question. So let me write down these questions in a proper form. This is some sort of theorem. Let V be a finite dimensional vector space over F. Let V = $W_1 \oplus W_2 \oplus \ldots \oplus W_k$.

Then there exists k operators $E_1, E_2, \ldots E_k$ such that (i) ${E_j}^2 = E_j$ (ii) $E_iE_j = 0$ for i $\neq j$ (iii) I = $E_1 + E_2 + \ldots + E_k$ (iv) Range of E_j is W_j, I mean to say W_j is the range of E_j. Now conversely if there are k operators, $E_1, E_2, \ldots E_k$ on V where V is the finite dimensional vector space over F.

(Refer Slide Time: 15:59)

$E_1, E_2, \ldots E_k$ satisfy the cond^n (i) to (iv)
then V is direct sum of $W_1, W_2, \ldots W_k$.

Pf:- Converse part: Given there are k LO $E_1, E_2, \ldots E_k$ satisfying (i) to (iv) claim V is direct sum of subspaces $W_1, W_2, \ldots W_k$.

We have for any $\alpha \in V$

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k, \quad \alpha_i \in W_i$$
$$= E_1\alpha + E_2\alpha + \cdots + E_k\alpha$$
$$\Rightarrow E_j(\alpha) = E_j(E_1\alpha) + E_j(E_2\alpha) + \cdots + E_j(E_k\alpha)$$
$$= E_j^2\alpha = E_j(\alpha) = \alpha_j \quad —$$

$\therefore$ for any expression of the form

The operators $E_1, E_2, \ldots E_k$ satisfy the condition (i) to (iv), then V is direct sum of $W_1, W_2, \ldots W_k$. So the problem is like this if V is a finite dimensional vector space over F, if

V is a direct sum of subspaces $W_1, W_2, \ldots W_k$, then there exists k linear operator $E_1, E_2, \ldots E_k$ which are basically projection operator on V satisfying four conditions like what we have given here.

Now conversely if there are k linear operators $E_1, E_2, \ldots E_k$ on the vector space V satisfying condition (i) to (iv) then V will also direct sum of subspaces $W_1, W_2, \ldots W_k$. When V is direct sum of $W_1, W_2, \ldots W_k$ already we have seen there are k linear operators $E_1, E_2, \ldots E_k$ which are basically projection operators of V and satisfying condition (i) to (iv). So the one part is done, now we are going to prove the second part, the converse part.

So I am going to prove the converse part. So given there are k linear operators $E_1, E_2, \ldots E_k$ satisfying (i) to (iv). Claim: V is direct sum of subspaces $W_1, W_2, \ldots W_k$. We have for any $\alpha \in$ V, $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k$ where $\alpha_i \in W_i$, this is because it is given to us I = $E_1 + E_2 + \ldots + E_k$, so E_i is projection operator, α operating on E_i it will give me $\alpha_i \in W_i$.

So that is why any element $\alpha \in$ V, $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k$. So this implies if I apply $E_j(\alpha) = = E_1\alpha + E_2\alpha + \ldots + E_k\alpha \Rightarrow E_j(\alpha) = E_jE_1\alpha + E_jE_2\alpha + \ldots + E_jE_k\alpha \Rightarrow E_j^2(\alpha) = E_j(\alpha) = \alpha_j$, since $E_jE_1 = 0$ according to given condition (ii) $E_iE_j = 0$ for i $\neq j$.

(Refer Slide Time: 21:31)

$0 = \alpha_1 + \alpha_2 + \cdots + \alpha_k$

$\Rightarrow E_j(0) = 0 = E_j(\alpha_1) + E_j(\alpha_2) + \cdots + E_j(\alpha_k)$

$= E_jE_1(\alpha_1) + E_jE_2(\alpha_2) + \cdots + E_jE_k(\alpha_k)$

$= E_j^2\alpha_j = E_j\alpha_j = \alpha_j$

$\Rightarrow 0 = \alpha_j$

Similarly, one can show that

$\alpha_i = 0$ for i = 1 to k

$\Rightarrow$ V is direct sum of subspaces $W_1, W_2, \ldots W_k$.

So for any expressions of the form, $0 = \alpha_1 + \alpha_2 + \ldots + \alpha_k \Rightarrow E_j(0) = 0 = E_j(\alpha_1) + E_j(\alpha_2) + \ldots + E_j(\alpha_k) = E_jE_1(\alpha_1) + E_jE_2(\alpha_2) + \ldots + E_jE_k(\alpha_k) = E_j^2(\alpha_j) = E_j(\alpha_j) = \alpha_j \Rightarrow 0 = \alpha_j$

So similarly one can show that α_i = 0 for i = 1 to k $\Longrightarrow$ V is direct sum of subspaces $W_1, W_2, \ldots\ W_k$. In fact, we are interested about the decomposition of this space as a direct sum of subspaces which are invariant under a linear operator defined on the space. So far, I have not used the direct sum decomposition of the space or direct sum of subspaces, whether subspaces are invariant under some operator or not that we have not utilized or discussed.

