Advanced Linear Algebra
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Lecture: 31
Application of Invariant Subspaces - 1

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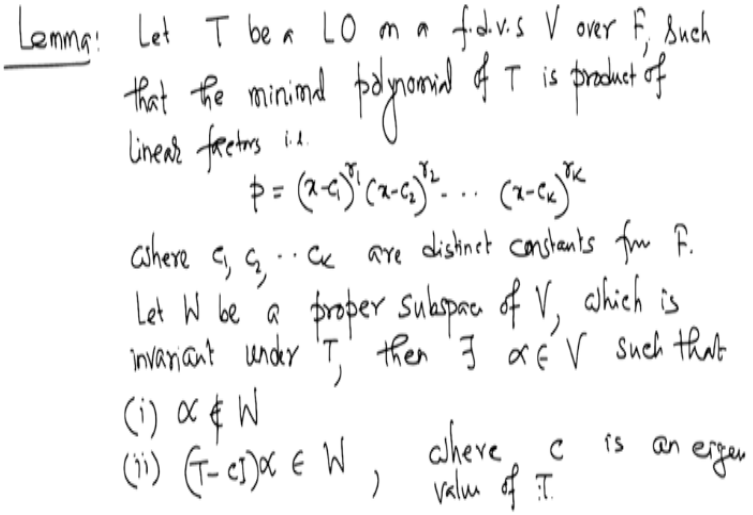

Welcome to lecture series on advanced linear algebra. In our last lecture, we have proved one result if T be a linear operator on a finite dimensional vector space V over the field F such that the minimal polynomial of the T is a product of linear factors that is if p is the minimal polynomial, $p = (x-c_1)^{r_1}(x-c_2)^{r_2} \ldots (x-c_k)^{r_k}$ where $\{c_1, c_2, \ldots c_k\}$ are distinct eigenvalues or you can say distinct constant from F.

And if W be a proper subspace of V which is invariant under T, then $\exists\ \alpha \in V$, certainly nonzero α because I have taken W is a proper subspace, so there is this a nonzero α in V such that the T-conductor of α is a linear polynomial of the form of (x–c). I mean to say (i) $\alpha \notin W$ (ii) $(T-cI)\alpha \in W$, where c is basically some eigenvalue of the operator T.

We shall utilize this result to prove the existence of simplified form of a linear operator defined over a finite dimensional vector space. I will talk about the triangular ability of the operator T.

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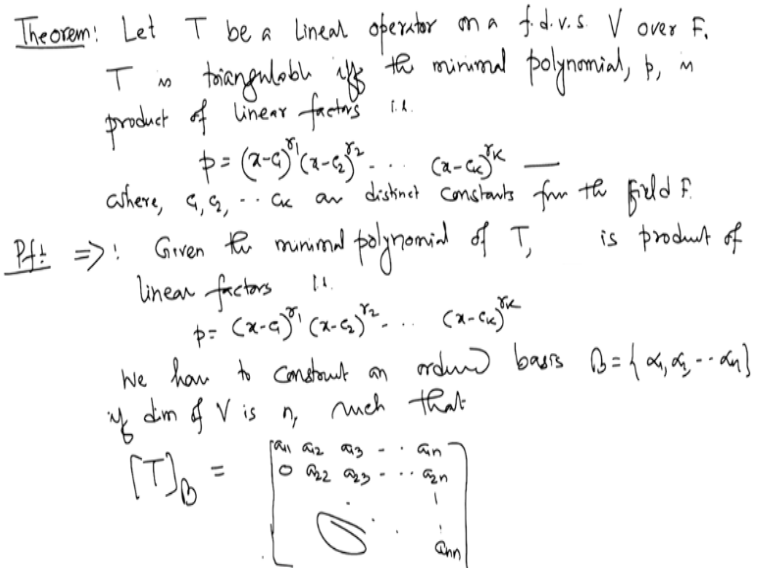

So this is a nice result I can write in terms of theorems like this. Let T be a linear operator on a finite dimensional vector space capital V over F, then T is triangulable if and only if the minimal polynomial say p is product of linear factors that is $p = (x - c_1)^{r_1} (x - c_2)^{r_2} \ldots (x - c_k)^{r_k}$. Certainly $\{c_1, c_2, \ldots c_k\}$ will be eigenvalue or characteristic value of the operator and r_i, i = 1 to k is the positive integers and $\{c_1, c_2, \ldots c_k\}$ are distinct eigenvalues.

