Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture: 33
Direct Sum Decompositions - 1

So welcome to lecture series on advanced linear algebra. Today, I will discuss about the decomposition of space. Since our objective is to find the simplified form of a linear operator defined over a finite dimensional vector space to achieve our goal, decomposition of the space is useful, so let me introduce the concept of decomposition of the space.

(Refer Slide Time: 01:11)

Consider V be a f.d v.s over the field F.
Let $W_1, W_2, \dots W_k$ be the subspaces of V.
$W_1, W_2, \dots W_k$ are said to be independent subspaces of V provided for
$0 = \alpha_1 + \alpha_2 + \dots + \alpha_k$, where $\alpha_i \in W_i$, $i = 1$ to k
implies $\alpha_i = 0$ for $i = 1$ to k.

This implies for $k = 2$, W_1 & W_2 are independent subspaces of V provided $W_1 \cap W_2 = \{0\}$
But for $k > 2$,
we have $W_j \cap W_1 + W_2 + \dots + W_{j-1} = \{0\}$ - for $j \geq 2$

Consider V be a finite dimensional vector space over the field F. Let $W_1, W_2, \dots W_k$ be the subspaces of V. $W_1, W_2, \dots W_k$ are said to be independent subspaces of V, provided for $0 = \alpha_1 + \alpha_2 + \dots + \alpha_n$, where $\alpha_i \in W_i$, i = 1 to k, $\Rightarrow$ $\alpha_i = 0$ for i = 1 to k. So subspaces $W_1, W_2, \dots W_k$ of the vector space V are said to be independent subspaces of V provided if I have an expression $\alpha_1 + \alpha_2 + \dots + \alpha_n = 0$ where $\alpha_i \in W_i$, i = 1 to k, then certainly $\alpha_i = 0$.

So this implies for k = 2, W_1 & W_2 are independent subspaces of V provided $W_1 \cap W_2 = \{0\}$, but for k ≥ 2 we have, $W_j \cap W_1 + W_2 + \dots + W_{j-1} = \{0\}$. So it is simply not that $W_1 \cap W_1 + W_2 + \dots + W_k = \{0\}$, it is defining this independence subspaces of a vector space, is basically saying that for any W_j, this is basically for j ≥ 2 So this is like this $W_j \cap W_1 + W_2 + \dots + W_{j-1} = \{0\}$.

(Refer Slide Time: 05:16)

<u>Lemma</u>: Let V be a f.d. v.s. over the field F. Let $W_1, W_2, \cdots W_K$ be Subspaces of V & let $W = W_1 + W_2 + \cdots + W_K$. The following are equivalent.

(a) $W_1, W_2, \cdots W_K$ are independent

(b) For each j $2 \leq j \leq k$ we have
$$W_j \cap (W_1 + W_2 + \cdots + W_{j-1}) = \{0\}$$

(c) If B_i is an ordered basis of W_i, for $i = 1$ to K, then $B = \{B_1, B_2, \cdots B_K\}$ is an ordered basis of W.

So this implies I can define small result like a lemma. Let V be a finite dimensional vector space over F. So from the definition, independent subspaces of the vector space V, I can write down this type of generalized results like this. If V be a finite dimensional vector space over the field F. Let $W_1, W_2, \dots W_k$ be subspaces of V & let W = $W_1 + W_2 + \dots + W_k$ then the following are equivalent. There are three statements which are equivalent type.

(a) $W_1, W_2, \dots W_k$ are independent (b) for each j, $2 \leq j \leq k$ we have, $W_j \cap (W_1 + W_2 + \dots + W_{j-1}) = \{0\}$ (c) $B_1, B_2, \dots B_k$ are the ordered basis for the $W_1, W_2, \dots W_k$ then B = $\{ B_1, B_2, \dots B_k \}$ is an ordered basis of W. So let us prove this result. This is a very simple result which is coming as a consequence of the definition of the independent subspaces of the vector space.

