Advanced Linear Algebra
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Lecture – 27
Cayley-Hamilton Theorem and Its Applications - I

We have come to vector series in Advanced Linear Algebra. We have discussed what is eigenvalue eigenvector definition of the operator, we also discussed the meaning of annihilating polynomial of a finite of a linear operator define over finite N dimension vector space or annihilating polynomial of for a nxn matrix defined over the field. We have also defined what is the meaning of minimal polynomials, characteristic polynomials?

We have seen that least degree Monic generator of the ideals of the all the polynomials which annihilates an operator T that is basically minimal polynomials for the operator T. Apart from the least degree polynomials there exist a polynomial this is called the characteristic polynomial evolves to define what is that. Cayley has given basically Cayley as well as Hamilton both of them has given one nice results.

That every linear operator define on a final dimensional vector space is a root of the its characteristic polynomials. So, that is basically Cayley Hamilton theorems. So, let me state the what are the statement of the theorems.

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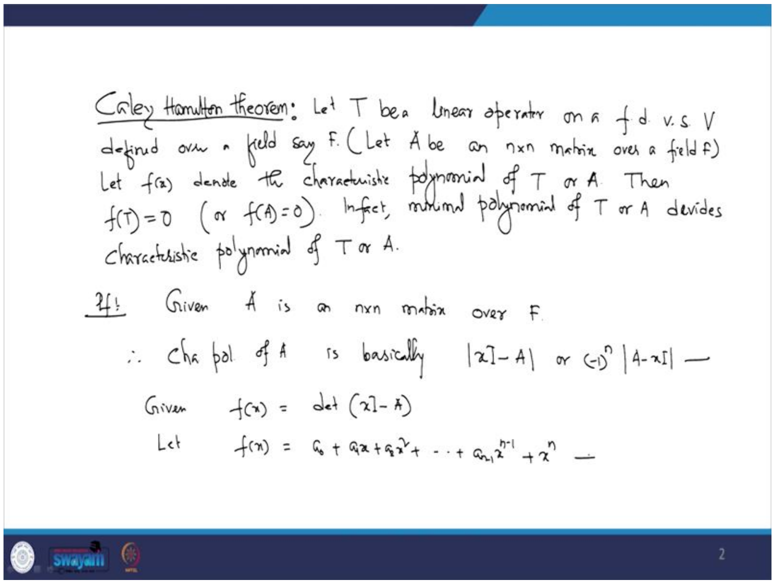

Let T be a linear operator on a finite dimensional vector space(f.d.v.s.) V which define over a field say F(let A be an nxn matrix over a field F). Let f(x) denote the characteristic polynomial of T or A then f(T) = 0 (or f(A) = 0). In fact it says that minimal polynomial of T or A divides characteristic polynomial of T or A.

So, statement of the theorem is like this say T be a linear operator defined over a finite dimensional Vector space V. Say for example the dimension of the V is say n and let A be a square matrix over the field F for if I consider V the final dimension of dimension n here I will consider a is a nxn matrix. Let F denote the characteristic polynomial of T or A then f(T) or f(A) = 0, I mean or if you concern instead of T matrix a then f(A) = 0.

So, this statement says that if the minimal polynomial of T or A is supposing p then certainly p **(()) (05:26)** divides f because according definition of the minimal polynomial and characteristic polynomials minimal polynomial the least degree monic generator of the ideals of all the polynomials from the f(x) and then field of polynomials facts which will basically any let the operator T or matrix A.

So, since f is also annihilating then definitely f will be multiple of p. So, that is why I am saying that in fact the minimal polynomial divides characteristic polynomial of the T or A. There are different ways to one can prove this result please. I have used the word that T be a linear operator on a final dimensional vector should be or have taken A is a square matrix over the field say F. Already I have established results remember any finite dimensional vector space is isomorphic to corresponding say n dimensional array space that is n double space if I consider as V is the final dimensional dimension n.

Then V isomorphic to F^n or F of V isomorphic to also F^{nx1} if V is the finite dimensional of Dimension n. Now since T is a linear operator defined over finite dimensional vector space V, I can consider standard order basis on V then based on if I consider V is basically say for instance this is I can say the see V the basically say F^n then we can have a matrix representation of the T with respect to standard order basis on the space.

