**Advanced Linear Algebra**
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**Lecture – 30**
**Invariant subspaces - II**

So, welcome to lecture series Already we have introduce to the concept of invariant of subspaces in my last lecture with couple of examples and the characteristic of the invariant of subspaces also we have seen it. We will continue that because our destination is to understand the linear operator in a more simplified form. So, for this let me introduce another terminology.

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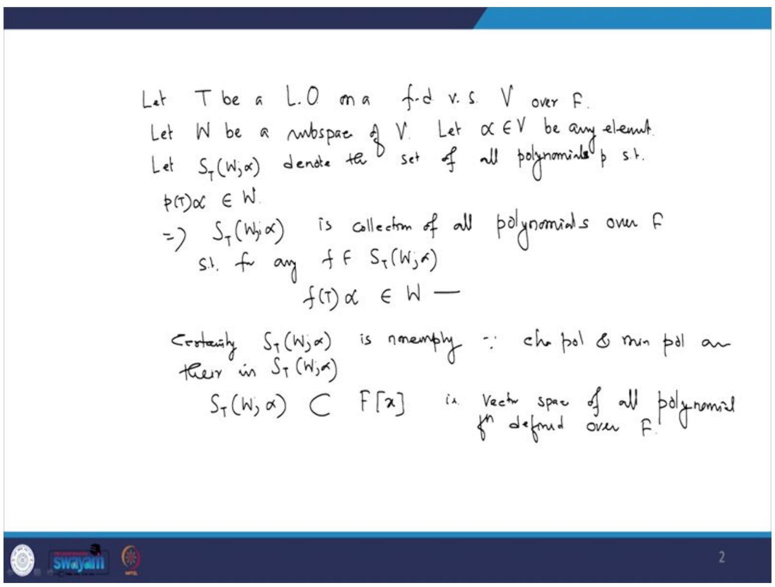


So, before that let me consider let T be a linear operator on a finite dimensional vector space V over the field F. Let W be a subspace of V. Let $\alpha \in$ V be any element. Let $S_T$(W;$\alpha$) denote the set of all polynomials p s.t. p($T$)$\alpha \in$ W. So, $\Longrightarrow$ $S_T$(W;$\alpha$) is collection of all polynomials over the field F s.t. for any f $\in$ $S_T$(W;$\alpha$), f(T)$\alpha$ $\in$ W.

Certainly, $S_T$(W;$\alpha$) is non empty. How can I say it is not empty? See if the dimension of the V is finite then definitely a finite degree polynomials over the field F will be there which will annihilate the operator T. So, if it annihilates the operator T, I mean f(T) = 0 operator then certainly f(T)$\alpha$ $\in$ W because W is a subspace of vector space V. So, characteristic polynomials, minimal polynomial all will be there in the $S_T$(W;$\alpha$).

So, since characteristic polynomial and minimal polynomial are there in $S_T$(W;$\alpha$). So, it is non empty it is not only non empty subset of the vector space of all polynomial function f(x). So, I can say that $S_T$(W;$\alpha$) will be subset of f(x) that is vector space of all polynomial functions defined over the field F because that is also field. So, I can say this will be also subset of this one. We will prove shortly that this is also a subspace of the vector space f(x) also.