If it is the characteristic polynomial, then I could say $\{r_1, r_2, \ldots r_k\}$ are the algebraic multiplicity of the eigenvalues $\{c_1, c_2, \ldots c_k\}$, where $\{c_1, c_2, \ldots c_k\}$ are distinct constants from F. How to prove this result? To prove this result we shall use our previous results that if T be a linear operator on a finite dimensional vector space and the minimal polynomial is product of linear factors.

And if you consider W the proper subspace of V, then $\exists\ \alpha \in V$ such that $(T–cI)\alpha \in W$. So, I shall use this result or this lemma repeatedly to get the corresponding ordered basis under which the operator will be triangulable. So let me consider if part, given the minimal polynomial of T is product of linear factors, it is given to us. That is $p = (x - c_1)^{r_1} (x - c_2)^{r_2} \ldots (x - c_k)^{r_k}$ this is given to us.

We have to construct an ordered basis B consisting of $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ if dimension of V is n such that $[T]_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ 0 & a_{22} & a_{23} & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$. So, I will have a triangular matrix like this. We have

to construct an ordered basis B consisting $\alpha_1, \alpha_2, \ldots \alpha_n$ provided the dimension of the space is n such that the matrix representation will be triangular matrix.

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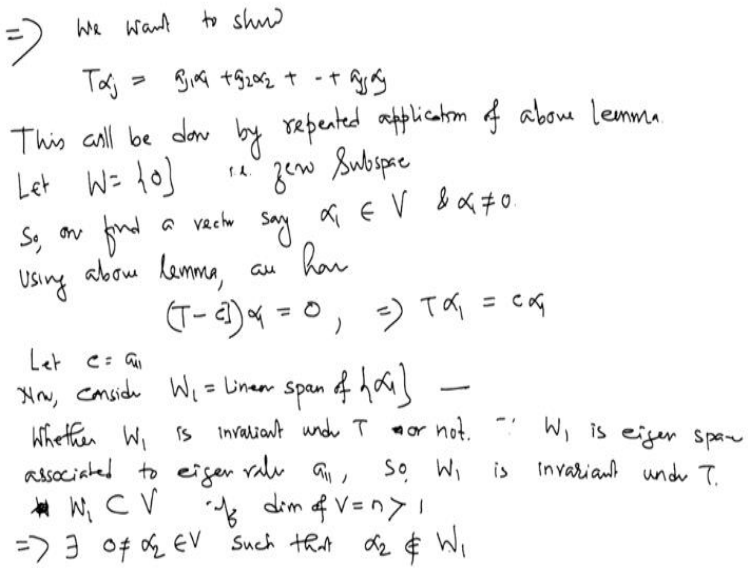

This implies we want to show that $T(\alpha_j) = a_{j1}\,\alpha_1 + a_{j2}\,\alpha_2 + \ldots + a_{jj}\alpha_j$. So we want to show that there exist an ordered basis $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ such that $[T]_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ 0 & a_{22} & a_{23} & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$ will be the characteristic values of the operator. So this will construct by repeated application of the previous results. This will be done by repeated application of above lemma.

So let W = {0}. So one can find $\alpha_1 \in$ V & $\alpha_1 \neq 0$, using above lemma, we have (T-cI)$\alpha_1 = 0$. So let c = a_{11}. This implies Tα_1 = $\alpha_1 c$, let c = a_{11}, just renaming the constant. Now consider W_1 as linear span of α_1.

W_1 is basically another subspace of vector space V which is like this, W_1 = linear spin of $\{\alpha_1\}$. Now question is whether W_1 is invariant under T? See the way W_1 has been constructed it is basically eigenspace associated to eigenvalue c = a_{11}. Since W_1 is eigenspace associated to eigenvalue a_{11}, so W_1 is invariant under T. Now W_1 if the dimension of the vector space n = 1 then we are through it here.

