Advanced Linear Algebra
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Lecture – 28
Cayley-Hamilton Theorem and Its Applications - II

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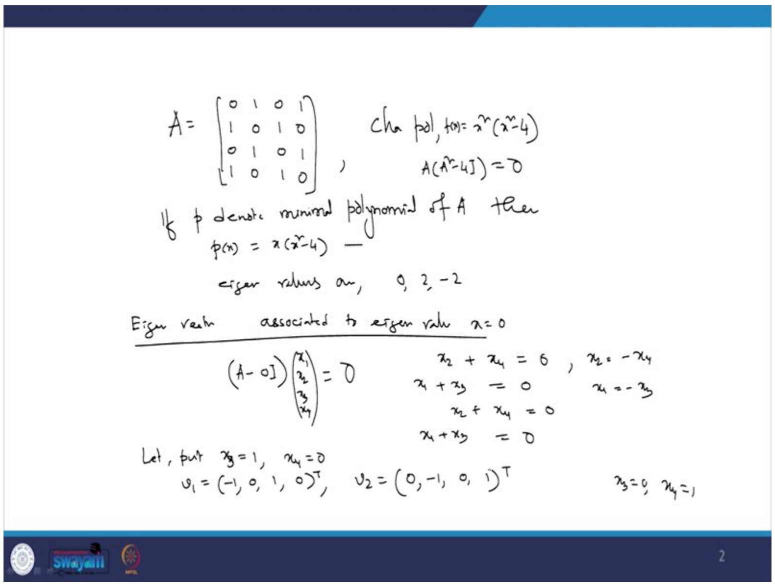

Welcome to lecture series in Advance Linear Algebra. In last lecture I took an example towards the end of my lectures considering Matrix representation of a linear operator which is defined over the 4x1 mathematics words of the form of recall this is something like $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. We have seen that the characteristic polynomials polynomial, $f(x) = x^2(x^2 - 4)$.

And we also obtained that $A^2(A^2 - 4I) = 0$, if P denote minimal polynomial polynomial of A then $p(x) = x(x^2 - 4)$. Already we have seen in the case of when an operator or Matrix is diagnosable there is minimal polynomial is basically product upon its linear factors. So, eigen values are here those are 0, 2 and -2. So, according to the theorem so, that $p(x) = (x – 0) (x – 2) (x + 2)$.

So, that one also we have seen it here fine. Now if I see the let us quickly calculate the eigen vectors related to eigenvalue 0 and 2 and -2. So, eigenvector associated to eigen value x = 0, can calculate it that is like this. So, it will be you know one has to calculate. So, (A-0I) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ = 0. So, I will have 4 equations that is I will have basically, $x_2 + x_4 = 0$, $x_1 + x_3 = 0$, $x_2 + x_4 = 0$, $x_1 + x_3 = 0$ $\Longrightarrow$ $x_2 = -x_4$ & $x_1 = -x_3$.

So, there are two free variables here, x_3, x_4. Let put $x_3 = 1, x_4 = 0$. So, then I will have say $v_1 = (x_1, x_2, x_3, x_4)^T = (-1, 0, 1, 0)^T$ and let $x_3 = 0, x_4 = 1$ then $v_2 = (x_1, x_2, x_3, x_4)^T = (0, -1, 0, 1)^T$.

