Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 3
Row-Reduced-Echelon Form and Its Applications

Welcome to my third lecture on advanced linear algebra. In my last lecture, we discussed about while discussing about the system of linear equations, we introduced a terminology called row-reduced form of a matrix. We have seen that for a mxn matrix A defined over a field F one can have a corresponding row-reduced equivalent matrix of order mxn and we have also seen that the system AX=0 and system RX=0 where R is the row-reduced matrix of A have the same solutions. Let me quickly take an example and say it support this.

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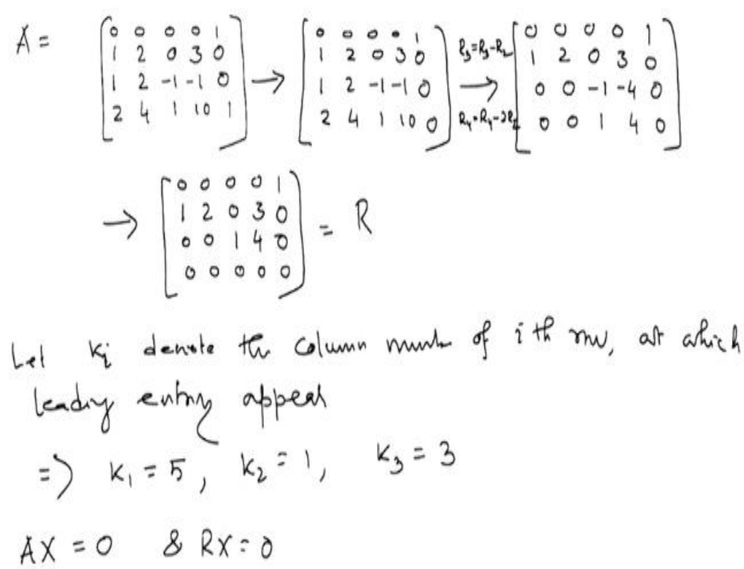

Let me consider a matrix A of 4x5 order which is given like this. Now this matrix it is not a row-reduced matrix because it does not satisfy the criteria of row-reduced matrix. So, first let me have a corresponding row-reduced matrix and to have the corresponding row-reduced matrix one has to prove this one, see from this I have to do the elementary row operations. So, the first a row is nonzero row and nonzero entry is 1, so I do not have to do anything for the first row.

However, I have to make the fifth column which contain the leading entry of first nonzero row I have to make all other entries 0. So for that I have to proceed like this. So, second row l will keep as it is 1 2 0 3 0 1 2 –1 –1 0. So, I will multiply –1 to the first row and then add it. So, if I do it I will have 2 4 1 10 0. So the first row, both the criteria of the row-reduced form are satisfied. Now, let me go to the second row.

Second row again nonzero rows, but the first entry appear on the first column, so therefore I have to make all other columns entries and all other entries have the same column first column as 0. So for this, what I will do I will write down quickly like this. The first row as it is, then second row also I will keep as it is 0, so I have to make first column third entry that is 0, so I will multiply –1 to the first row. So basically, I am saying that $R_3 = R_3 - R_2$.

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So if I do it, I will have 0 0 –1 –4 0. And similarly for the fourth row, I have to make this as 0, it is 2, so I have to replace basically $R_4 = R_4 - 2R_2$. So if I do it then I am getting 0 0 1 4 0. Now, we have a new matrix, but still it is not a row-reduced matrix because we see that fifth column which contained the leading entry of the first row, all other entries 0 that is fine.

Similarly for the second row leading entry appear on first column and all other entries are 0, but if I see the third row, third row is also nonzero row and third row's first nonzero entry is –1, it appear in your third column. So, if I do this operation again, elementary row operations, so keeping first second as it is, so 0 0 0 1 1 2 0 3 0 and so since I have to make it 1, so I can divide with –1, I will have the result, so 0 0 1 4 0.