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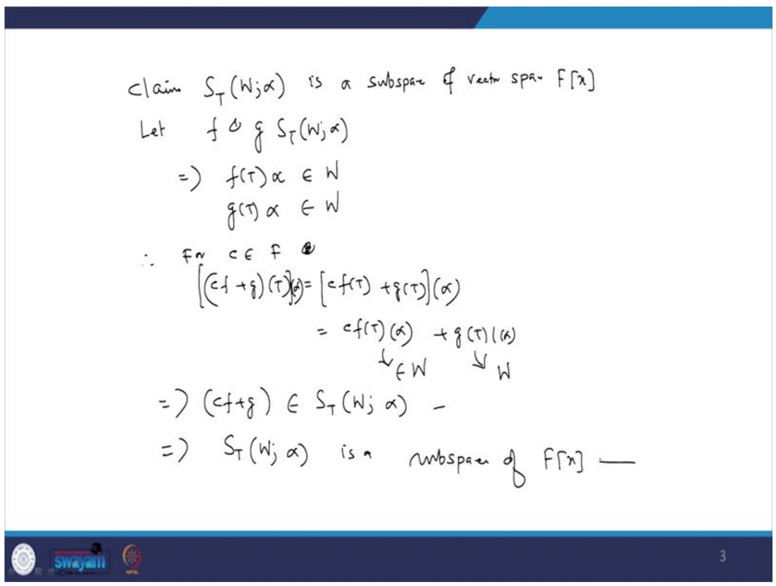


Claim $S_T$(W;$\alpha$) is a subspace of vector space F[x], I consider the operators T define what vector space T which is defined over the field F. Now with respect to that field we have defined another vector space or you can say field of all polynomial function that is f(x) this one. This is also a vector space. So, this subset $S_T$(W;$\alpha$) what we have construct what I have already constructed here for a given W and $\alpha$ we see that this is a subset of f(x).

And now I want to prove that it is not only subset it is also a subspace of f(x) also subspace also. Let me consider in the very beginning here W be an invariance of space under T, now $S_T$(W;$\alpha$) is subspace of a vector space,I have to prove it please, how? Let f & g $\in$ $S_T$(W;$\alpha$). So, this implies f(T)$\alpha$ $\in$ W and g(T)$\alpha$ $\in$ W.

So, for $c \in F$, $[(cf + g)(T)]\alpha = [cf(T) + g(T)]\alpha = cf(T)\alpha + g(T)\alpha$. Now this each of them will belongs to W and because f & g $\in S_T(W;\alpha)$ and its $f(T)\alpha \in W$ this implies that $(cf+g) \in S_T(W;\alpha)$.

So, this implies the set $S_T(W;\alpha)$ is a subspace in the subspace of the vector space F[x]. I am not talking that is how space of the vectors will be T it is a series of space of vector space F[x].

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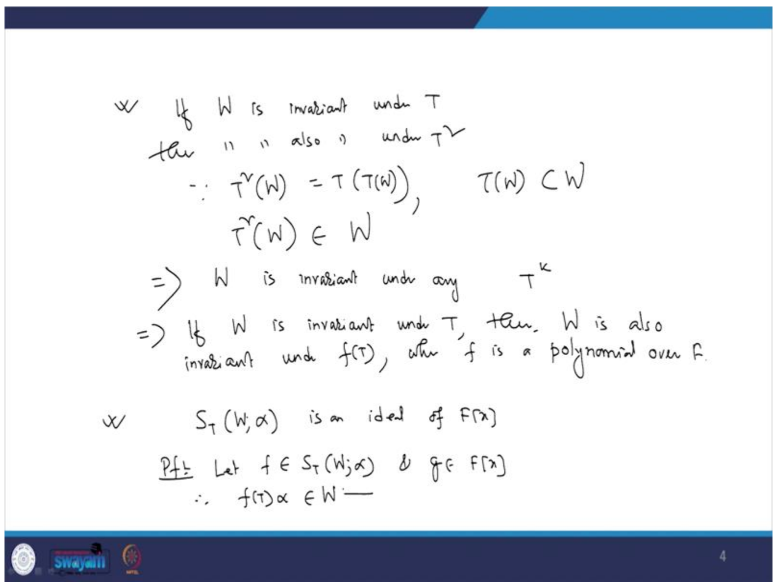


Another interesting is that if W is invariant under T then W is also invariant under $T^2$ because $T^2(W) = T(T(W))$, $T(W) \subset W \implies T^2(W) \in W$. $\implies$ if W is invariant under T, it is also invariant under $T^2$ and it is also invariant under any $T^k$ this mean W is invariant under any $T^k$.