And that if i say that is basically my A then talking characteristic of the operator T talking characteristic matrix a both are same place. So, instead of operator I will talk basically prove for the matrix Square matrix A. So, given A is an nxn matrix over the field F according to definition of the characteristic polynomials. So, I can say characteristic polynomial of A is basically |xI-A| or $(-1)^n$|A-xI| both are same place.

Because from the definition of the determinant of the matrix these 2 are same statement please given it is also given to us f(x) = det(xI-A). So, let since the characteristic polynomial for the matrix A where a is nxn matrix will be polynomial in x of degree at the most n. So, let f(x) = $a_0+a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n$, the power n because it is monic polynomials.

So, the coefficient of highest power must be equal to one. So, that is why f(x) equal to of this structure base if you recall the definition or some property of the adjunct matrix.

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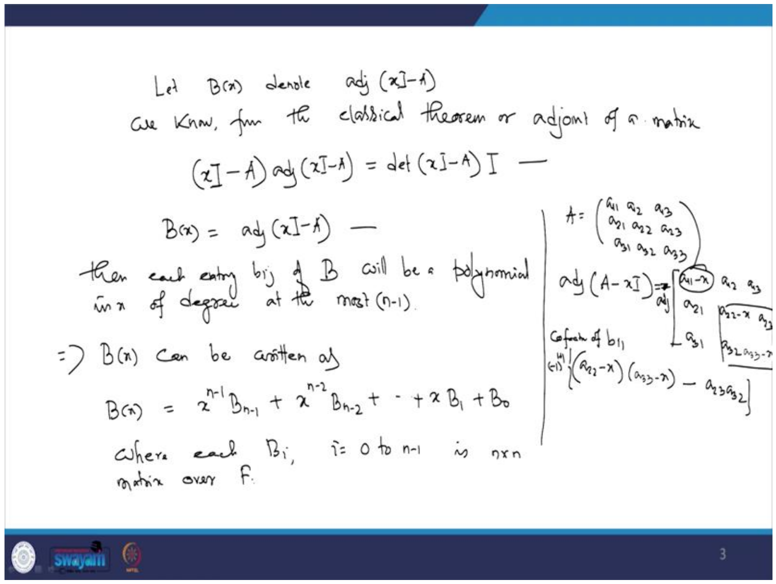

Let B(x) denote adj(xI-A) then we know from the classical theory or adjoint of a matrix, adj(xI-A)adj(xI-A) = det(xI-A) this this is the classical results we know. So, I have considered here B(x) = adj(xI-A) but what is this. So, to have a quick idea about this let me also parallel it take some examples to let a equal to let me concern here, A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. So, based on that if I proceed it see here I will have basically what I will have you know I can write simply determinants of ok original matrix. So, let me write this one please. So, adj(A-xI) = adj$\begin{bmatrix} a_{11}-x & a_{12} & a_{13} \\ a_{21} & a_{22}-x & a_{23} \\ a_{31} & a_{32} & a_{33}-x \end{bmatrix}$, cofactor of b_{11} = $(-1)^{1+1}\{(a_{22}-x)\,(a_{33}-x) - a_{23}a_{32}\}$. So, anyhow I am getting this is a polynomial in x of degree 2. Similarly if I go for the a_{12} I will have polynomial degree in that case I will have polynomial degree only x other cases also x.

So, I will have the co factors of each b_{ij} will be basically polynomial of degree at the most increase then each entry b_{ij} of B matrix will be a polynomial in x of degree at the most at the most (n -1) place because when I consider a is equal to 3x3 matrix I am getting the maximum degree of this polynomial is 2 only. So, one can rewrite. So, this implies B(x) can be written as B(x) = $x^{n-1}B_{n-1} + x^{n-2}B_{n-2} + \cdots + xB_1 + B_0$. Where each B_i, i = 0 to (n-1) is nxn matrix over the field F. So, I see that the adjoint of the (xI –A) is basically nxn matrix written like this p(x). So, if I say this is star.