And then if I just simply add it to this fourth row I am getting 0 0 0 0 0. So, now this is a row-reduced matrix because I see it satisfied both the criteria of the definition of the row-reduced form say R. So let K_1 denote the column number or K_i denote the column number of ith row at which leading entry appears. This implies K_1 =5, K_2=1, K_3 = 3.

Now, since our objective is to answer the question that is whether a system of linear equations has solution or not and if it at all it has solution, how to find a solution of it? So, we are looking for a suitable procedure through which we can answer these questions. So, here I am saying that if got a system AX=0 and RX=0 both the systems have same solution space. Solving AX=0 is same as solving RX=0.

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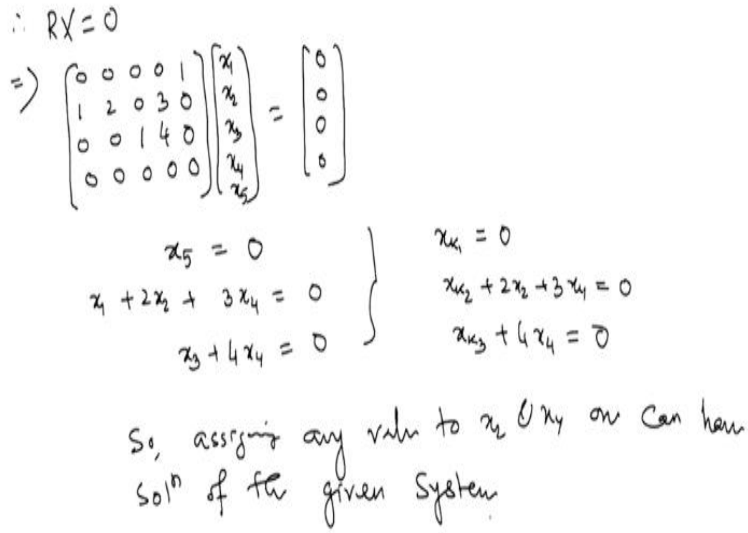

Now, based on that now I am getting RX=0, So, RX=0 implies I am getting 0 0 0 0 1 1 2 0 3 0 and 0 0 1 4 0 and then fourth one is 0 0 0, so this is my then I am saying that X= x_1, x_2 , x_3, x_4 and x_5. So this is equal to 0 0 0 0. So my system is like this. So, I can write down this system RX=0 in linear equation form, then I am getting what this implies x_5=0. Then second equation, x_1+2 x_2+ 3x_4=0 and x_3+4x_4=0.

So, I am getting three equations like this I can also write down these ones x_{k_1}=0 And again I can write down again x_{k_2}+2x_2+3x_4=0 and x_{k_3}+4x_4 =0. So, this implies that I can write down the solutions for the $x_{k_1}, x_{k_2}, x_{k_3}$ as a function of x_2 and x_4. So assigning any value to x_2 and x_4 one can have solution of the given system, but this is for small matrix pulse system is alright.

When you are talking about the very big matrix, very big system assigning to you even to take the help of computer then we have to give proper algorithms for that.

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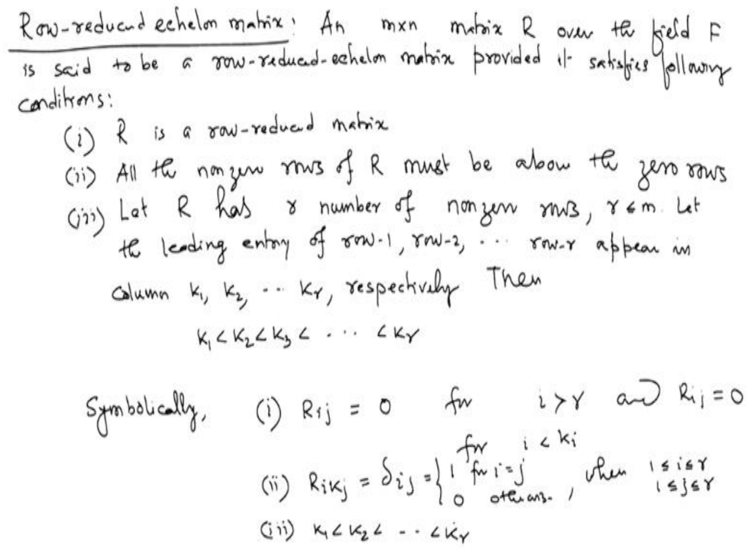