Now $W_1 \subset$ V if dimension of V = n >1. So, this implies $\exists\ 0 \neq \alpha_2 \in$ V such that $\alpha_2 \notin W_1$.

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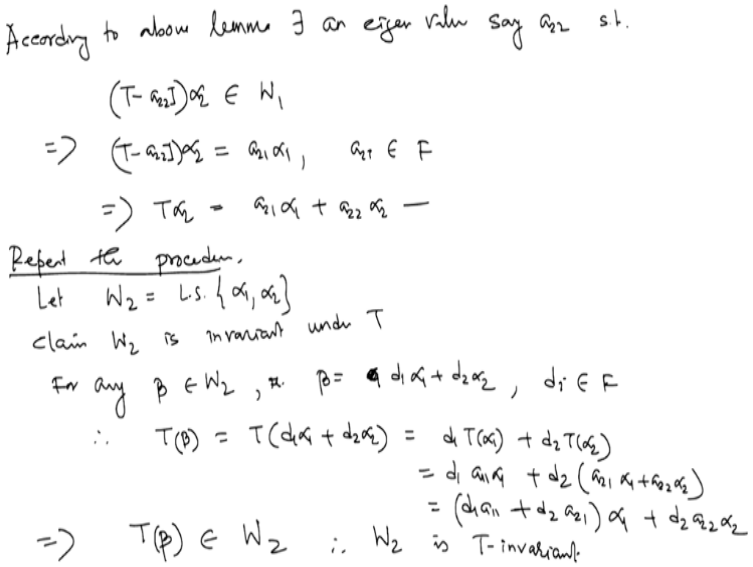

So according to above lemma, there exist an eigenvalue say a_{22} such that $(T\text{-}a_{22}I)\alpha_2 \in W_1 \Longrightarrow (T\text{-}a_{22}I)\alpha_2 = a_{21}\alpha_2$ where $a_{21} \in F \quad \Longrightarrow T(\alpha_2) = a_{21}\alpha_1 + a_{22}\alpha_2$. Repeat the procedure. But before repeating procedures I have to say that let W_2 = linear spin of $\{\alpha_1, \alpha_2\}$. Claim W_2 is invariant under T. How it is invariant on T? For any $\beta \in W_2$, $\beta = d_1\alpha_1 + d_2\alpha_2$, $d_i \in F$. So $T(\beta) = T(d_1\alpha_1 + d_2\alpha_2) = d_1T(\alpha_1) + d_2T(\alpha_2) = d_1a_{11}\alpha_1 + d_2(a_{21}\alpha_1 + a_{22}\alpha_2) = (d_1a_{11} + d_2a_{21})\alpha_1 + d_2a_{22}\alpha_2$. So this implies $T(\beta) \in W_2$. So W_2 is T invariant. So now I can repeat the same procedure.

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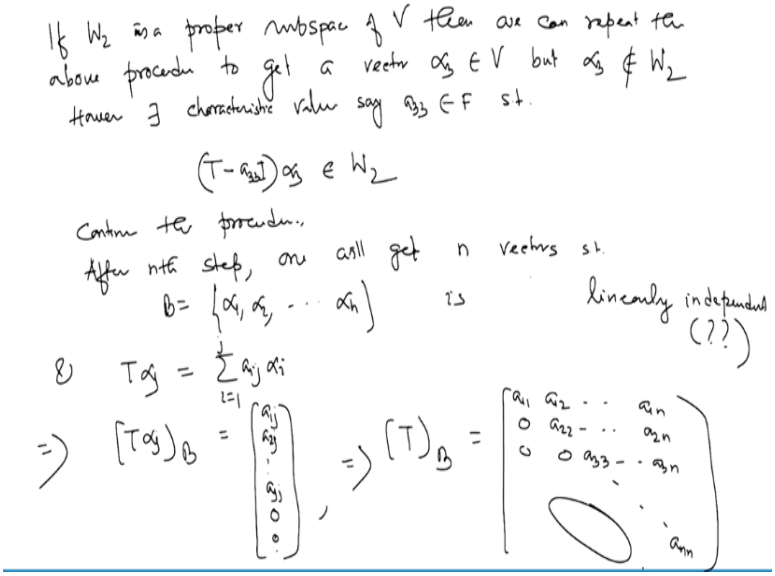