(Refer Slide Time: 09:20)

Pf:- (a) $\Rightarrow$ (b)

Given $W_1, W_2, \dots W_k$ are independent subspaces of V

Let $\alpha_j \in W_j$ & $\alpha_j \in (W_1 + W_2 + \dots + W_{j-1})$

$\Rightarrow \alpha_j \in W_j \cap (W_1 + W_2 + \dots + W_{j-1})$

$\therefore \alpha_j = \alpha_1 + \alpha_2 + \dots + \alpha_{j-1}$,

$\Rightarrow \alpha_1 + \alpha_2 + \dots + \alpha_{j-1} + (-1)\alpha_j = 0$

or $\alpha_1 + \alpha_2 + \dots + \alpha_{j-1} + (-1)\alpha_j + 0 + \dots + 0 = 0$

But $W_1, W_2, \dots W_k$ are independent subspaces of V

$\Rightarrow \alpha_1 = \alpha_2 = \dots \alpha_j = 0$

$\Rightarrow W_j \cap (W_1 + W_2 + \dots + W_{j-1}) = \{0\}$

First, we will prove that (a) $\Longrightarrow$ (b) that is given $W_1, W_2, \dots W_k$ are independent subspaces of vector space V, it is given to us. We have to show that this statement is equivalent to the statement the second one (b) $W_j \cap (W_1 + W_2 + \dots + W_{j-1}) = \{0\}$. Let $\alpha_j \in W_j$ & $\alpha_j \in ($ $W_1 + W_2 + \dots + W_{j-1})$, I have to prove that $\alpha_j = 0 \Longrightarrow \alpha_j \in W_j \cap W_1 + W_2 + \dots + W_{j-1}$.

So $\alpha_j = \alpha_1 + \alpha_2 + \dots + \alpha_{j-1}$, this is the definition of the sum of subspaces $W_1 + W_2 + \dots + W_{j-1}$ implies that an element of $W_1 + W_2 + \dots + W_{j-1}$ will be of the form of $\alpha_1 + \alpha_2 + \dots + \alpha_{j-1}$. So $\alpha_j = \alpha_1 + \alpha_2 + \dots + \alpha_{j-1}$, $\Longrightarrow \alpha_1 + \alpha_2 + \dots + \alpha_{j-1} + (-1)\alpha_j = 0$ or $\alpha_1 + \alpha_2 + \dots + \alpha_{j-1} + (-1)\alpha_j + 0 + \cdots + 0 = 0$.

See up to α_j is there, $\alpha_j \in W_j$, so there will be $W_{j+1} + W_{j+2} + \dots + W_k$, I have taken zero element only so that I can write down the same expression like this. But $W_1, W_2, \dots W_k$ are independent subspaces of vector space V $\Longrightarrow \alpha_1 = \alpha_2 = \dots + \alpha_j = 0 \Longrightarrow W_j \cap (W_1 + W_2 + \dots + W_{j-1}) = \{0\}$. So we see that (a) $\Longrightarrow$ (b).

(Refer Slide Time: 13:12)

(b) $\Rightarrow$ (a)

Given $W_j \cap (W_1+W_2+\cdots+W_{j-1}) = \{0\}$

We have to show if

$\alpha_1+\alpha_2+\cdots+\alpha_k = 0$, (*) then $\alpha_i = 0$ for i = 1 to k

Let j be the maximum value of i for which $\alpha_i \neq 0$

$\therefore$ (*) can be rewritten as

$$\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_j = 0$$

$$\Rightarrow \alpha_j = (-1)\alpha_1 + \underbrace{(-1)\alpha_2}_{W_2} + \cdots + (-1)\alpha_{j-1}$$

$$\Rightarrow \alpha_j \in W_j \;\&\; \alpha_j \in (W_1+W_2+\cdots+W_{j-1})$$

$$\therefore (b) \Rightarrow \alpha_j = 0$$

Now let us see (b) $\Longrightarrow (a)$ given $W_j \cap W_1 + W_2 + \ldots + W_{j-1} = \{0\}$, it is given to us. I have to prove that to show if $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 0$ then $\alpha_i = 0$ for i = 1 to k that is what I have to prove. Let j be the maximum value of i for which $\alpha_i \neq 0$. So, the expression star(*) can be written as $\alpha_1 + \alpha_2 + \ldots + \alpha_j = 0 \Longrightarrow \alpha_j = (-1)\alpha_1 + (-1)\alpha_2 + \ldots + (-1)\alpha_j \Longrightarrow \alpha_j \in W_j$ & $\alpha_j \in (W_1 + W_2 + \ldots + W_{j-1})$

