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Application of Invariant Subspaces - 2

So, welcome to lecture series on advanced linear algebra. We have seen the importance of invariant subspace to give a simplified structure of the linear operator defined over a finite dimensional space in terms of triangulability of the operator. This concept also can be utilized to a linear operator defined over a finite dimensional vector space whether it is diagonalizable or not that can be also proved by use of the concept of invariant subspace. So what is that? It is like this.

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Let T be a linear operator on a finite dimensional vector space V over F. Then T is diagonalizable if and only if the minimal polynomial say p is product of linear factors of degree one, i.e. $p = (x-c_1)(x-c_2)\dots(x-c_k)$ where $\{c_1, c_2, \dots c_k\}$ are distinct constants from F or I can say distinct eigenvalues of the operator T. Already we have seen if T is diagonalizable then minimal polynomial $p = (x-c_1)(x-c_2)\dots(x-c_k)$.

So already we have seen if T is diagonalizable then the minimal polynomial of T is of the form of * i.e. $p = (x-c_1)(x-c_2)\dots(x-c_k)$ where $\{c_1, c_2, \dots c_k\}$ are distinct eigenvalues of T. Now we want to prove other way.

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Now, we want to prove other way, i.e. if
$$p = (x-c_1) \cdots (x-c_k)$$
then the operator T is diagonalizable.

Let $W_1, W_2, \cdots W_k$ be the eigen spaces associated to eigen value $c_1, c_2 \cdots c_k$, respectively.

Let $W = W_1 + W_2 + \cdots + W_k$

Let $W \subset V$ (is a proper subspace of V)

Let $0 \neq \alpha \in V$ s.t. $\alpha \notin W$ —

So, according to last lemma; there exist an eigen value

say c_j s.t.

$\quad (T-c_jI)\alpha \in W$ —

Let $\quad \beta = (T-c_jI)\alpha \in W$

Proof other way that is p = $(x-c_1)(x-c_2) \dots (x-c_1)$, then the operator T is diagonalizable, so you want to prove this one. So, I shall prove this result with the help of concept of our last class lemma that is if T be a linear operator defined over a finite dimensional vector space and the minimal polynomial is product of linear factors, therefore any subspace W which is invariant under T if you consider any vector $\alpha \in V$, $\alpha \notin W$, the T conductor of α will be linear factor, linear polynomial. $(T-c_jI)\alpha \in W$

I mean to say $(T-c_jI)\alpha \in W$. So I will use this result to prove the reverse one. Let $W_1, W_2, \dots W_k$ be the eigenspaces associated to eigenvalue $\{c_1, c_2, \dots c_k\}$ respectively. I mean to say W_i is the eigenspace associated to eigenvalue c_i, i = 1 to k. Let $W = W_1 + W_2 + \cdots + W_k$. Let W $\subset$ V. Otherwise the operator T will be diagonalizable because the space is spanned by the eigenvectors of the operator T.

So, I have assumed that W the sum of eigenspaces is a proper substance of V, I will say that it is not possible. So, let $0 \neq \alpha \in V$ s.t. $\alpha \notin W$. So according to last lemma, what is that? If T be a linear operator defined over the finite dimensional vector space V which minimum polynomial is product of linear factors and if $W \subset V$ or any subspace of V such that if $\alpha \in V$ s.t. $\alpha \notin W$, then there exists an eigenvalue c such that $(T-c_jI)\alpha \in W$. So I am not writing the statement of the lemma. So I can simply say that according to last lemma there exists an eigenvalue say c_j, so certainly c_j will be from one of the $\{c_1, c_2, \dots c_k\}$ such that $(T-c_jI)\alpha \in W$. So let $\beta = (T-c_jI)\alpha \in W$.