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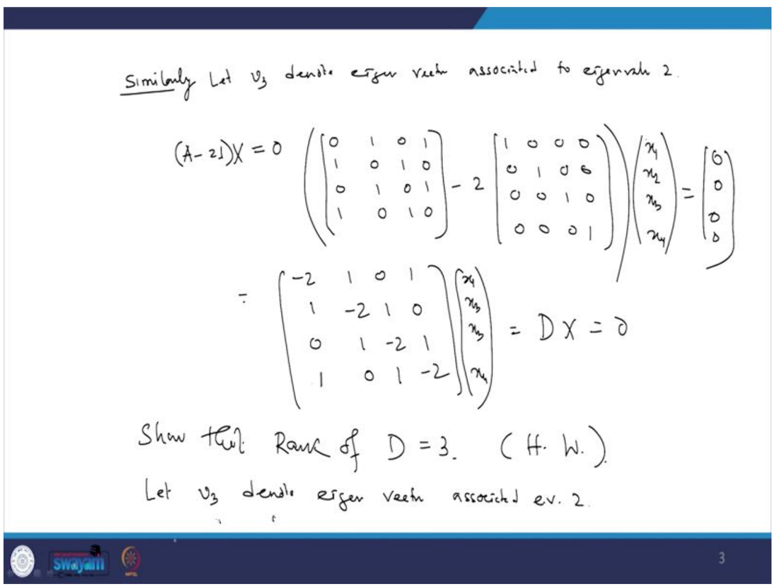

Similarly, let v_3 denote eigen vecto r associated to eigen value 2 then (A-2I) = 0 $\Longrightarrow$ $\left\{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$ DX = 0 so, that rank of D =3. These are the homework please you can consider this one and so, that I will have only one free variables. So, one can have only one linearly independent eigenvector let v_3 denote eigenvector associated to eigenvalue 2.. So, already written okay and similarly one can calculate v_4 denote the eigen vector associated eigen values say -2 then I will have what.

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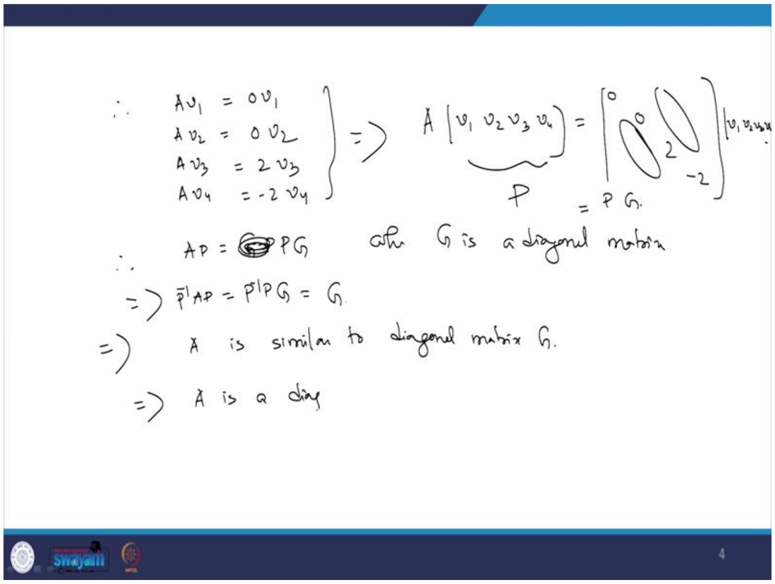

So, I will have $Av_1 = 0v_1,\ Av_2 = 0v_2,\ Av_3 = 2v_3,\ Av_4 = -2v_4$. So, this implies that this implies what I can write down say $A[v_1, v_2, v_3, v_4] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} [v_1, v_2, v_3, v_4]$, and let P = $[v_1, v_2, v_3, v_4]$, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ = G. So, AP = PG, where G is a diagonal matrix. So, this implies and this P is invertible because each of them are linearly independent column. So, I will have multiplying P^{-1} from the left side , I will have $P^{-1}AP = P^{-1}PG = G$.

So, $\Rightarrow$ A is similar to diagonal matrix G and $\Rightarrow$ A is diagonalizable matrix.