So if W_2 is a proper subspace of V, then we can repeat the above procedures to get a vector $\alpha_3 \in V$, $\alpha_3 \notin W_2$. However, ∃ characteristic eigenvalue say $a_{33} \in F$ s.t. $(T\text{-}a_{33}I)\alpha_3 \in W_2$. Continue the procedure. After n-th step one will get n vectors $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$. How it is

linearly independent? Already I have given the proof the way we have constructed the vector $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$. I have utilized the concept that $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$ are linearly independent. So, we will get n linearly independent vectors $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$ which are linearly independent and $T(\alpha_j) = \sum_{i=1}^{j} a_{ij}\,\alpha_i$. So, I will have a column matrix like this, $[T(\alpha_j)]_B = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{jj} \\ 0 \end{bmatrix}$. So, this implies

$[T]_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ 0 & a_{22} & a_{23} & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$.

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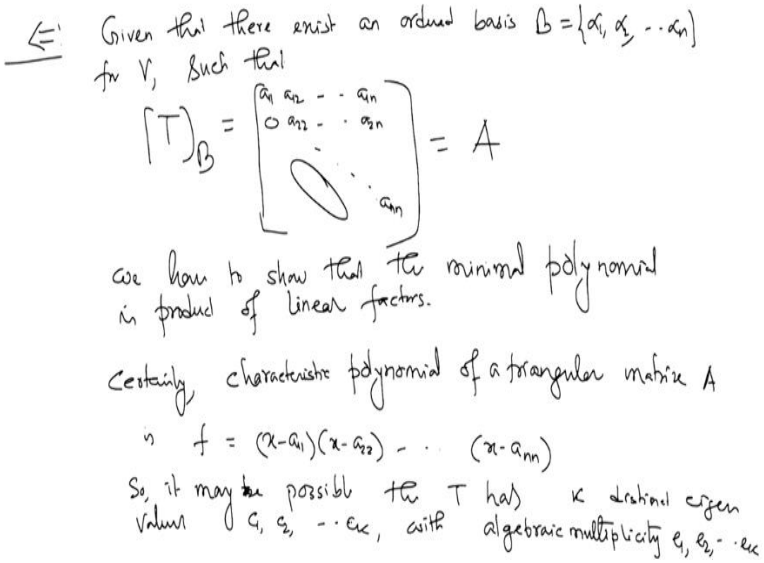

Now it is other way, only if. Given that there exist an ordered basis B = $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$ for vector space V such that $[T]_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ 0 & a_{22} & a_{23} & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$ = A. We have to show that the minimal polynomial is product of linear factors. Certainly, characteristic polynomial of triangular matrix this is equal to say A, is f = $(x - a_{11})(x - a_{22}) \ldots (x - a_{nn})$. It may be possible that these characteristic values are repeated. So, it may be possible that T hats k distinct eigenvalues $\{c_1, c_2, \ldots\ c_n\}$ with algebraic multiplicity say $\{e_1, e_2, \ldots\ e_n\}$.

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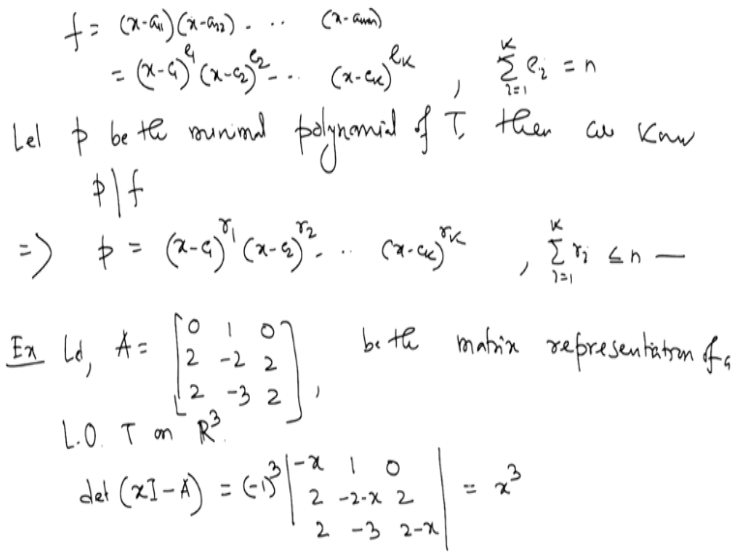