So (b) $\Longrightarrow \alpha_j = 0$ Since $\alpha_j = 0$ then again from this expression I will have that $\alpha_1 + \alpha_2 + \ldots + \alpha_{j-1} = 0$ and use the same principles, repeat it, and show that each $\alpha_j = 0$, I mean $\alpha_i = 0$ for i = 1 to k.

(Refer Slide Time: 15:57)

$\therefore$ we will have

$$\alpha_1+\alpha_2+\cdots+\alpha_{j-1} = 0$$

Using same logic, we can show, $\alpha_{j-1} = \alpha_{j-2} = \cdots = \alpha_1 = 0$

$\therefore$ (b) $\Rightarrow$ (a)

Now we want to show (a) $\Rightarrow$ (c)

Given $B_1, B_2, \ldots B_k$ are the ordered bases of $W_1, W_2, \ldots W_k$, respectively.

Claim $B = \{B_1, B_2, \ldots B_k\}$ is an ordered basis for W.

Certainly B spans $W = W_1+W_2+\cdots+W_k$

B is also a L.I. set of V

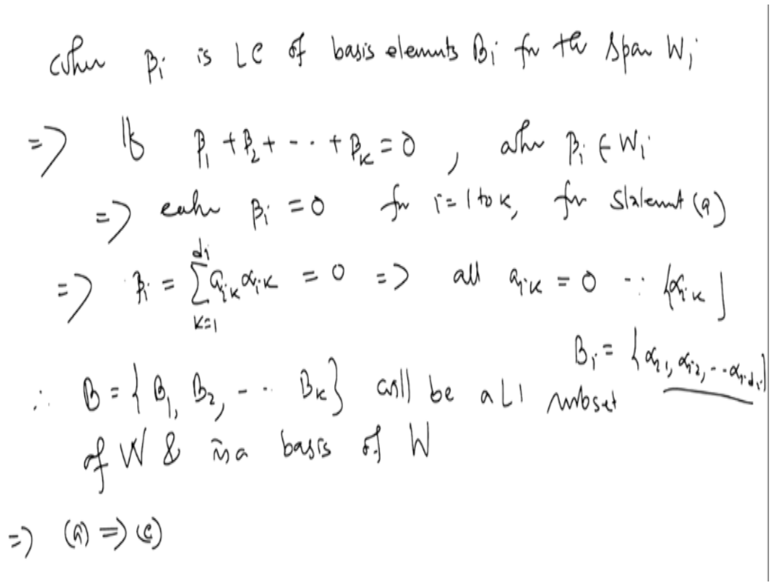
Any linear combination of elements of $B_1, B_2 \ldots B_k$ will of the form of

$$\beta_1+\beta_2+\cdots+\beta_k$$

So we will have $\alpha_1 + \alpha_2 + \ldots + \alpha_{j-1} = 0$. Now using same logic we can show $\alpha_{j-1} = \alpha_{j-2} = \cdots = \alpha_1 = 0$. So (b) $\Longrightarrow (a)$. Now these two statements (a) and (b) are equivalent. Now we want to show (a) $\Rightarrow (c)$, Given $B_1, B_2, \ldots B_k$ are the ordered basis of $W_1, W_2, \ldots W_k$ respectively. Claim B = { $B_1, B_2, \ldots B_k$} is an ordered basis for W. I mean to say they span the space W and they are also linearly independent.

Certainly, B spans W = $W_1 + W_2 + \ldots + W_k$. So now we want to show B is also a linearly independent set of V. Now if I consider any linear combination of the elements of $B_1, B_2, \ldots B_k$ will be of the form of $\beta_1 + \beta_2 + \ldots + \beta_k$.