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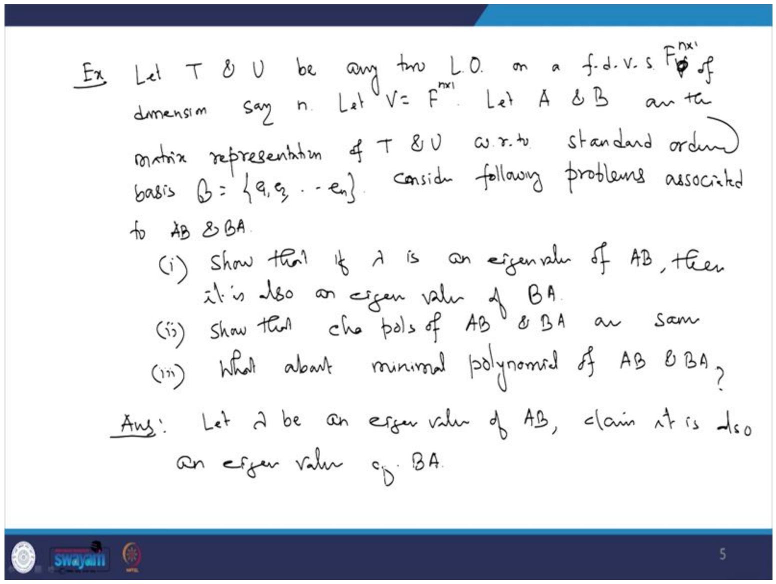

Let me take another example let T and U be any two linear operators on a finite dimensional Vector space say V of dimension say small n. Let V = F^{nx1} and so, that I can give immediately you know matrices representation also and next with respect to standard order basis on F^{nx1}.

Let A and B are the Matrix representation of T and U with respect to standard order basis B = $\{e_1, e_2, \ldots e_n\}$. Consider following problems associated to AB and BA. (i) Show that if λ is an eigenvalue of AB then it is also an eigen value of BA. (ii) Eigen values are same it does not mean that characteristic polynomial will be exactly same.

So, that is why let me write down again. So, that characteristic polynomial of AB and BA are same. Of course second implies one but one does not imply second place. (iii) What about minimal polynomial of AB and BA?. Let me answer one by one please. Let λ be a eigen value of AB. Claim it is also an eigen value of BA, how?

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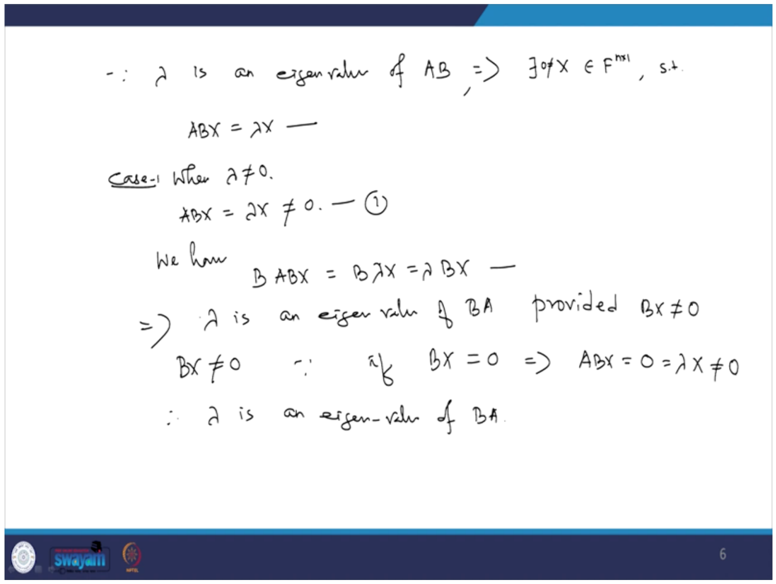

Since λ is an eigen value of AB, $\Longrightarrow$ $\exists$ $0 \neq X \in F^{nx1}$, s.t. ABX = λX, there are two possibilities $\lambda = 0$ or $\lambda \neq 0$. Case(i) when $\lambda \neq 0$ means ABX = λX $\neq 0$, we have BABX = BλX = λBX because λ is scalar quantity I can take it out, so I can write like this in place. So, this implies λ is an eigen value of BA $\neq 0$.

Since if BX = 0 $\Longrightarrow$ ABX = 0 = λX $\neq 0$, which contradicts that $\lambda \neq 0$. So, BX $\neq 0$. So, λ is an eigen value of BA. So, if λ is the AB then also eigen value of BA.