So, let me introduce another terminology called a row-reduced echelon matrix. What is that? An mxn matrix R over the field F is said to be a row-reduced echelon matrix provided it satisfies following conditions. What are the conditions? The first condition R is a row-reduced matrix, means it has to satisfy two criteria of the row-reduced matrix. Second all the nonzero rows of R must be above the zero rows. Third one let R has small r number of nonzero rows.

So, certainly r is less than equal to m. Let the leading entry of row 1, row 2, row r appear in column k_1 , k_2 ,…k_r respectively. Then $k_1 < k_2 < k_3 ... < k_r$. So, an mxn matrix is said to be a row-reduced echelon matrix provided it satisfies these three conditions. Symbolically I can say your R_{ij}=0 for i $> r$ and also R_{ij}=0 for $i < k_i$, so this is first one.

Second one $R_{ik_j} = \delta_{ij} = \begin{cases} 1, & for\ i = j \\ 0, & otherwise \end{cases}$ when $1 \le i \le r, 1 \le j \le r$

and third one $k_1 < k_2 ... < k_r$. These three conditions can be written also in the same form.

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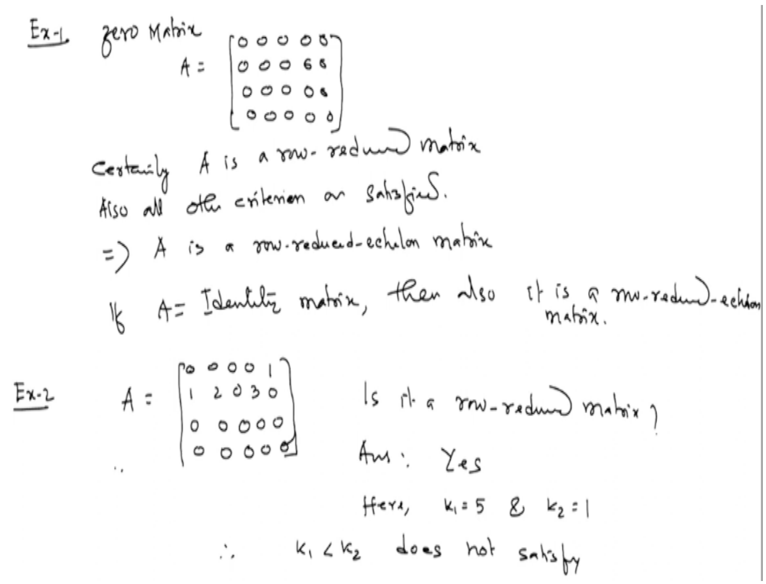

So, let me take an example first, as usual zero matrix will be set certainly a row-reduced echelon matrix. First is zero matrix. This is a trivial example zero matrix that is if A = 0 0 0 0, suppose you consider this one, then certainly A is a row-reduced matrix that is first criteria satisfied and also all other criteria are satisfied So, this implies A is a row-reduced echelon matrix. Similarly, I can take if A equal to let me take identity matrix then also it is a row-reduced echelon matrix.

Now let me take some nontrivial examples. Let me consider A equal to say 0 0 0 0 1 1 2 0 3 0 0 0 0 0 0 0 0 0 0 0 okay. So, I have taken again say 4x5 matrix. Let me check does it satisfy the definition of my row-reduced echelon form. Is it a row-reduced matrix? Answer is yes. We see that it satisfies all these criteria, this one. So next is does it satisfy two more criteria, okay here leading entry on the first nonzero appears in k_1 equal to fifth column, so k_1 =5 and k_2 =1. So, this implies $k_1 < k_2$ does not satisfy. So, therefore this A cannot be row-reduced echelon matrix.