(Refer Slide Time: 19:09)



Where β_i is linear combination of basis elements B_i for the space W_i $\Rightarrow$ if $\beta_1 + \beta_2 + \ldots + \beta_k = 0$, where $\beta_i \in W_i$ $\Rightarrow$ each $\beta_i = 0$ for i = 1 to k, from statement (a) and β is basically linear combination of the basis element of B_i $\Rightarrow$ $\beta_i = \sum_{k=1}^{d_i} a_{ik}\,\alpha_{ik} = 0 \Rightarrow$ all $a_{ik} = 0 \ldots \{\alpha_{ik}\}$

Since α_{ik} that is basically I can say the set that is $\alpha_{i1}, \alpha_{i2}, \ldots \alpha_{id_i}$ this is a basis element for the W_i and if I consider $B_i = \{\alpha_{i1}, \alpha_{i2}, \ldots \alpha_{id_i}\}$, then I am saying that since β_i equal linear combination of these set of elements so definitely the coefficient has to be 0, all $a_{ik} = 0$. So this implies B = { $B_1, B_2, \ldots B_k$ } will be a linearly independent subset of V and is a basis of W. Of course, this is a linearly independence subset of W also, so this one. So $\Rightarrow$ (a) $\Rightarrow (c)$ also.

(Refer Slide Time: 22:21)

Direct sum of Subspaces: Let V be a f.d.v.s. over the field say F. Let $W_1, W_2, \dots W_k$ be the subspaces of V, which are also independent. Then the sum of subspaces $W_1 \dots W_k$

$$W = W_1 + W_2 + \dots + W_k$$

is said to be direct sum of subspaces $W_1, W_2, \dots W_k$

$$W = W_1 \oplus W_2 \oplus \dots \oplus W_k$$

Ex Let V be a f.d. v.s. over F. Let dim of V is n.
Let W_i denote one dim. subspaces of V - i.e. if
$B = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ be is an ordered basis of V
$W_i = LS\{\alpha_i\}$ then
$V = W_1 \oplus W_2 \oplus \dots \oplus W_n$

So based on the definition of the independent subspaces, let me introduce one terminology this is called direct sum of subspaces. Let V be a finite dimensional vector space over the field say F. Let $W_1, W_2, \dots W_k$ be the subspaces of V which are also independent. Then the sum of subspaces k given by W = $W_1 + W_2 + \dots + W_k$ is said to be direct sum of subspaces $W_1, W_2, \dots W_k$. And mathematically I mean we can write to give a special identity.

So I can write down W = $W_1 \oplus W_2 \oplus \dots \oplus W_k$. So $W_1, W_2, \dots W_k$ which are the subspaces of the vector space V which satisfied one of the conditions, I mean three equivalent statements what we have done just now that (a), (b), (c) so that $W_1, W_2, \dots W_k$ will be independent subspaces of V, then the sum of subspaces of $W_1, W_2, \dots W_k$ if I write as W, then W is said to be direct sum of subspaces $W_1, W_2, \dots W_k$.

For example, let V be a finite dimensional vector space over field F and let dimension of V is say n. Let W_i denote one dimensional subspaces of V that is if B = $\alpha_1, \alpha_2, \dots \alpha_n$ is an ordered basis of V and $W_1 =$ LS$\{\alpha_i\}$, then V = $W_1 \oplus W_2 \oplus \dots \oplus W_k$. This is a very trivial example I have taken, I mean if V has dimension n, if I consider an ordered basis $\{\alpha_1, \alpha_2, \dots \alpha_n\}$.

Then subspace span of α_1 if i denote W_1, for subspace span of α_2 if i denote $W_2$and in general subspace span by α_i if i denote W_i then certainly V = $W_1 \oplus W_2 \oplus \dots \oplus W_n$ because this $\alpha_1, \alpha_2, \dots \alpha_n$ if it collect altogether this will be basis for here subspace V and W = V. In our last example I took subspace W that is $W_1 + W_2 + \dots + W_k$, here W = V.