So then f = $(x - a_{11})(x - a_{22}) \ldots (x - a_{nn}) = (x - c_1)^{e_1} (x - c_2)^{e_2} \ldots (x - c_k)^{e_k}, \sum_{i=1}^{k} e_i = n.$ So the characteristic polynomial is like this thing and this $\{c_1, c_2, \ldots c_n\}$ are coming from a_{ii}, i = 1 to n. Let p be the minimal polynomial of T, then we know $\frac{p}{f}$, so this implies p = $(x - c_1)^{r_1} (x - c_2)^{r_2} \ldots (x - c_k)^{r_k}, \sum_{i=1}^{k} r_i \leq n.$

So we see that if T is a linear operator defined over a finite dimensional vector space over the field F and if its minimal polynomial is product of linear factors, I mean to say the minimal polynomial is factorizable over the corresponding field, then the linear operator is triangulable that is there exist an ordered basis under which T can be represented by a triangular matrix.

So for examples let A = $\begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$ be matrix representation of a linear operator T on R^3. One can easily calculate, det(xI–A) = $(-1)^3 \begin{bmatrix} x & 1 & 0 \\ 2 & 2-x & 2 \\ 2 & -3 & 2-x \end{bmatrix} = x^3$.

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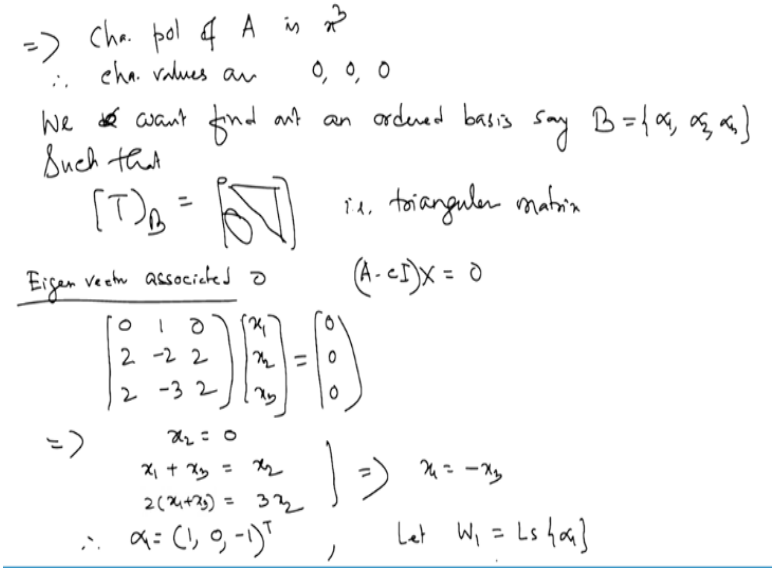

So this implies characteristic polynomial of A is x^3. So characteristic values are 0, 0, 0. Now we want to find out an ordered basis consisting $\{\alpha_1, \alpha_2\ \alpha_3\}$ such that the matrix representation of T is a triangular matrix. We want to find out an ordered basis say B $= \{\alpha_1, \alpha_2, \alpha_3\}$ such that

$[T]_B = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{1n} \\ 0 & 0 & a_{23} & a_{2n} \\ 0 & 0 & 0 & a_{3n} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, i.e. triangular matrix. How to find it? So first I have to find out the eigenvector associated to eigenvalue equal to 0 that will give me the α_1. So eigenvector associated to 0 eigenvalue. So this means that I will have, (A-cI)X = 0 this mean $\begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

So this implies, I will have the system $x_2 = 0,\ x_1 + x_3 = x_2,\ 2(x_1 + x_3) = 3x_2$, this implies $x_2 = -x_3$. So α_1 I can construct is like this, $x_1 = 1$, so $x_2 = 0,\ x_3 = -1,\ \alpha_1 = (1, 0, -1)^T$. So I am working for the matrix A, so this is basically associated to the matrix A only, Now second one I have to obtain let W_1 = LS$\{\alpha_1\}$ that is a scalar multiple of $(1, 0, -1)^T$.