But if I exchange first row to two second row and second row to first row, then certainly you will get as a row-reduced echelon matrix case. Now, as in the case of any mxn matrix over the field F each row equivalent row-reduced matrix, here also any mxn matrix over the field F is row equivalent to a row-reduced echelon matrix case. Similar to row-reduced matrix here also we can say every mxn matrix defined over a field F is row equivalent to a row-reduced matrix.

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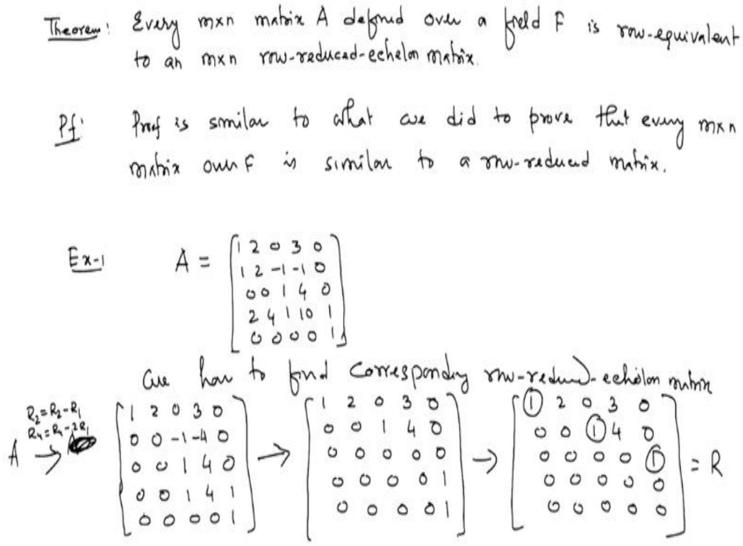

Theorem it can be like this that every mxn matrix A defined over a field F is row equivalent to an mxn row-reduced echelon matrix. Proof is similar to what we did to prove that every mxn matrix over F is similar to a row-reduced matrix. So, I am leaving this one as a homework, so you can try this one please. So, let me consider problems and show that one can have corresponding row-reduced echelon matrix.

So, let me take an example, say capital A equal to 2 4 1 10 and 1 0 0 0 00. So, let me consider this matrix, this is basically 5x5 matrix. Now, let me see how to get the corresponding row-reduced echelon matrix. So, for this again we have to do the elementary row operations. So, if I proceed it so we have to find corresponding row-reduced equivalent matrix. So, for this let me do elementary row representation of 1 to first row.

Since the first nonzero entry in the first nonzero row is 1, so I will leave first row as it is. So 1 2 0 3 0 and then since the first column contained leading entry of the first row, so I have to make all other entries 0, so for that I have to make it 0, so this is basically $R_2 - R_1$. So, if I do it 0 0 –1 –4 and 0, then 4 0 and fourth row I have to basically replace fourth row by fourth row minus two times of first row.

If I do it then it will be 0 0 and this is 1 and again 10 – 6 = 4 and here I am getting 1. So, then it is basically 0 0 0 0 1. So, A goes to this metric space now. So by elementary row operation that is we have done here $R_2 = R_2 - R_1$ and $R_4 = R_4 - 2R_1$. so I have got this one. Then I see that it still even not a row-reduce matrix also. So, for that I have to do another step. So the first row I will leave it as it is because not violating the rules for the definition of the row-reduced echelon form, so 1 2 0 3 0.

And second row I will divide by –1, so it will be 1 4 0. And then third row I have to add this one also to work, so it will be 0 0 0 0 0 and similarly fourth row 0 0 0 0 then it is 1, then 0 0 0 0 1. Still it is not a row-reduced echelon form. So, one more step is required. think that is 1 2 0 3 0 and 0 0 1 4 0. Since all the nonzero rows must be above the zero rows, let me exchange to the third row, so it will be 0 0 0 0 1.