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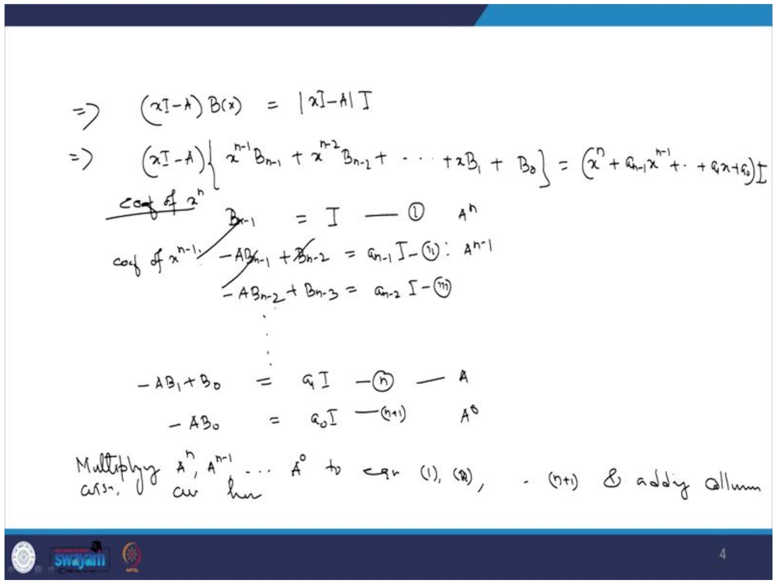

$\Longrightarrow$ (xI-A)B(x) = |xI-A|I. So, this implies I will have, (xI-A){ $x^{n-1}B_{n-1} + x^{n-2}B_{n-2} + \cdots + xB_1 + B_0$} = ($x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$)I. So, that is why we tell this one now find out the coefficient of different power of x. So, let us first find out the coefficient of x^n what is this? So, on the both side if I have and compare then I will have see this side is I will have only $B_n = I$. So, this will be equal to certainly you know identity matrix because the coefficient of x^n, $B_{n-1} = I$.

So, the first equation like this thing is now coefficient of x^{n-1}, $-AB_{n-1} + B_{n-2} = a_{n-1}$, $-AB_{n-2} + B_{n-3} = a_{n-2}$, similarly $-AB_1 + B_0 = a_1 I$, $-AB_0 = a_0 I$. Now multiply see the first equation $(B_{n-1} = I)\,A^n$, $(-AB_{n-1} + B_{n-2} = a_{n-1}I)A^{n-1}$, $(-AB_{n-2} + B_{n-3} = a_{n-2}I)A^{n-2}$, similarly $(-AB_1 + B_0 = a_1 I)A$, $(-AB_0 = a_0 I)A^0$.

So, multiplying $A^n, A^{n-1}, \ldots.A^0$ to equations (i), (ii), …. (n+1) & adding column wise we have what do we have. So, if I multiply this once A^n and this is again A^n this side. So, this is I multiplying from the left side place multiplying from left side. So, I will have A^n B^n and this will the power also A^n B^n will cancel out this will cancel out this will all cancel out please.

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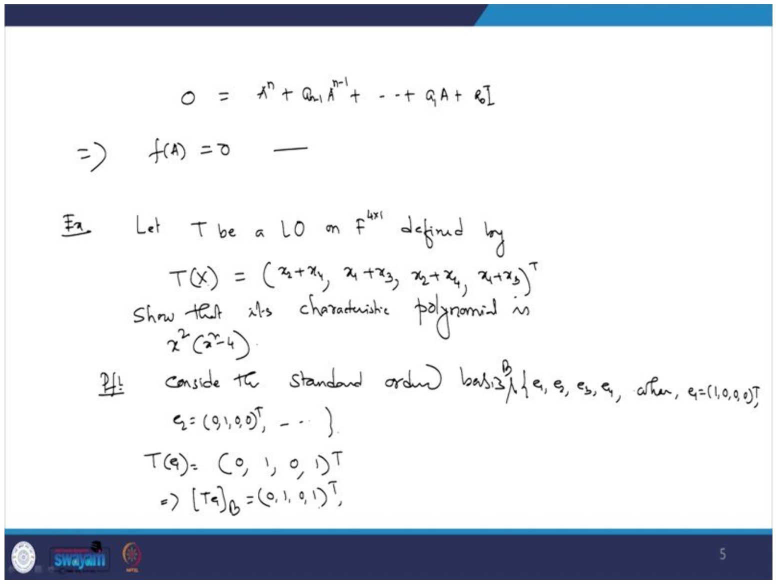