(Refer Slide Time: 27:41)

Ex-2 Let V be the set of all nxn matrix over $\mathbb{R}$

Then, if we consider any matrix $A \in V$

$$A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right) = A_1 + A_2$$

then A_1 is a symmetric matrix (H.W)

whereas A_2 is a skew-symmetric matrix

$\Rightarrow$ If W_1 denotes the set of all symmetric matrices over $\mathbb{R}$ & W_2 " " " " skew-symmetric " " $\mathbb{R}$

then $V = W_1 \oplus W_2$

Let me take another example. Let V be the set of all nxn matrix over the real number R. Then, if we consider any matrix A ∈ V, I will be able to write, A = $\left(\frac{A + A^T}{2}\right) + \left(\frac{A - A^T}{2}\right) = A_1 + A_2$ then A_1 is a symmetric matrix, where as A_2 is a skew-symmetry matrix. See this one as a homework that show that any matrix A can be written as sum of symmetry and skew-symmetry matrix.

So this implies that if W_1 denote the set of all symmetry matrix over R and W_2 denote a set of all skew-symmetric matrix over R, then this V = $W_1 \oplus W_2$. I mean any element of V can be written as an element from W_1 + element of W_2 that already we have seen that any matrix can be written as sum of symmetry matrix and skew-symmetry matrix.

(Refer Slide Time: 30:26)

Ex-3 Let T be a L.O. defined on a f.d.v.s. V over F.

Let $c_1, c_2, \ldots c_k$ denote the eigenvalues of T &

$W_1, W_2, \ldots W_k$, " the corresponding eigenspaces.

Then $W = W_1 \oplus W_2 \oplus \cdots \oplus W_k$

$\because W_j \cap (W_1 + W_2 + \cdots + W_{j-1}) = \{0\}$

* Projection: Let V be a vector space over the field F.

A projection E of V is a linear operator on V, such that $E^2 = E$ —

Let R & N denote the range space & null space of the the projection operator E

$E: V \to V$

Let me consider another nice example. Let T be a linear operator defined on a finite dimensional vector space V over F. Let $c_1, c_2, \ldots c_k$ denote the eigenvalues of T and $W_1, W_2, \ldots W_k$ denote the corresponding eigenspaces. Then W = $W_1 \oplus W_2 \oplus \ldots \oplus W_k$ because we know the eigenspaces W_1 and W_2 if you intake the intersection that is zero subspace.

In fact, here also $W_j \cap W_1 + W_2 + \ldots + W_{j-1} = \{0\}$already we have seen this one. So based on that experience I can say the sum of subspaces $W_1, W_2, \ldots W_k$ will be a direct sum of subspaces $W_1, W_2, \ldots W_k$. Let me define a terminology called projection. Let V be a vector space over the field say F. A projection say E of V is a linear operator V such that $E^2 = E$.

What is the meaning of this? So let R and N denote the range space and null space of the projection operator E. We have E as basically operator from V to V.

(Refer Slide Time: 34:08)

If $\alpha \in R$, then $\alpha = E\beta$ for some $\beta \in V$

$\Rightarrow E\alpha = EE\beta = E^2\beta = E\beta = \alpha$

$\therefore$ If $\alpha \in R \Rightarrow E\alpha = \alpha$ —

Also, $\alpha = E\alpha \Rightarrow \alpha \in R$, i.e. range of E.

We have for any $\alpha \in V$

$$\alpha = \underbrace{E\alpha}_{\in R} + \underbrace{(\alpha - E\alpha)}_{N}$$

$\because E(\alpha - E\alpha) = E\alpha - E\alpha = 0$

$\Rightarrow V = R \oplus N$ —

If $\alpha \in$ R, then $\alpha = E\beta$ for some $\beta \in V \implies E\alpha = EE\beta = E^2\beta = E\beta = \alpha$. So if $\alpha \in R \Longrightarrow E\alpha = \alpha$. Also, $\alpha = E\alpha \implies \alpha \in$ R, i.e. range of E. We have for any $\alpha \in$ V, $\alpha = E\alpha + (\alpha - E\alpha)$.

So $E\alpha \in$ R and $(\alpha - E\alpha) \in$ N, because $E(\alpha - E\alpha) = E\alpha - E\alpha = 0$. So we see that $\Longrightarrow$ V = R $\oplus N$ this implies any element of V can be written as a sum of element of the range space of E and element from the null space of E. So this means that V = R $\oplus N$ where R is the range of the projection operator E and N is the null space of the projection operator E.

