Advanced Linear Algebra
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Lecture – 29
Invariant subspaces - I

So welcome to lecture series Advance Linear Algebra. We have already seen some linear operator are diagonalizable but most of them are not diagonalizable. I mean one can find out in order basis corresponding space under which the operator can be repaired by a diagonal matrix or the matrix is similar to a diagonal matrix at this place but it is not true for the most of the cases. So, what the next simplified form we are looking for that one please.

So, for that let me introduce one terminology that is called invariance of space. So, what is that what is invariant subspace?

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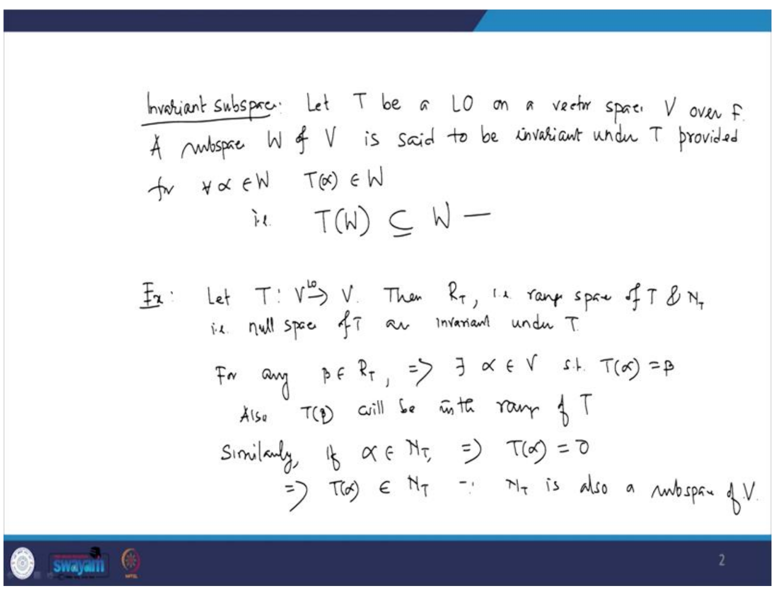

Let me define it in variant subspace let T be a linear operator(L.O.) on a vector space V over the field F a subspace W of V is said to be invariant under T provided for $\forall\ \alpha \in W$, $T(\alpha) \in W$, i.e. $T(W) \subseteq W$, then only I can say that this half space W of the vector space V is invariant under T, for example let $T: V \to V$ that I am denoting as do not need to be then R_T that is range space of

T and N_T that is null space of T are invariant under T, how? For any $\beta \in R_T$ implies that $\exists$ α $\in$ V such that T(α) = β and also T(β) will be in the range ofm T, because it will be either it will β = 0 then also beta will be in the range space because range space is subspace. So, it must include 0 if T(β) $\neq$ 0 then by default T(β) will be in the R_T. Similarly if α $N_T \Longrightarrow$ T(α) = 0.

So, $\Longrightarrow$ T(α) $\in$ N_T beacuse N_T is also a subspace of V. So, by default we see that drain space as well as null space of the operator are invariant under T.

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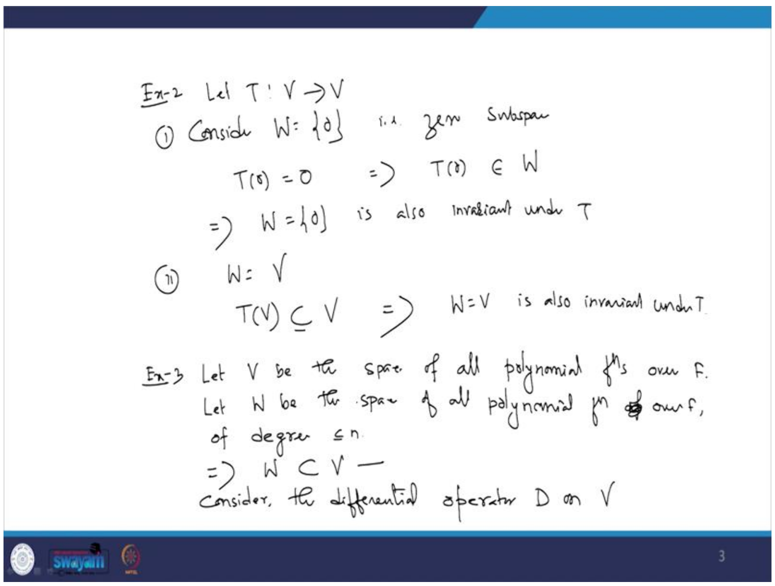