But now I want to use this concept that the subspaces will be invariant under a linear operator defined over the space. Using this information, you will see that it gives more simpler structures.

(Refer Slide Time: 24:41)

Let V be a vector space of finite dimension over F.
Let T be a LO on V. Let $V = W_1 \oplus W_2 \oplus \cdot \oplus W_k$
where W_i is invariant under T, for i = 1 to k.

Consider the restriction of T on W_i & let it is denoted by T_i $\therefore$ T_i is restriction of T on W_i

Given for any $\alpha \in V$

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k \text{ —}$$

$$T(\alpha) = T(\alpha_1) + T(\alpha_2) + \cdots + T(\alpha_k)$$
$$= T_1(\alpha_1) + T_2(\alpha_2) + \cdots + T_k(\alpha_k) \text{ —}$$

Let V be a vector space over F. Let T be a linear operator on V. Let V $= W_1 \oplus W_2 \oplus \ \ldots \oplus W_k$ where W_i is invariant under T, so it is a invariant subspace of V for i = to k. Now, let us see this information how it is helping to split the operator T. Now consider the restriction operator, restriction of T on W_i i and let it is denoted by say T_i. So this implies that T_i is the basically restriction, so T_i is restriction of T on W_i.

So, T_i is a linear operator whose domain is W_i. It is given for any $\alpha \in$ V, $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k$. So, T($\alpha$) $= T(\alpha_1) + T(\alpha_2) + \ldots T(\alpha_k) = T_1(\alpha_1) + T_2(\alpha_2) + \ldots T_k(\alpha_k)$. So, I can write down the image of any element α under T as sum of image of different component $\alpha_1, \alpha_2, \ldots\ \alpha_k$ under restriction of T into $W_1, W_2, \ldots\ W_k$. So let us see what way this will be useful.

(Refer Slide Time: 28:24)

Ex Let T be a LO on a f.d.v.s V &

$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ — , V is defined over F.

Let $B_1, B_2, \ldots B_k$ be the basis of $W_1, W_2, \ldots W_k$ respectively.

Given that W_i is invariant under T

$\Rightarrow$ $T_i = T_{|W_i}$ & let dim of W_i be d_i

$[T_i]_{B_i} = A_i$ $d_i \times d_i$ matrix over F.

$\Rightarrow$ If $B = \{B_1, B_2, \ldots B_k\}$

$$[T]_B = \begin{bmatrix} [A_1]_{d_1 \times d_1} & & & \\ & [A_2]_{d_2 \times d_2} & & \\ & & \ddots & \\ & & & [A_k]_{d_k \times d_k} \end{bmatrix}$$

So for this let me consider example. Let T be a linear operator on a finite dimensional vector space V and V = $W_1 \oplus W_2 \oplus \ldots \oplus W_k$. Let $B_1, B_2, \ldots B_k$ be the basis of $W_1, W_2, \ldots W_k$ respectively. Given that W_i is invariant under T. So this implies if I consider $T_i = T_{|_{W_i}}$ & let dimension of W_i be d_i. $[T_i]_{B_i} = A_i$ where $d_i \mathrm{x} d_i$ matrix over F $\Longrightarrow$ if B = { $B_1, B_2, \ldots B_k$ }, then

$$[T]_B = \begin{bmatrix} [A_1]_{d_2 \mathrm{x} d_2} & 0 & 0 & 0 \\ 0 & [A_2]_{d_2 \mathrm{x} d_2} & 0 & 0 \\ 0 & 0 & [A_3]_{d_3 \mathrm{x} d_3} & 0 \\ 0 & 0 & 0 & [A_k]_{d_k \mathrm{x} d_k} \end{bmatrix}.$$

So we see that through this example when V = $W_1 \oplus W_2 \oplus \ldots \oplus W_k$ and each W_i is invariant, then the matrix representative T is a diagonal matrix, diagonal entry is basically again each of them are square matrix of the size $d_i \mathrm{x} d_i$, so it is a very simplified structure. So, definitely this structure is much simpler than if I consider full matrix. So, we will see that how this invariant concept helps to simplify the operator T. This we will discuss in the next class.