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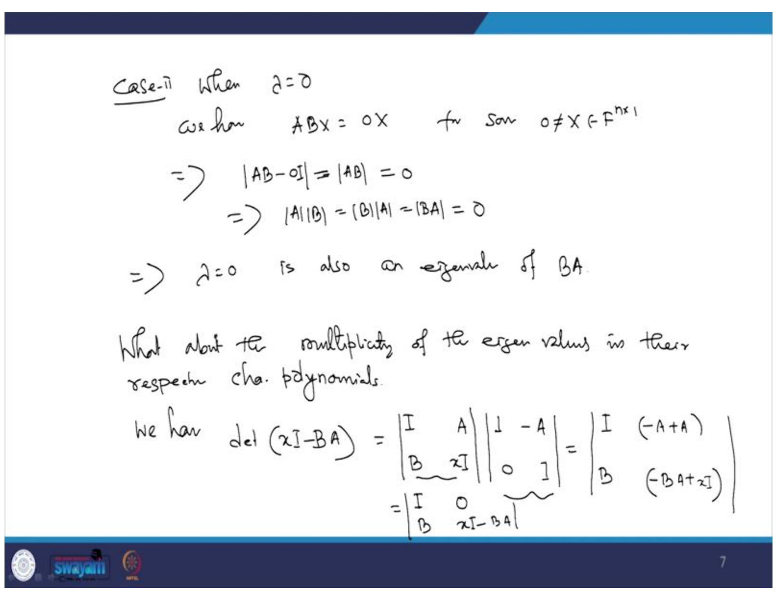

Now case (ii) when $\lambda = 0$, we have ABX = 0X for some $0 \neq X \in F^{nx1}$. So, $\Rightarrow$ |AB-0I| = |AB| = 0 $\Rightarrow$ |A| |B| = |BA| = 0. So, $\Rightarrow$ $\lambda = 0$ is also an eigenvalue of BA. So, we have seen that if λ is a eigenvalue of the Matrix AB or operator TU then it is also eigenvalue of UT or eigenvalue of BA please. So, what about the multiplicity of the eigenvalues in the in their respective characteristic polynomial if f_1 and f_2 are the characteristic polynomial AB and BA then whether f_1 will be equal to f_2 or not that is the question.

Basically I mean saying that what about the multiplicity of the eigenvalues in their respective polynomials means restrict characteristic dependent means whether characteristic polynomial of AB will be exactly equal to characteristic polynomial of BA or not that we have to check it please. So, this one we can do using the concept or again determinant place we have determinant of you know det(xI-BA) = $\begin{vmatrix} I & A \\ B & xI \end{vmatrix} \begin{vmatrix} I & -A \\ 0 & I \end{vmatrix} = \begin{vmatrix} I & (-A+A) \\ B & (BA+xI) \end{vmatrix} = \begin{vmatrix} I & 0 \\ B & xI - BA \end{vmatrix}$.

So, I am getting the same thing please this one, det(xI-BA) = $\begin{vmatrix} I & A \\ B & xI \end{vmatrix} \begin{vmatrix} I & 0 \\ (-\mathrm{x}^{-1}B) & I \end{vmatrix}$ = $\begin{vmatrix} I - \mathrm{x}^{-1}AB & A \\ B - B & xI \end{vmatrix}$ = $\begin{vmatrix} I - \mathrm{x}^{-1}AB & A \\ 0 & xI \end{vmatrix} = |xI - AB| = \det(xI - AB)$.