So, this implies if W is invariant under T then W is also invariant under f(T) where f is a polynomial over the field F. So, you see that if W is independent under T then it will remain invariant under any operator f(T) is basically polynomial over the field F when the W is a vector space of space of a vector subspace V to define what the field F. So, we see that the way we have constructed a set of polynomials $S_T(W;\alpha)$, we see that is a subspace of the polynomial space I mean vector space of polynomial functions f(x).

In fact I will say that this $S_T$(W;$\alpha$) is an ideal of the polynomial field effects please. This is a free I am saying field it is also a vector space. So, $S_T$(W;$\alpha$) is an ideal of F[x], I mean both in the case of say W is invariant. So, I will not consider left ideal or right ideal and the ideal general uses to say when left ideal equal to right ideal then I will say that it is a simply ideal place I do not want to confuse it. So, simply I will say that is a subset of the field F[x] will be an ideal if it is basically satisfy the definition of the group with respect to one operations that is vector additions.

And of course it satisfied Cruiser property with respect to multiplications and it satisfied that if I consider f ∈ $S_T$(W;$\alpha$) & g ∈ F[x] then g ∈ $S_T$(W;$\alpha$) then only I can say $S_T$(W;$\alpha$) will be an ideal. So, let me prove this once. Let f ∈ $S_T$(W;$\alpha$) & g ∈ F[x]. So, f(T)$\alpha$ ∈ W.

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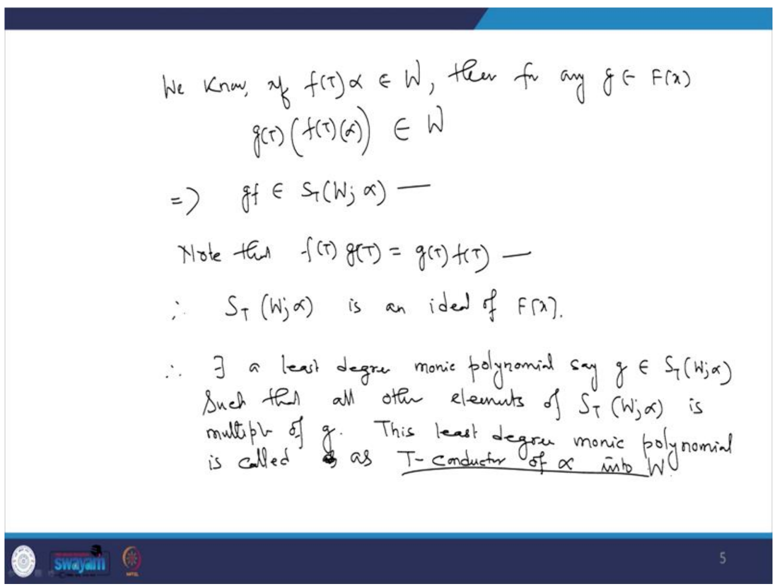


We know if f(T)$\alpha$ ∈ W then for any g ∈ F[x], g(T)(f(T)($\alpha$)) ∈ W already have prove it if W is invariant under T it will be invariant under any polynomial in T. So, since this is a element in W. So, this implies gf ∈ $S_T$(W;$\alpha$). So, what I found we have seen that if a function polynomial function gf ∈ $S_T$(W;$\alpha$).

Here I have multiplied from the left. Now note that f(T) g(T) = g(T) f(T). So, therefore left multiplying from left or right is a irrelevant place. So, I can say that $S_T$(W;$\alpha$) is an ideal of F[x]. Since it is an ideal definitely there is a generator for this I mean to say least degree monic polynomial. So, there exists a least degree monic polynomial say g $\in S_T(W;\alpha)$ such that all other elements of $S_T(W;\alpha)$ is multiple of g this least degree monic polynomial is called as T-conductor of $\alpha$ into W.