And fifth row if I subtract from the third row, you will have basically 0 0 0 0 0 and 0 0 0 0 0. So, I will have this as the final form. Now, this matrix is a row-reduced, this is a row-reduced echelon matrix and the leading entries here appearing in k_1 I can say that.

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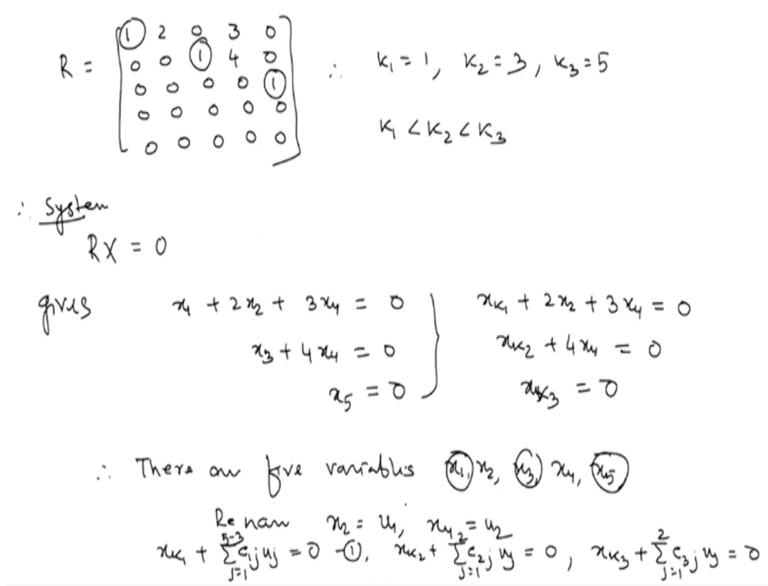

So let me write down R equal to as 1 2 0 3 0. Then second one is 0 0 1 4 0 and third one is 0 0 0 0 1, fourth one is 0 0 0 0 0 and fifth one also same okay, so this is the matrix. So, leading entries appear in the, so k_1 =1 and k_2 =3, and k_3 =5. So, we see that it is a row-reduced matrix and it satisfied the conditions that all the nonzero rows will be above the zero row that is also true.

And then the last one that here $k_1 < k_2 < k_3$ is also satisfied. So, I have a row-reduced echelon form. So, previously we have seen that every mxn matrix defined over a field F is a row equivalent row-reduced matrix. Then we have seen the corresponding system of equations because as you have discuss here also the system AX=0 and system RX=0 where R is row-reduced echelon form, both have the same solution.

So, accordingly now if I talk in the sense of solution point of view, let me write down again the system RX=0. The system RX=0, this gives x_1+2x_2 + 3x_4=0, then x_3+4x_4 =0 and x_5=0. So, I can rewrite this one as x_{k_1}+2x_2+3x_4=0 and x_{k_2}+4x_4=0 and x_{k_3}=0. There are 5 five variables x_1, x_2, x_3, x_4, and x_5 . Now, I see that this x_{k_1}, x_{k_2}, x_{k_3}, we can express explicitly as a function of rest of the variables.

So, this x_1 is a coming and x_{k_1}, x_3, x_5 . So, if I will consider these three variables in one set and others are basically another set, then I can rename say x_2=u_1, x_4=u_2. So, then I can write down the system as x_{k_1}+$\sum_{j=1}^{5-3} c_{1j}\, u_j = 0$, x_{k_2}+$\sum_{j=1}^{2} c_{2j}\, u_j = 0$, x_{k_3}+$\sum_{j=1}^{2} c_{3j}\, u_j = 0$.

So, I can write down this system of equation in this form also and x_{k_1} as I explicitly assigning the value of u_1, u_2; u_1, u_2 is nothing, u_1 is my rename of the variable x_2 and x_4 . Here x_2 and x_4 are treated as free variable. So, to assign one value to x_2 and x_4 ,we will get one solution. So, from this I can also clearly say that this system has solution, not only solutions, it has nonzero solution also.