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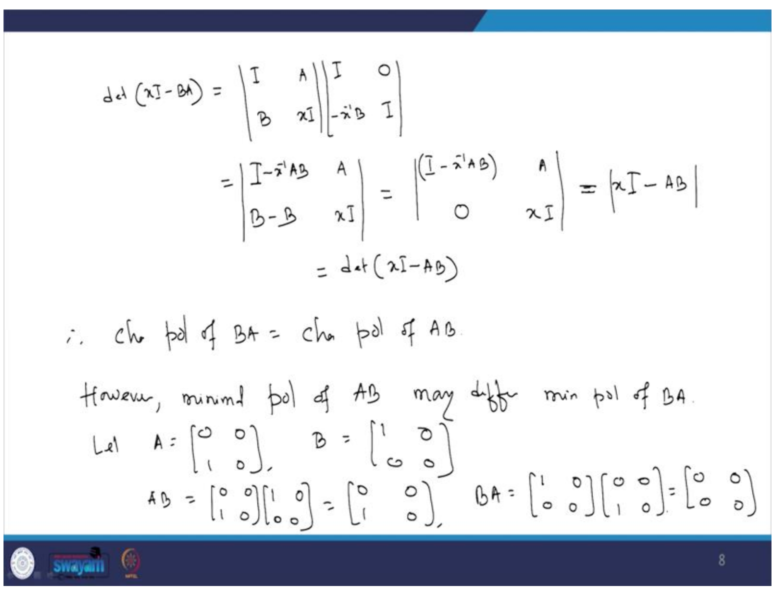

So, characteristic polynomial of BA equal to characteristic polynomial of AB. So, I am getting both of them having same characteristic polynomials using this simple Algebra I can prove it.

Please but what about the minimal polynomials. So, minimal polynomial case it is not true we can sight one example. However minimum polynomial of AB may differ minimal polynomial of BA. So, let me take these examples that when $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So, certainly the characteristic a minimum polynomial of AB and BA value difference because B is zero matrix. So, x = 0 is the minimum polynomial for BA but for AB it is not true.

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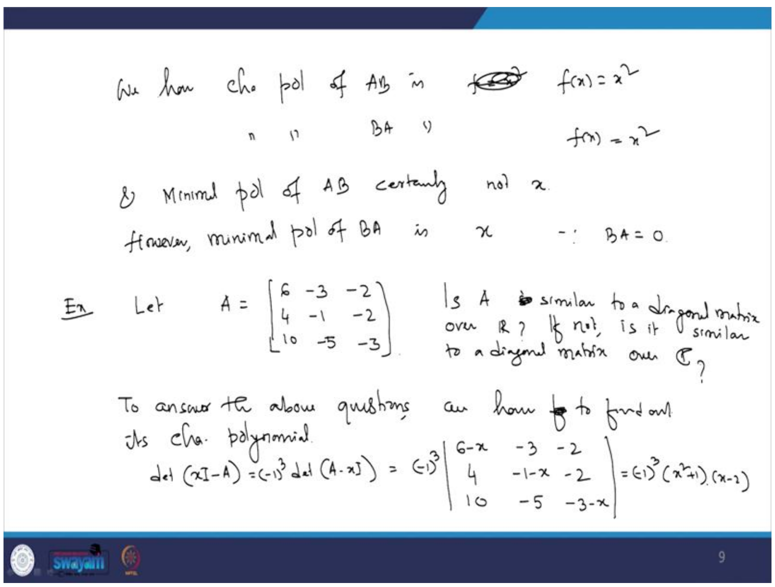

We have characteristic polynomial of AB is $f(x) = x^2$, we have also characteristic polynomial of BA is $f(x) = x^2$ that is true and minimal polynomial of AB certainly not x, because means AB has zero matrix but AB is not a zero matrix however minimal polynomial of BA is x, since BA = 0.

So, we see that here is even though both AB and BA have same characteristic polynomials I mean same eigenvalues but the minimal polynomial for the both the matrixes are different place. Let me consider one more nice problem please. So, let me consider a matrix $A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$ or equivalent to saying that consider a linear operator T define on F^{3x1} whose matrix representation is the given matrix A is a similar to a diagonal matrix over R.