So, T conductor of alpha into W is least degree monic polynomials which is basically generator for the ideal $S_T(W;\alpha)$.

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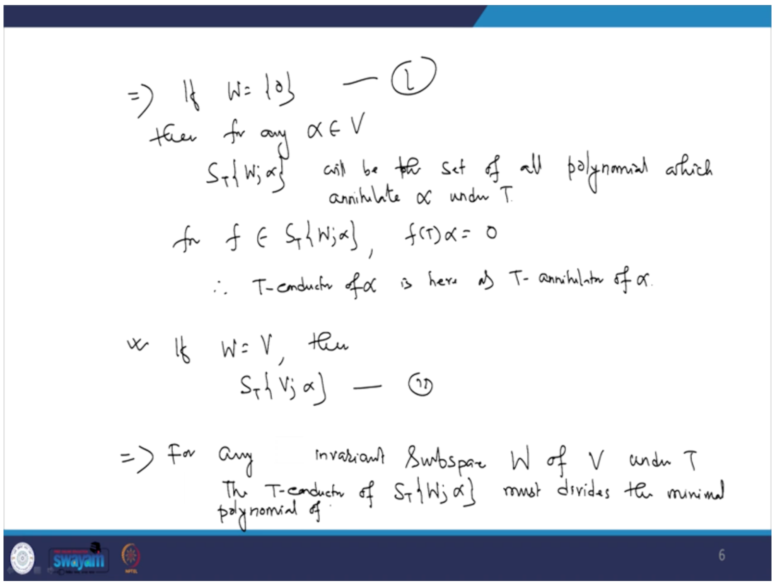


So, this implies if $W = \{0\}$ subspace then for any $\alpha \in V$, $S_T(W;\alpha)$ will be the set of all polynomials which annihilate $\alpha$ under T, I mean to say for $f \in S_T(W;\alpha)$, $f(T)\alpha = 0$, if I consider W is a zero subspace of the vector space. So, one simple case when $W = \{0\}$ subspace then for any element $\alpha \in V$, $S_T(W;\alpha)$, I mean the T-conductor of $\alpha$.

So, T-conductor of $\alpha$ is here at T annihilator there of $\alpha$. Now the question is this when W = 0 subspace then the characteristic polynomial there or not because characteristic polynomial annihilate $\alpha$. So, characteristic polynomial will be in this set also what about the minimal polynomial minimum polynomial also will be in this set also. Now let me take extreme case if W = V then we know V is also invariant under T.

So, in this case what about the $S_T(W;\alpha)$ will be what here the polynomial will be I mean little much bigger set this $S_T$(W;$\alpha$) will be much more bigger set. So, if I consider the least degree monic generator of this one definitely this will be much smaller compared to the least degree monic generator of the $S_T$(W;$\alpha$), W = {0} subspace. I mean these are the two extreme this is a two and other one is so one.

So, both of them are ideals but the monic generator of the second one will be certainly include your monic generator of first one but reverse is not true. So, the monic generator of second one must divide the monic generator of the first ones this implies for any invariant of space subspace W of V under T, the T-conductor of $S_T$(W;$\alpha$) maust divides the minimal polynomial of the operator T because minimal polynomial will by default in each of the set.

So, based on this concept we can write one nice results which I will use to simplify the operators. So, before applying that concept let me define one terminology.