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$\Rightarrow \beta = \beta_1 + \beta_2 + \cdots + \beta_k$, where $\beta_i \in W_i$

$\quad T\beta = T\beta_1 + T\beta_2 + T\beta_k$

$\quad\quad = c_1\beta_1 + c_2\beta_2 + c_k\beta_k$

For any polynomial $h(x)$ over F

$\quad h(T)\beta = h(c_1)\beta_1 + h(c_2)\beta_2 + \cdots + h(c_k)\beta_k$ ——

For minimal polynomial p, we have

$\quad p = (x-c)q(x)$ ——

$\Rightarrow \quad q(x) - q(c) = (x-c)h(x)$

$\Rightarrow \quad \left(q(T) - q(c)I\right)\alpha = (T-cI)h(T)\alpha$

$\quad\quad\quad = h(T)(T-cI)\alpha$

$\quad\quad\quad = h(T)\beta \in W$

$\because \quad q(x) = q(x) - q(c)$
$\quad\quad q(c) = q(c) - q(c)$
$\quad\quad\quad = 0$

$\because \quad (T-cI)\alpha = \beta$

So this implies $\beta = \beta_1 + \beta_2 + \ldots + \beta_k$ where $\beta_i \in W_i$ and $T(\beta) = T(\beta_1) + T(\beta_2) + \ldots + T(\beta_k) = c_1\beta_1 + c_2\beta_2 + \ldots + c_k\beta_k$. In general for any polynomial h(x) over F if you consider, then $h(T)\beta = h(c_1)\beta_1 + h(c_2)\beta_2 + \ldots + h(c_k)\beta_k$. So, I will have h(T) for any polynomial h over the F, h to this operator operating on β will be of this structure and it will be in your W. For minimal polynomial $p = (x-c_j)q(x)$. Based on our hypothesis, I will get the contradictions that $(x-c_j)$ divides q(x). I will prove that $(x-c_j)$ this is a factor of the q(x), so that our hypothesis that T is not diagonalizable will be wrong.

So $p = (x-c_j)q(x) \implies q(x) - q(c_j) = (x-c_j)h(x)$. Since if I consider say $g(x) = q(x) - q(c_j)$ then $g(c_j) = q(c_j) - q(c_j)$, so that is why I am saying that g have $(x-c_j)$ as a factor. So I can write down, $q(x) - q(c_j) = (x-c_j)h(x) = (q(T) - q(c_j)I)\alpha = (T-c_jI)h(T)\alpha = h(T)(T-c_jI)\alpha = h(T)\beta \in W$. Where $(T-c_jI)\alpha = \beta$.

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$$\Rightarrow q(T)\alpha - q(\varsigma)\alpha = h(T)\beta \in W \quad — \quad (2)$$

$$0 = p(T)\alpha = (T-\varsigma I)q(T)\alpha$$

$\Rightarrow q(T)\alpha$ is an eigen vector provided $q(T)\alpha \neq 0$.

If $q(T)\alpha = 0 \Rightarrow \alpha \in W$

$\therefore q(T)\alpha \neq 0$.

$\therefore q(T)\alpha \in W$

$$\Rightarrow (1) \quad q(\varsigma)\alpha = q(T)\alpha - h(T)\beta \in W$$

$$\Rightarrow q(\varsigma)\alpha \in W, \quad \text{But } \alpha \notin W$$

$$\Rightarrow q(\varsigma) = 0 \ , \Rightarrow (x-\varsigma) \text{ must divides } q(n)$$

So this implies $q(T)\alpha - q(c_j)\alpha = h(T)\beta \in W$. My claim is that $q(T)\alpha \in W$, how? We have $0 = p(T)\alpha = (T-c_j I)q(T)\alpha \implies q(T)\alpha$ is an eigenvector provided $q(T)\alpha \neq 0$. If $q(T)\alpha = 0 \implies \alpha \in W$, where $q(T)$ is a product of $(T-c_1 I)$ $(T-c_2 I)$ like that.