Certainly each equation satisfied 0 0 0 0 as one of the solutions that is in it each of them represents basically a plane in five dimensional space that we used to say a basically a hyperplane So, each of them represent a hyperplane in five dimensional space and each plane passes through the origin, therefore they intersect the point 0, 0 is the origin. So, in the origin there will be by default a solution, so 0 0 0 0 will be there by default solution.

But apart from zero solution, one can have nonzero solution that is nontrivial solution here for these problems we see by assigning that 5 – 3 variables and we have the solution. So, let me generalize this concept.

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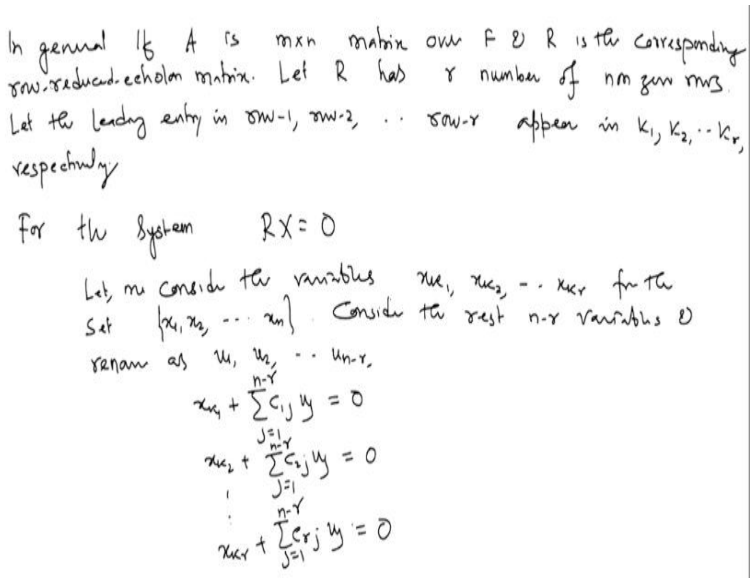

In general, if A is an mxn matrix over F, F is a set of field basically, and R is the corresponding row-reduced echelon matrix. Let R has r number of nonzero rows and let the leading entry in row 1, row 2, row r appear in column k_1, k_2 ... k_r respectively. This implies the system RX=0 will have basically r equations basically as in the case of previously I have seen through the examples even though r is a 5x5 matrix, but it has only 3 nonzero rows, so we have only 3 equations.

So, I will have small r number of equations. Let me consider the variables $x_{k_1}, x_{k_2}, \dots x_{k_r}$ they are the variables from the set $x_1, x_2, \dots x_r$. So, I will take out these variables, so rest will be n – r variables. Consider the rest n – r variables in these n number of variables and rename as u_1, u_2,.. u_{n-r}. Then the system RX=0 can be written as you know by the system equations like this first is as x_{k_1}+$\sum_{j=1}^{n-r} c_{1j}\, u_j = 0$,

Then similarly x_{k_2}+$\sum_{j=1}^{n-r} c_{2j}\, u_j = 0$ and x_{k_3}+$\sum_{j=1}^{n-r} c_{rj}\, u_j = 0$. So, l will have a new system here having r equations and so I have n – r three variables assigning to anything with n – r variables to any constant value, one can have solution. See this one for homogeneous system. One you can also generalize this concept for the nonhomogeneous system also.