Let me take other examples again consider the same let T: V → V (i) consider W = {0}, i.e. is zero subspace what can I say about this zero subspace whether will it be invariant under T or not certainly it will be, yes. Then we see T(0) = 0 there is a property of the linear operator. So, this implies T(0) ∈W. So, this implies W = {0} is also invariant under T.

Now (ii) if I consider W = V, what can you say about this ones then also we have, T(V) ⊆ V. So, this implies that W = V is also invariant under T. So, there they are all trivial examples now let me consider some examples which is not trivial please. So, this is say example-2, this is example-3 let V be the space of all polynomial function functions over the field F.

So, we know V is a vector space let W be the space of all polynomial function over F of degree

$\leq$ n. So, this implies that W ⊂ V. Now consider the differential operator D on V.

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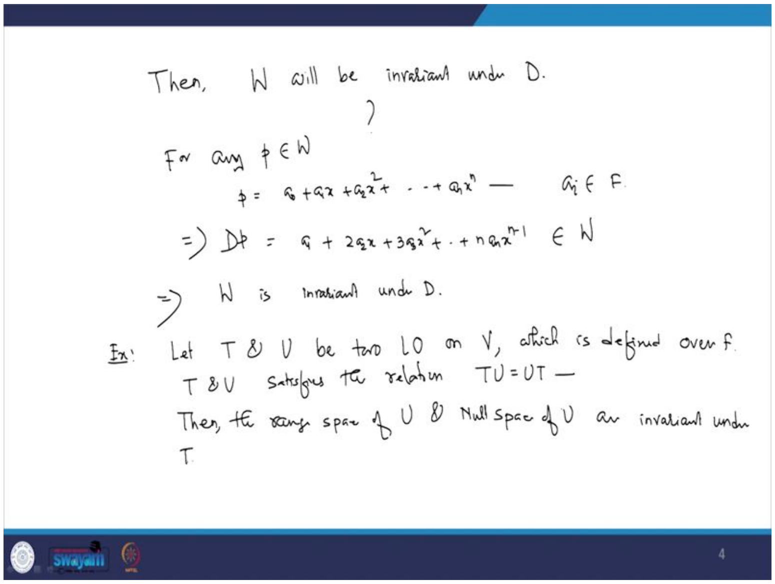

Then W will be invariant under D, how? For any for any p $\in$ W, p = $a_o + a_1$x + a_2x$^2 + \cdots + a_n$x^n where a_i $\in$ F because the polynomial function of degree $\leq$ n. So, p will be of this form. So, this implies when the differential operator Dp = $a_1 + 2a_2$x + $3a_3$x$^2 + \cdots + na_n$x^{n-1} $\in$ W.

So, this implies that W invariant under D. So, let me consider another but interesting results and interesting examples like this. Let T and U be two linear operator linear operator on vector space V which is defined over the field F, I am not talking about the dimension of the vector space V B please here it may finite infinite value like that the operator T and U such that T and U satisfies the relation that is our property you can say like that TU = UT.

I mean to say the operator T and U that commutative then the range space of U and null space of U are invariant under T how.

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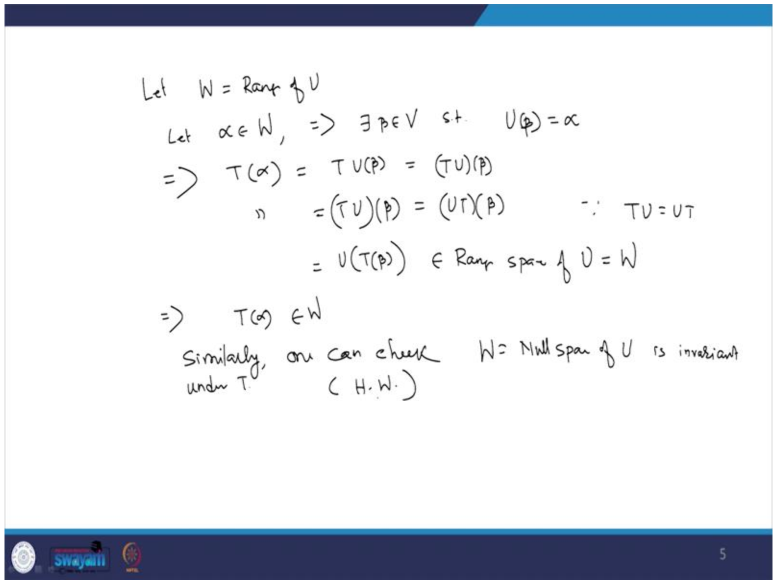