So this means that α will be annihilated by this, but I can say that then $q(T)\alpha = 0 \implies \alpha \in W$, because α will be eigenvector of the operator T in that case, but $\alpha \notin W$, so therefore $q(T)\alpha \neq 0$. It is eigenvector, so $q(T)\alpha \in W$ because it is an eigenvector and W is the eigenspace, sum of eigenspaces.

So this implies if I go to this expression from I, we have what? We have $q(c_j)\alpha = q(T)\alpha - h(T)\beta \in W$ because $q(T)\alpha \in W$, $h(T)\beta \in W$. So this implies $q(c_j)\alpha \in W$, but $\alpha \notin W$, so this implies what? The $q(c_j) = 0$. So, this implies $(x-c_j I)$ must divide $q(x)$.

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which is not possible.

∴ T is diagonalizable

Ex1: Every matrix A over the real line R, satisfying
the relation $A^2 = A$ is diagonalizable over R.

Pf: If $A^2 = A$ ⟹ the minimal polynomial of A

is $p(x) = x^2 - x = x(x-1)$

∴ p is product of linear factors of degree one

∴ A is diagonalizable.

Which is not possible because q(x) is basically the part of the minimal polynomial p excluding the factor $(x-c_j I)$ and the minimum polynomial p having only one degree factor having a degree one only. So T is diagonalizable. So let us consider some problems. First problem for example type every matrix A, I mean square matrix of course, over the real line R satisfying the relation $A^2 = A$ is diagonalizable over R, how it is?

If $A^2 = A$ this implies that the minimal polynomial of A is $p(x) = x^2 - x = x (x-1)$. So p is product of linear factors of degree one. So according to last theorems A is diagonalizable.

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Ex-2 Consider the indefinite integral operator T defined over the
space continuous f^n in C[0,1], & is given by

$$(Tf)(x) = \int_0^x f(t)\, dt \quad —$$

Let W be the space of all polynomial f^ns over IR.
So, W ⊂ C[0,1].
Then what can we say about the T-invariant of the W?

Certainly W will be invariant under T, ∵ the indefinite
integral of a polynomial f^n gives again a polynomial
f^n.

Let W_1 be the space of polynomial f^n over F
which are zero at $x = \frac{1}{2}$
Then what can we say about T-invariant of W_1?

Let me consider some more problems. Consider the indefinite integral operator T defined over the space of continuous function in C[0,1]. Suppose it is given by $(Tf)(x) = \int_0^x f(t)dt,$ so this is the my integral operator I am taking this one which is indefinite integral operator T. Let W

be the space of all polynomial functions over real line R. You will see that this W is a subset of space of all continuous function, so I can give its name as C[0,1].

So, W $\subset$ C[0,1]; not only subset it will be also subspace of C[0.1]. Then what can we say about the T invariant of the subspace W? I mean the space of polynomial function when you integrate will it remain again a polynomial function? Certainly, because if I integrate a polynomial function; I will get again a polynomial over the corresponding field. So therefore, W will be invariant under T.

Certainly, W will be invariant under T since the integration of a polynomial function, when the indefinite integral of course, integral of a polynomial function gives again a polynomial function so that is why W will be invariant under T. Let me modify it. Let W_1 be the space of all polynomial functions which are zero at $x = \frac{1}{2}$, I mean to say the $(x-\frac{1}{2})$ is a factor of the each polynomial function. Then do you think that W_1 will be invariant under T? Then what can we say about T invariance of W_1?

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Answer is no, why? Suppose if you take, $(Tf)(x) = \int_0^x (t - \frac{1}{2})dt = \left[\frac{t^2}{2} - \frac{t}{2}\right]_0^x = \frac{t}{2}[t - 1]_0^x = \frac{x(x-1)}{2}$. So, $\left(x-\frac{1}{2}\right) \nmid \frac{x(x-1)}{2} \Rightarrow \left(x-\frac{1}{2}\right) \nmid W_1$. So you can solve many other problems which are given in the tutorial sheets and see the applications of concept of invariance of space in diagonalization or triangulation of the operator.