So if it is, then I will have many solutions no doubt, but I can take scalar quantity c = 1. I can have like this.

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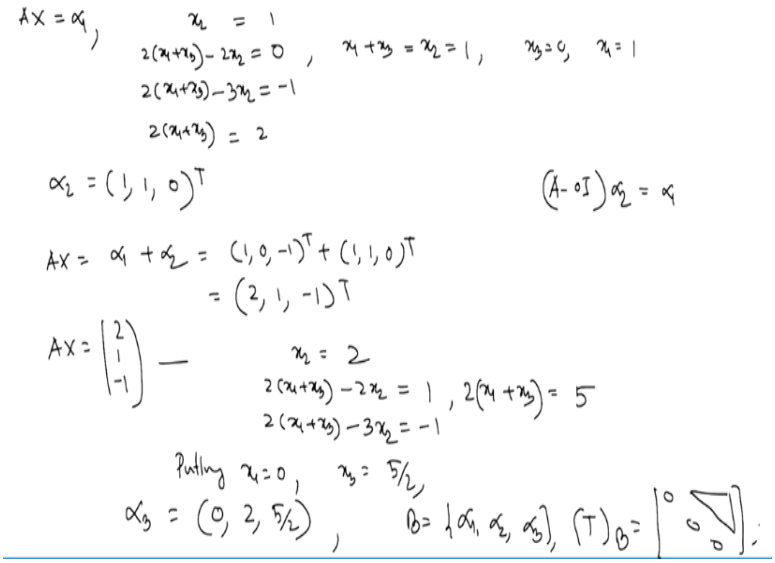

Again system, I will have basically AX = α_1, $x_2 = 1$, $2(x_1 + x_3) - 2x_2 = 0$ this implies $x_1 + x_3 = x_2 = 1$, $x_3 = 0$, $x_1 = 1$, $2(x_1 + x_3) - 3x_2 = -1$, So this implies that I will have $x_2 = 1$ and this is giving me , $2(x_1 + x_3) = 2$. So I will have $x_3 = 0$ and $x_1 = 1$. So this means α_2, I can construct as 1, $\alpha_2 = (1,1,0)^T$. So the construction that α_2 such that here I have see (A–0I)$\alpha_2 = \alpha_1$.

Now α_2 is known, α_3 is also known. Now let me consider again AX = $\alpha_1 + \alpha_2 = (1,0,-1)^T + (1,1,0)^T = (2,1,-1)^T$. So AX = $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Now I have to solve this system. If I solve this system, I will have what $x_2 = 2$, $2(x_1 + x_3 0 - 2x_2 = 1$ and $2(x_1 + x_3 0 - 3x_2 = -1$. Now I have to solve this system. So if I substitute $x_2 = 2$, this implies that $2(x_1 + x_3) = 5$, Third one also gives me, so put $x_2 = 2$, so again same equation. So what I will do? I will take putting $x_1 = 0$, I have $x_3 = \frac{5}{2}$.

So I will have $\alpha_2 = (0, 2, \frac{5}{2})^T$. You can check A$\alpha_3$ = $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ linear combination of α_1 and α_2.

Now B = $(\alpha_1, \alpha_2, \ldots \alpha_n)$, $[T]_B = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{1n} \\ 0 & 0 & a_{23} & a_{2n} \\ 0 & 0 & 0 & a_{3n} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So what is the triangular matrix? So diagonal entry will be 0 because eigenvalue is 0 and this one will be what? We have to see what is the coefficient of T(α_1) and T(α_2) and T(α_3), so that it will give me a triangular matrix.

So you have seen that one can find an ordered basis under which the operator can be represented by a triangular matrix. But this representation is not unique because it depends on the constant scalar quantity. You take different scalar having coefficient 1 and 1, if you take different coefficient I will have different solution, but it is possible to have an ordered basis under which one can have the triangular representation of the operator. So, we will continue the applications of invariance of space in our next class also. Thank you.