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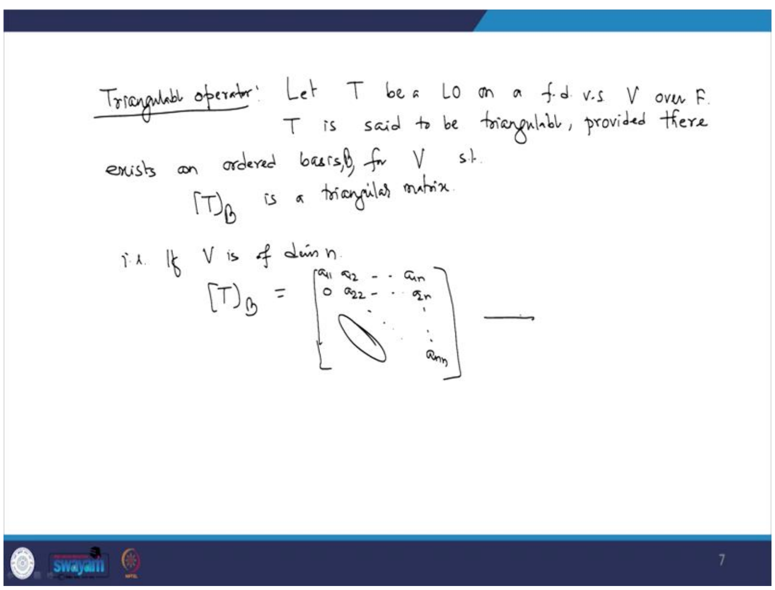


It is called triangulable operator. Let T be a linear operator on a finite dimensional vector space capital V over the field F. T is said to be triangulable T is said to be triangulable, provide it there exist an order basis B, for vector space V s.t. $[T]_B$ is a triangular matrix. I mean to say that is if

V is of dimension n, then $[T]_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ 0 & a_{22} & a_{23} & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$.

So, the structure of this matrix represent of the T with respect to order of B is will be something like this triangulable matrix space then only I will say that a given operator is triangulable. I will see under what conditions a linear operator defined over a finite dimension vector space is triangulable. So, to understand that concept let me introduce a small results in terms of Lemma it is like this.

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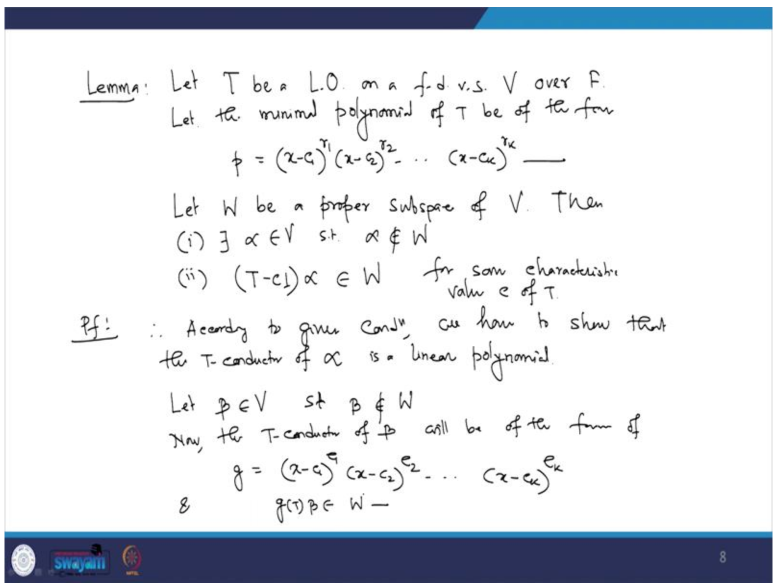


Let T be a linear operator on a finite dimensional vector space V over F. let the minimum polynomial of T be of the form p = $(x-c_1)^{r_1}(x-c_2)^{r_2} \ldots (x-c_k)^{r_k}$. Let W be a proper subspace be a proper subspace of V then ∃ $\alpha \in$ V s.t. $\alpha \notin$ W (ii) (T-$c_i$)$\alpha \in$ W for some characteristic value c of T.

So, it is given me that W be a proper space and it also given that the minimum polynomial is product of linear factors. So, in this case result says that for any $\alpha \in$ V but $\alpha \notin$ W there exist a characteristic value c of T such that (T-$c_i$)$\alpha \in$ W. So, according to given condition basically we have to show have to show that the T-conductor of $\alpha$ is a linear polynomial.