So, it will basically we have $0 = A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I$, this implies f(A) = 0. So, this is the Cayleys-Hamilton theorem please. So, we can take some examples. So, let me take an examples let T be a linear operator defined by T(x) = $(x_2 + x_4, x_1 + x_3, x_2 + x_4, x_4 + x_3)^T$. So, that its characteristic polynomial is $x^2(x^2 - 4)$. I am going to say I have to show that A^2 and e^2 – 4 = 0 to prove this one I have to immediately translate this operator into the corresponding matrix form. So, consider the standard order basic, $B = \{e_1, e_2, e_3, e_4\}$ where $e_1 = (1,0,0,0)^T$, $e_1 = (0,1,0,0)^T$ and so on. So, we see that $T(e_1) = (0,1,0,1)^T$, this implies $[T(e_1)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

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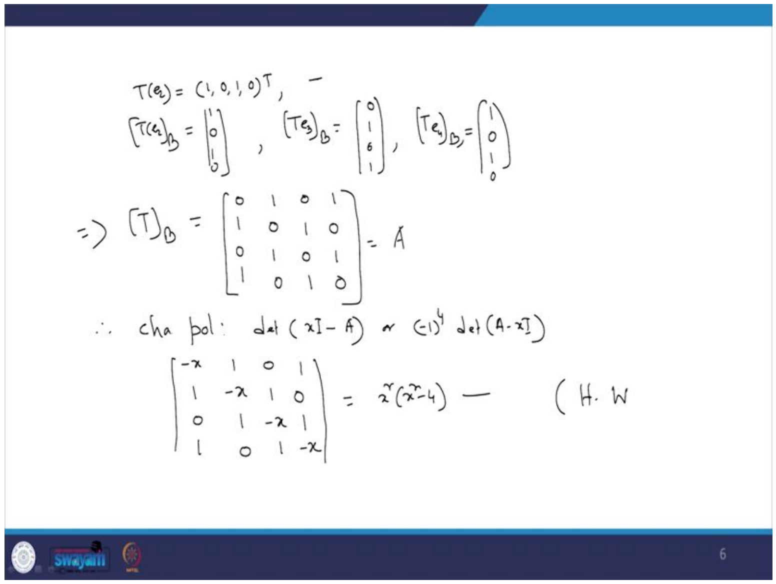

and $T(e_2) = (1,0,1,0)^T$, $[T(e_2)]_B = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$, $[T(e_3)]_B = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$ and $[T(e_4)]_B = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$. So, this one so, this implies $[T]_B = \begin{bmatrix}0 & 1 & 0 & 1\\1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & 1 & 0\end{bmatrix}$. So, one can find out the characteristic polynomials. So, characteristic polynomials = det(xI-A) = $(-1)^4$det(A-xI). So, if I do it, it will basically you know I can say $\begin{vmatrix}-x & 1 & 0 & 1\\1 & -x & 1 & 0\\0 & 1 & -x & 1\\1 & 0 & 1 & -x\end{vmatrix}$ = $x^2(x^2-4)$. So, this thing we can check it. So, leave it as a homework this one but this thing I will show that I also by showing that calculating different power of A and showing that a $x^2(x^2-4)$ equal to 0. Since this is a matrix of 4x4. So, I need a characteristic polynomial has to be 4 degree polynomial piece. So, let me quickly calculate what is a square is.

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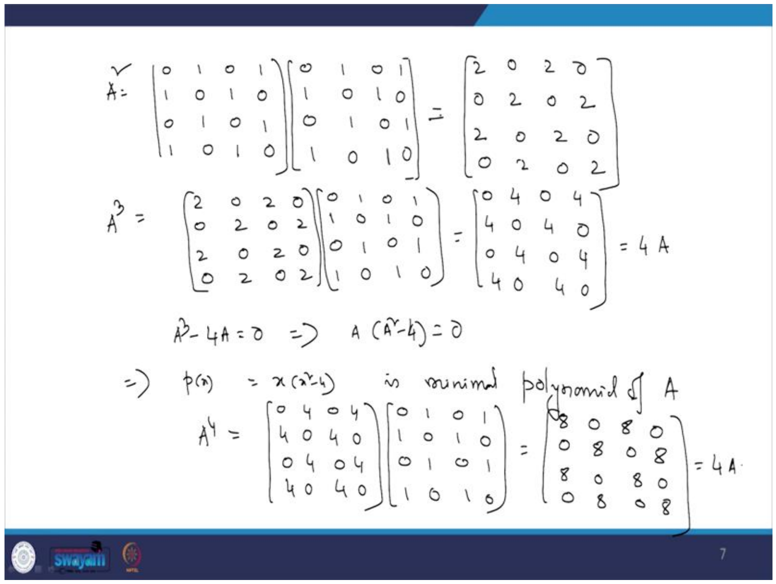