So, let us quickly check how they range space of U will be invariant under T, let W = Range of U. So, let $\alpha \in$ W $\Longrightarrow \exists\ \beta \in$ V s.t. U(β) = α $\Longrightarrow$ T(α) = T U(β) = $(TU)(\beta)$ = (UT)(β) = U(T(β)) $\in$ Range space of U = W, since TU = UT $\Longrightarrow$ T(α) $\in$ W.

We see that range space of U is invariant under the operator table is similarly one can check that W = Null space of U is invariant under T. This I am giving as a homework. So, we have seen couple of examples of invariants subspaces.

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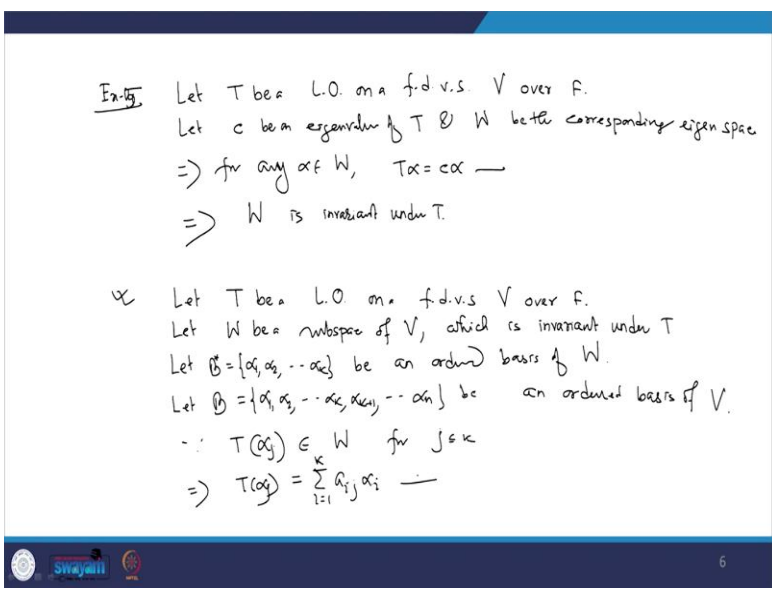

Let me give another very standard example which I shall use it frequently in the coming lectures. So, therefore let me introduce that one also let T be a linear operator on a finite dimensional Vector space V over the field R. let c be a and eigen value of T and W be the corresponding eigen space. So, this implies that for any $\alpha \in$ W , Tα = cα.

So, $\Longrightarrow$ W is invariant under T. So, eigenspace of a linear operator is always invariance of space of that operator piece. Now let us see what way this concept of invariance will helpful to understand the characteristic polynomials, minimal polynomial of the operator piece. So, let T be a linear operator on a finite dimensional vector space V over the field F. Let w be a subspace of V which is invariant under the operator T.

Let $B^* = \{\alpha_1, \alpha_2, \dots \alpha_k\}$ be an order basis of W. So, I can extend B^* by adding some more linear independent element of V to be a basis for the entire space V. So, let B $=\{\alpha_1, \alpha_2, \dots \alpha_k, \alpha_{k+1}, \dots . \alpha_n\}$ be an order basis order basis of V. Since $T(\alpha_j) \in W$ for $j \leq k$. Now when I represent, so, implies for $T(\alpha_j) = \sum_{i=1}^{k} a_{ij}\, \alpha_i$ can be represents by the order basis element $\{\alpha_1, \alpha_2, \dots \alpha_k\}$.

and W is the invariant under T. So, that is why $T(\alpha_i)$ for $i \leq k$, I will be able to express with the basis element of W. So, $T(\alpha_j)$ will be basically linear combination of $\{\alpha_1, \alpha_2, \dots \alpha_k\}$. So, I need only k scalars.