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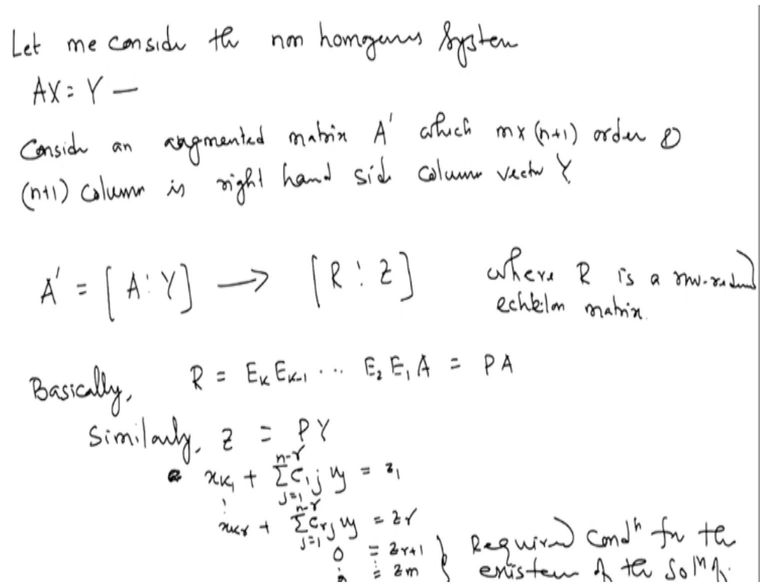

So, in that case I will have, so for that let me consider a nonhomogeneous system say AX=Y. Consider in an augmented matrix and A' which is mx(n+1) order and the n+1 column is right hand side column vector Y. And then one has to do the elementary row operations. So, one will get from this you know $A'=[A : Y] \rightarrow [R : Z]$, this is my this matrix.

So, this matrix when you translate through the elementary row operation, then multiplying each time elementary matrices I will have suppose through the sequence of process I will have R on Z type. So, I will have this type of matrix, so where R is a row-reduced echelon matrix. So basically, R is obtained as multiplying a finite number of elementary matrices from the left side to the A. So, it is like this.

So, R is equal to something like say we have multiplied suppose says small k number of elementary matrices, so $R = E_k \ E_{k-1} \ \ldots E_2 \ \ E_1 \ A \ \ = P \ A$ Since each of them are invertible, so I can say this is equal to something P into A. So, row-reduced echelon form R is basically obtained multiplying invertible P matrix to the left hand side of the A. Similarly, the Z=PY.

Now, if R has exactly r number of nonzero rows, then it had m – r zero rows. So, in that case I will have $x_{k_1}+\sum_{j=1}^{n-r} c_{1j}\, u_j = z_1$ and $x_{k_r}+\sum_{j=1}^{n-r} c_{rj}\, u_j = z_r$. Then I will also have 0 $=z_{r+1}$and 0=z_m, the required condition for the existence of the solution of original system AX=Y.

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original System AX=Y

$$x_1 + 2x_2 = 1 \qquad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$x_1 + 2x_2 = 4$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

$$x_1 + 2x_2 = 1$$

$$0 = 3 \quad \times$$

See if you look these things in a bit geometrical way, so what does it mean that z_{r+1}= 0 to z_m= 0? This will basically introduce a parallel plane if I do not consider z_{r+1}= 0. Any of these nonzero that will introduce a plane this will be so that in our system AX=Y which basically consists of m plane that there are some plane which are parallel. See for example if I take say x_1+2x_2=1 and if I consider again say x_1+2x_2=4, then if I do the elementary row operations, I am basically writing this is 1 2 1 2 x_1 x_2= 1 4.

So this augment the process 1 2 0 0 1 and here I will have basically 3. So, I will have this type of system. So, this means that I will have system x_1+2x_2=1and 0 = 3 which is meaningless. So, I have to make that our z_{r+1}= 0 to z_m= 0. So, that is the basic criteria for existence of the solution for the system of equation AX=Y. So, let us quickly summarize what we have learnt.

So, we have seen that concept row-reduced matrix helps to find the solution of the system, to talk about whether the system has solution or not, but more proper way one has to introduce the row-reduced echelon form of the given matrix space and through that one can say whether system AX=Y has solution or not. If it has solution, then by assigning the value of the free variables one can have the different solutions of the system AX=Y. Thank you.