If not is it similar to a diagonal matrix over the complex number C. So, to answer this question. So, first of all we have to find out what is the characteristic polynomials of this Matrix. Above questions we have to first we have to find out its characteristic polynomial. So, characteristic polynomial means $\det(xI\text{-}A) = (-1)^3\det(A\text{-}\ xI) = (-1)^3\begin{vmatrix} 6-x & -3 & -2 \\ 4 & -1-x & -2 \\ 10 & -5 & -3-x \end{vmatrix} = (-1)^3(x^2+1)(x-2)$.

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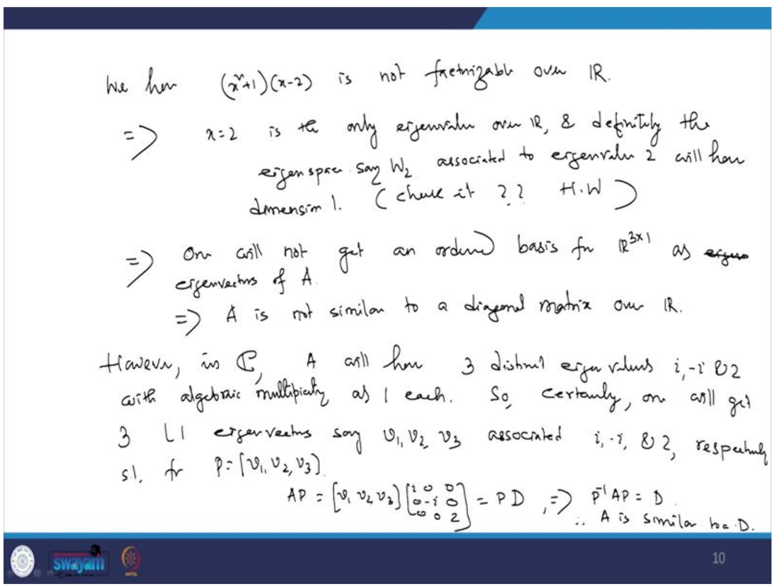

We have $(x^2+1)(x-2)$ is not factorizable is able over R. So, this implies your since we are not able to factorize over the (R) , therefore only we are getting x = 2. So, this implies x = 2 is the only the only eigenvalue over R and definitely eigenspace say W_2 definitely the eigen space say W_2 associated to eigen value to will have dimension 1. But check it please as a your homework.

So, you will see that for x = 2 the dimension of the eigen space associated 2 that will be 1. So, this implies that I will not have a basis where I do not have an order basis for this space your R^{3x1}, I will not get an order basis for are 3x1 as eigenvector eigenvectors of given matrix A. So, this implies A is not similar to a diagonal matrix over R. The dimension of the W_2 here I have asked you that check it is a basically one you can immediately use the results.

The geometric multiplicity of an eigen value will never exceed the algebraic multiplicity of the corresponding eigen value please. Because some of the algebraic multiplicity of the different eigen values of the corresponding matrix or operator will be basically the dimension of the corresponding space. So, here dimension of the space is here 3 here I am getting basically x = 2 and other analysis x = $\pm i$.

So, each of them will have eigenvectors and but that will be over the complex plane. So, but over the real and only one so therefore it is not similar to a this one. However in complex plane in C a will have 3 distinct eigen values i, -i, 2. and with algebraic multiplicity and one each and we know if diagonal values are different then the corresponding eigen spaces are linearly independent. So, certainly one will get 3 linearly independent eigen vectors say v_1, v_2, v_3 associated to i, -i, and 2 respectively.

Such that for P $= [v_1, v_2, v_3]$, AP = $[v_1, v_2, v_3] \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 2 \end{bmatrix}$ = PD P^{-1}AP = D, so this implies A is similar to a diagonal matrix D. In this way you can solve many other problems also to understand this concept which is will be given in the assignment set please.

Hope you have understood the concept of eigen values and eigen vector minimal polynomial algebra multiplicity geometric multiplicity and relation between the geometric multiplicity and algebra multiplicities basically you will use these results for our next lecture also. So, you should understand very clearly this one please, thank you.