So, let $\beta \in$ V such that $\beta \notin$ W. Now T-conductor of $\beta$ will be of the form of say g = $(x-c_1)^{e_1}(x-c_2)^{e_2} \ldots (x-c_k)^{e_k}$. Why I did like this we know that the T-conductor of $\beta$ must divide the minimal polynomial and minimal polynomial having structure like this. So, the structure of g has to be like this such that g divides p that is why the structure of the T-conductor of $\beta$ will be like this $g(T)\beta \in$ W, it is given to us.

I mean we have assume please now since g is like this structures what can you say about the $e_1, e_2, \ldots e_n$ certainly $e_1$ cannot cross , $e_2$ cannot cross $r_2$ and $e_k$ cannot cross $r_k$ that is true because g divides p. Is it possible that all $e_i$ will be equal to 0.

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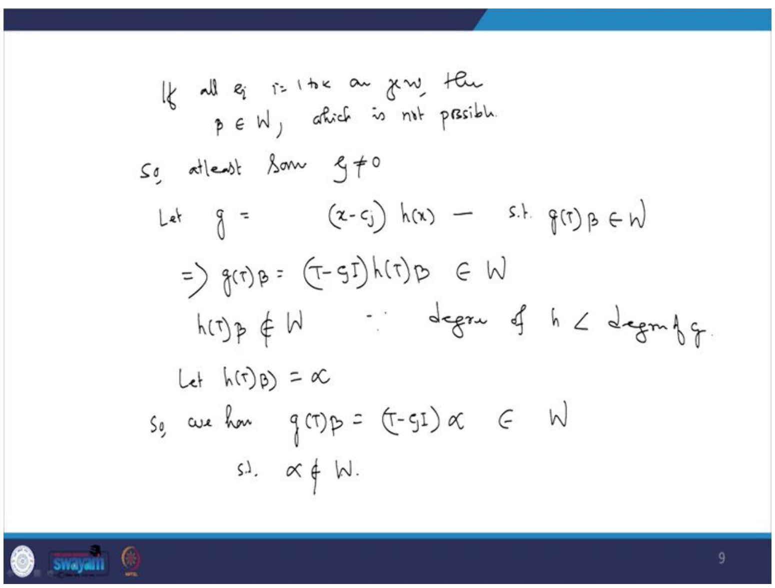


If all $e_i$, i = 1 to k are zero then I will have $\beta \in$ W which is not possible according to our hypothesis which is not possible. So, at least some $e_j \neq 0$. So, let g = (x-$c_i$)h(x) s.t. $g(T)\beta \in W$, So this implies $g(T)\beta$ = (T-$c_j I$)h(T)$\beta$ $\in$ W.

I have taken $e_j \neq 0$, so, $e_j$ is basically more than one I mean it will be like two three four like that. So, that is why I wrote, g = (x-$c_i$)h(x) where h is a polynomial of degree less than the degree of the g. Now , $h(T)\beta \notin$ W because degree of h is less than degree of g and g the least degree monic generator.

So, therefore $h(T)\beta$ cannot be in W. So, let $h(T)\beta = \alpha$. So, we have we have $g(T)\beta$ = (T-$c_jI)\alpha$ $\in$ W s.t. $\alpha \in$ W. So, we see we have proved that the T -conductor of $\alpha$ is basically here as a linear polynomials (x-$c_j$), i shall use this concept to prove the nice result that is called triangularization of a linear operator.

So, what we see here that if minimal polynomial of an operator is product of linear factors and the operator is defined over a finite dimensional vector space then for any proper subspace W for any element $\beta \in$ V such that $\beta \notin$ W, we see that the T-conductor of $\beta$ will be basically a linear polynomial. I mean linear polynomial of the form of something (x-c) where c is basically a characteristic value of the operator. I shall use this concept to prove the triangulation of a linear operator also. So, this will be done in the next class.