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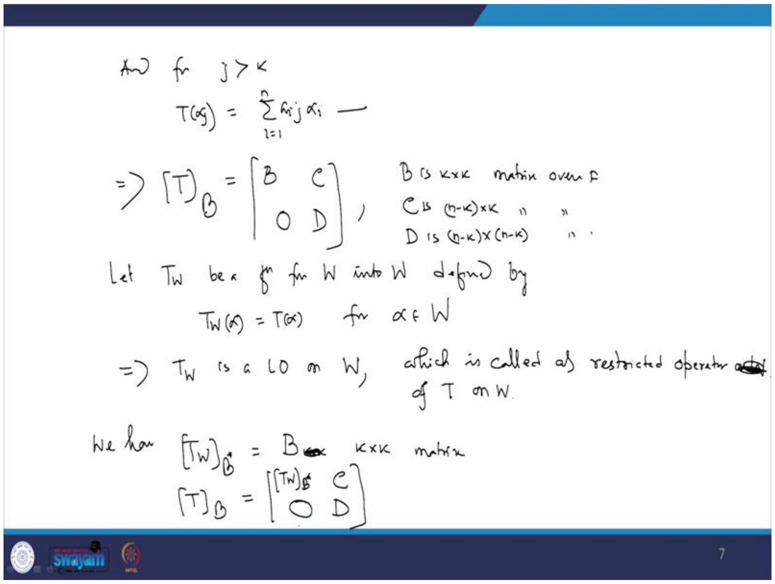

And for $j > k$, $T(\alpha_j) = \sum_{i=1}^{n} a_{ij}\, \alpha_i$. So, here I need to express α_j by the basis element of the vector space V. So, I need n constant. So, this implies the matrix representation of T with respect to this order basis B, $[T]_B = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, where B is kxk matrix o ver F, C is (n-k)xk matrix over F and your D is will be (n-k)x(n-k) Square matrix about this one piece.

So, the matrix representative likes this thing. So, if I simplify it let me introduce one more operators Define like this let T_W I mean I want to induce a linear operator on W that is T_W be the be a function from your w into W defined by $T_W(\alpha) = T(\alpha)$ for $\alpha \in W$. So, this implies that T_W is a linear operator on w which is called as restricted operator restricted operator on W I mean basically restricted operator of T on W.

So, T_W is nothing it is a basically W induce a linear operator which is a restriction of T on W that is restricted operator of T on W that is T_W. So, we have, $[T_W]_{B^*} =$ B, so one can write down, $A = [T_W]_B = \begin{bmatrix} [T_W]_{B^*} & C \\ 0 & D \end{bmatrix}$. So, it is like this thing please.

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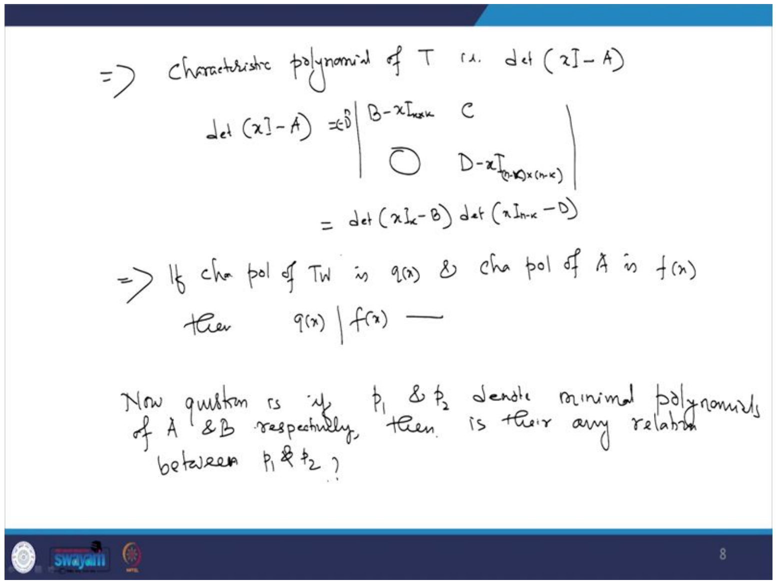