(Refer Slide Time: 36:43)

Ex: Find a projection operator E which projects $\mathbb{R}^2$ onto the subspace spanned by $(1,-1)$ along the subspace spanned by $(1,2)$

Sol^n We have to define the projection operator E on the subspace spanned by $(1,-1)$ $\Rightarrow R = L.S.\{(1,-1)\}$

Along the subspace spanned by $(1,2)$ $\Rightarrow N = L.S.\{(1,2)\}$

$\therefore \; E(1,2) = (0,0)$ & $E(1,-1) = (1,-1)$ —

consider standard ordered basis $B = \{e_1 = (1,0), e_2 = (0,1)\}$

$e_1 = (1,0) = c_1(1,-1) + c_2(1,2)$ $\qquad e_2 = (0,1) = -\frac{1}{3}(1,-1) + \frac{1}{3}(1,2)$

$= \frac{2}{3}(1,-1) + \frac{1}{3}(1,2)$

$\Rightarrow E(e_1) = \frac{2}{3}E(1,-1) = \frac{2}{3}(1,-1)$, $E(e_2) = -\frac{1}{3}(1,-1)$

Now to understand more clearly let me consider one example. So see that this space can be written as a direct sum of range of E and null space of E. Here I can also slightly modify one thing. The definition of my projection operator we have defined like this that a linear operator on V such that $E^2 = E$, here R and N basically denote the range space and null space of the projection operator E, I have defined.

I will say that E is a projection operator on R along the null space your N. So let me consider one example. Find a projection operator E which projects this vector space V = R^2 onto the subspace spanned by (1,-1) that is the R = LS{(1,-1)} along the subspace spanned by (1, 2) $\Rightarrow$ $N = LS\{(1,2)\}$. So E(1,2) =(0,0) & E(1,-1) = (1,-1). Consider the standard ordered basis B = { $e_1 = (1,0),\; e_2 = (0,1)$. If I know the image of e_1 & e_2 under E, then we will be able to find the image of any element (x, y) in R^2. So let us know what is the image of e_1 & e_2 under the operator E.

So to find that one I have to express $e_1 = (1,0) = c_1(1,-1) + c_2(1,2) = \frac{2}{3}(1,-1) + \frac{1}{3}(1,2) \Rightarrow E(e_1) = \frac{2}{3}E(1,-1) = \frac{2}{3}(1,-1),\; e_2 = (0,1) = -\frac{1}{3}(1,-1) + \frac{1}{3}(1,2) \Rightarrow E(e_2) = -\frac{1}{3}E(1,-1)$.

(Refer Slide Time: 43:01)

$\Rightarrow$ For any element $\alpha = (x, y) \in \mathbb{R}^2$

$$(x, y) = x e_1 + y e_2$$

$$\Rightarrow E(x, y) = x E(e_1) + y E(e_2)$$

$$= x \frac{2}{3}(1,-1) + y\left(\frac{-1}{3}\right)(1,-1)$$

$$= \frac{2}{3}(x, -x) - \frac{1}{3}(y, -y)$$

$$= \left(\frac{2}{3}x - \frac{1}{3}y, -\frac{2}{3}x + \frac{y}{3}\right)$$

So, this implies for any element α = (x, y) $\in$ R^2, (x, y) = xe_1 + y e_2 $\Rightarrow$ E(x, y) = x E(e_1) + y E (e_2) = x $\frac{2}{3}$ (1,-1) +y (-$\frac{1}{3}$) $(1,-1)$ = $\frac{2}{3}$ (x,-x) - $\frac{1}{3}$ $(y, -y)$ = ($\frac{2x}{3}$ - $\frac{y}{3}$, -$\frac{2x}{3} + \frac{y}{3}$), anyhow some this type of structure you will get. Now you can also solve some problems related to the projection operator which are given the assignment sheet and we shall utilize this concept of projection operator to simplify our operator. So, we will continue in our next class.