We have, $A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix} = 4A$. So, I am getting $A^3 - 4A = 0 \implies$

$A(A^2 - 4A) = 0$. So, $\implies$ $p(x) = x(x^2 - 4x)$ is the minimal polynomial of A. You can check it again what about, $A^4 = \begin{bmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \\ 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \end{bmatrix} = 4A^2$.

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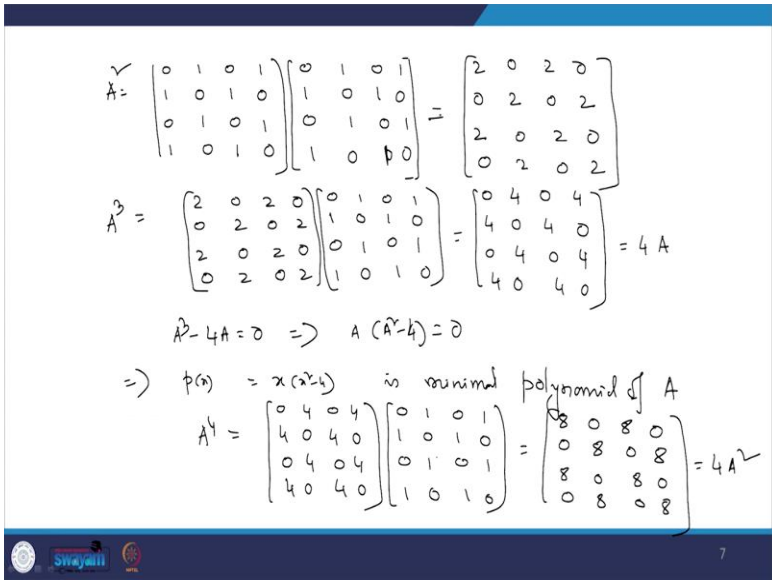

So, this implies $A^4 - 4A^2 = 0$. So, this implies $f(x) = x^4 - 4x^2$ characteristic polynomial of A and I also see that because the 4 degree polynomials and $x(x^2 - 4) = p(x) / f(x) = x^4 - 4x^2 = x^2(x^2 - 4)$. So, you see that the minimal polynomial divides characteristic polynomial also characteristic polynomial.

So, I will have x = 0.0 and x = 2,-2. So, I will define to another terminology that is called algebraic multiplicity and geometric multiplicity. What is algebraic multiplicity? In general if $f(x) = (x - c_1)^{d_1}(x - c_2)^{d_2} \ldots (x - c_k)^{d_k}$, where $\{c_1, c_2, \ldots c_k\}$ are the distinct eigen values then d_i is known as algebraic multiplicity complicity of eigen value that is your c_i.

Let W_i denote the null space of your $(A - c_i I)$, if I consider this once this matrix then that is basically eigen space to the eigen value c_i then dimension of W_i is called as geometric Multiplicity of eigen value c_i. So, in this example x= 0 having okay, x = 2, -2 having geometric multiplicity is one but what about the geometric multiplicity of x = 0. So, in that case I have to basically calculate the rank of this matrix a because x = 0. So, this rank of this matrix here I have to calculate please because I can see it here a is equal to the matrix is 0 1 0 1. So, it is repeated. So, only 2 linearly independent rows will be there. So, therefore definitely rank of the your angle x=0 here in this case will be 2. So, since rank is 2. So, null space of the (A - 0I) will be dimension 2.

So, definitely geometric multiplicity for the eigen value x=0 will be again 2 bits. So, I hope you have understood this concept of Cayley-Hamilton theorems and the concept of geometric multiplicity algebraic Multiplicity.