This implies the characteristic polynomial characteristic polynomial of T, i.e. det(xI-A) = $(-1)^n$ $\begin{vmatrix} B - xI_{kxk} & C \\ 0 & D - xI_{(n-k)x(n-k)} \end{vmatrix}$ = det(xI_k-B) det(x$I_{(n-k)}$-D). So, this implies if characteristic polynomial of T_W is q(x) and characteristic polynomial of A is say f(x) then $\frac{q(x)}{f(x)}$. So, characteristic polynomial of restricted operator T_W divides characteristic polynomial of A. Now what can you say about the minimal polynomial of T_W and minimum polynomial of the A is there any relation between these two minimum polynomials. Now, to answer this question again so, let us see what is different power of A.

Now question is if p_1 and p_2 denote minimal polynomial of A and B respectively then is there any relation between p_1 and p_2 another question if p_1 and p_2 denote minimal polynomials of A and B respectively then is there any relation between p_1 and p_2.

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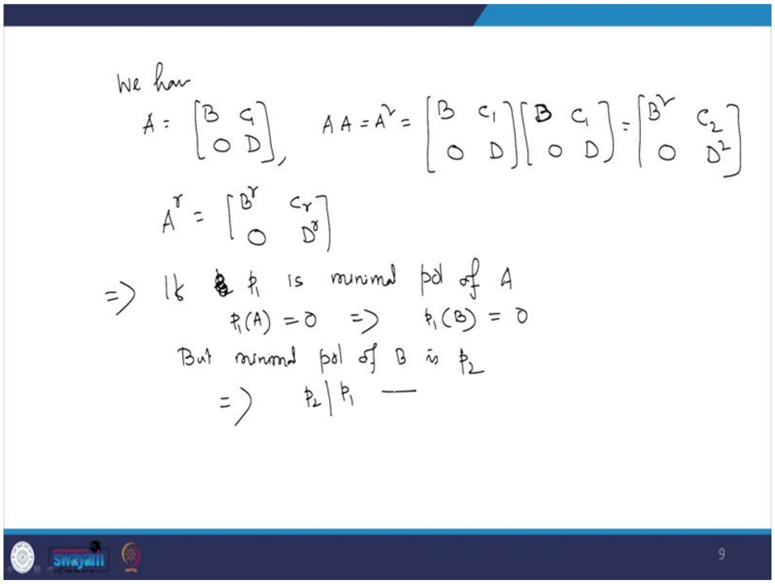

See for this we have, A = $\begin{bmatrix} B & C_1 \\ 0 & D \end{bmatrix}$, AA = A^2 = $\begin{bmatrix} B & C_1 \\ 0 & D \end{bmatrix}\begin{bmatrix} B & C_1 \\ 0 & D \end{bmatrix} = \begin{bmatrix} B^2 & C_2 \\ 0 & D^2 \end{bmatrix}$.

So, in general you will have, $A^r = \begin{bmatrix} B^r & C_r \\ 0 & D^r \end{bmatrix}$, this implies if p_1 is minimal polynomial of A. So, $p_1(A) = 0 \Longrightarrow p_1(B) = 0$ but minimal polynomial of B is given p_2.

So, that is the least degree polynomic polynomials which annihilate B. So, this implies $\frac{p_2}{p_1}$, that is minimal polynomial of B must divide minimum polynomial of original matrix A. So, we have seen that if W be a invariant of space of a vector space V then it simplifies the matrix representations. It also helps to understand the characteristic polynomial minimum polynomials. And you see it has many more applications to simplify the operator.

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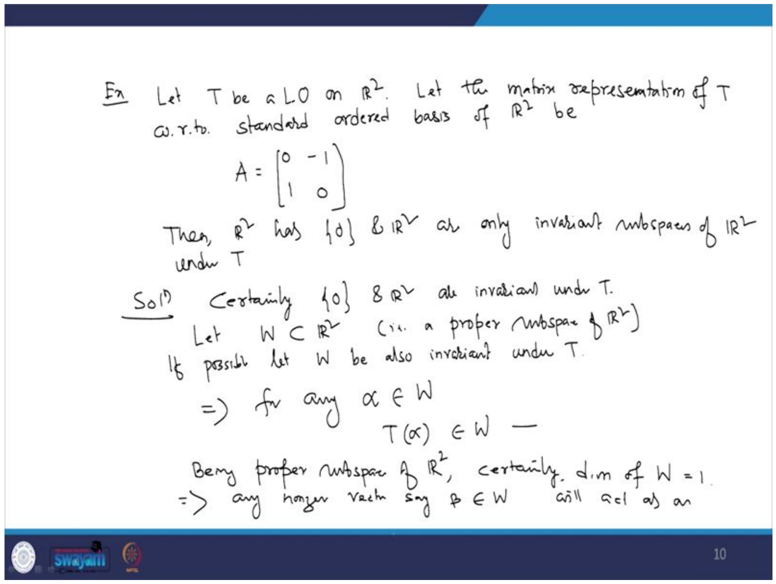

Now let me consider examples. Let T be a linear operator on R^2. Let the matrix representation of T with respect to standard order basis of R^2 be $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then R^2 has $\{0\}$ and R^2 are only invariant subspaces of R^2 under the operator T. here I have considered T V linear pattern R^2, certainly R^2 over R^i mean field is the real number. I have to show that for this case the only subspaces which are T invariant or invariant under T are the $\{0\}$ and R^2 are invariant under T.

Let W is a proper subspace of R^2, if possible. Let W be also invariant under the operator T. So, this implies for any $\alpha \in W$, $\mathrm{T}(\alpha) \in \mathrm{W}$, again being proper subspace of R^2 certainly W has to be a one dimensions of space only dimension of W = 1. So, this implies any non-zero vector say $\beta \in$ W, will act as a basis of W.

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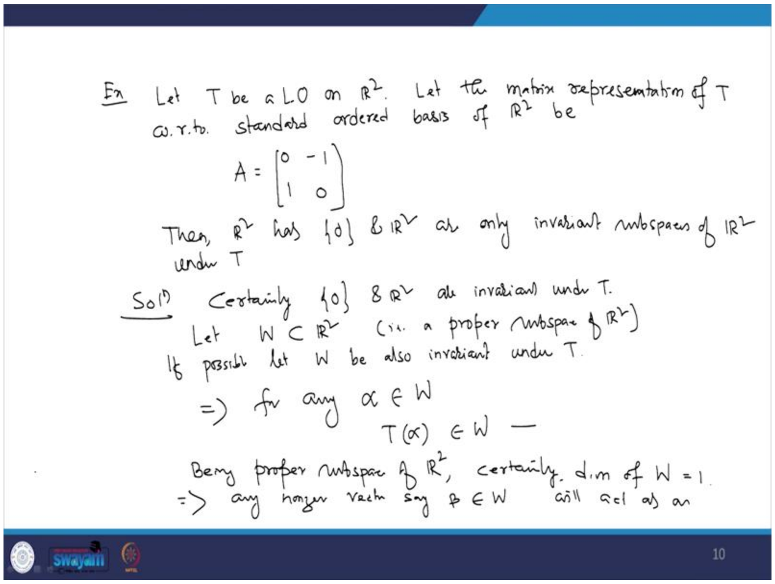

So, this implies W $= LS\{\beta\}$. So, this implies if I consider $\alpha \in W$, $\alpha = c\beta$ & $T(\alpha) \in W$ $\Longrightarrow$ $T(\alpha) = d\beta = \frac{d}{c}\,\alpha = t\,\alpha$, where we let $\alpha \neq 0$, and $t \in F$. So, this implies t is an eigenvalue of the operator T but T does not have any eigenvalue.

Because the since characteristic polynomial that is $x^2 + 1$ is not factorizable over the field R if W is invariant under T then W cannot be a proper subspace of V, but if you change the suppose if I consider the operator T instead of defining over R^2 if I consider C^2 then this correspond matrix is remained same but it will be then I will see that there will be proper subspace which will be also invariant under T.

Because in that case you will have eigenvalues as $\pm i$ and we will get the corresponding eigen space which will be T invariant. So, we will continue to use this concept of invariant of space for simplification of the operators. So, that will be done in the next class.