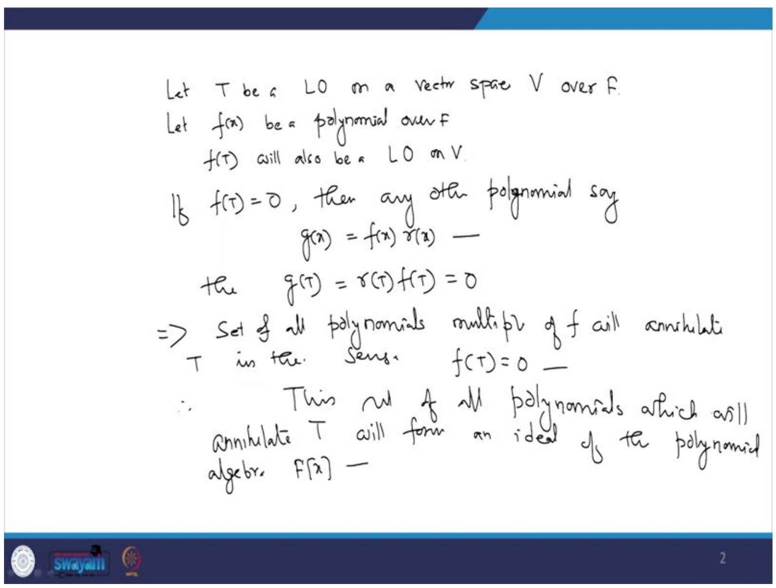
Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 26
Annihilating Polynomial of Linear Operator

So, welcome to Advance Linear Algebra. Today we will discuss about the annihilating polynomial of linear operator. What is this what is the meaning of annihilating polynomial or linear operator and what is the use of it we will discuss in today's lecture also.

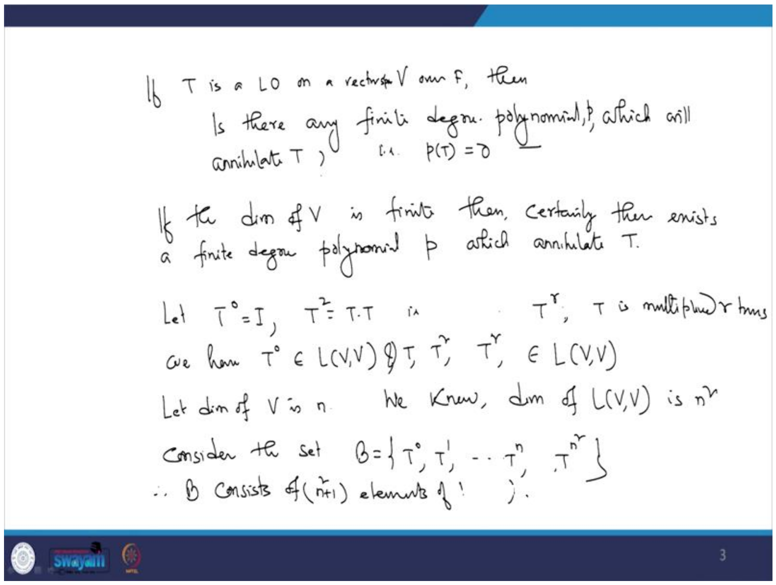
(Refer Slide Time: 00:48)



Let T be a linear operator on a Vector space V over F. Let f(x) be a polynomial over F. If T is a linear operator on a vector space V then f(T) will also be a linear operator on the V. If f(T) = 0 then any other polynomial say g(x) = f(x)r(x) then, g(T) = r(T)f(T) = 0. So, this implies set of all polynomial multiple of f will and he let T in the sense f(T) = 0.

So, this set of polynomial, we basically ideal of the field F[x] , I mean this field of basically polynomial algebra please. So, this set of all polynomials which will annihilate the will form an ideal of the polynomial algebra you know F[x]. This is basically set of all the polynomials defined over the field F we know its form a is again a field place. Now over this set this instead of all the Polynomial which annihilate T will begin an ideal of this set place or this field also not to confuse field because f is field itself and f(x) is the again affiliate it is basically consists of all the polynomials define over the f(x).

(Refer Slide Time: 04:36)



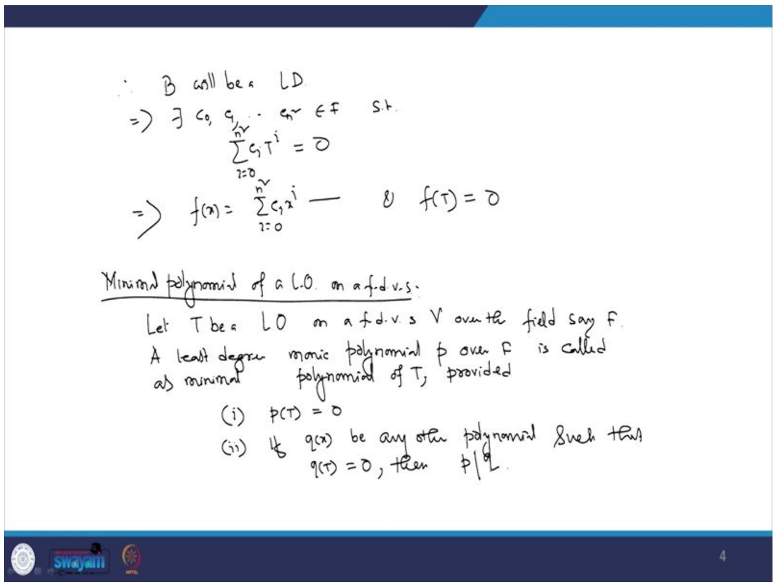
So, there is this again ideal means there will be some generators the least degree polynomials which will also annihilate the operator T please. If the vector space is of dimension infinity the vector space may be Dimension infinity or maybe finite also. The question is if T is an operator is a linear operator on a vector space V over F then is there any finite degree polynomial which will annihilate. Let T is there any finite degree polynomial say p which will annihilate T , i.e. p(T)= 0.

So, the meaning of the annihilating of the operator T please may be it may not have we do not know if the vector space is of infinite ourselves is of infinite dimensions. But if the dimension of V is finite then certainly there exist a finite degree polynomial p which annihilate right T this is the answer I am saying. But the question is how can he say that there exists a finite degree polynomial which will annihilate T.

Let $T^0 = I$, $T^2 = T.T$ and T^r, its means T is multiplied r times, what do you mean by the operator is Multiplied at times basically applied r times you can say. So, we have $T^0 \in L(V,V)$ & and T, T^2, $T^r \in L(V,V)$, let dimension of V is n.

Then we know dimension of $L(V,V) = n^2$. Consider the set $B = \{T^0, T^1, \ldots T^n, T^{n^2}\}$. So, B consists of $(n^2 + 1)$ elements of L(V,V) certainly this set will be linear and independent.

(Refer Slide Time: 09:48)



So, B will be a linearly dependent not independent because there are more than n^2. So, this implies $\exists \{c_1, c_2, \ldots c_{n^2}\} \in F$ s.t. $\sum_{i=0}^{n^2} c_i T^i = 0 \implies f(x) = \sum_{i=0}^{n^2} c_i T^i$ is a finite degree polynomials which annihilate T & we see that, f(T) = 0.

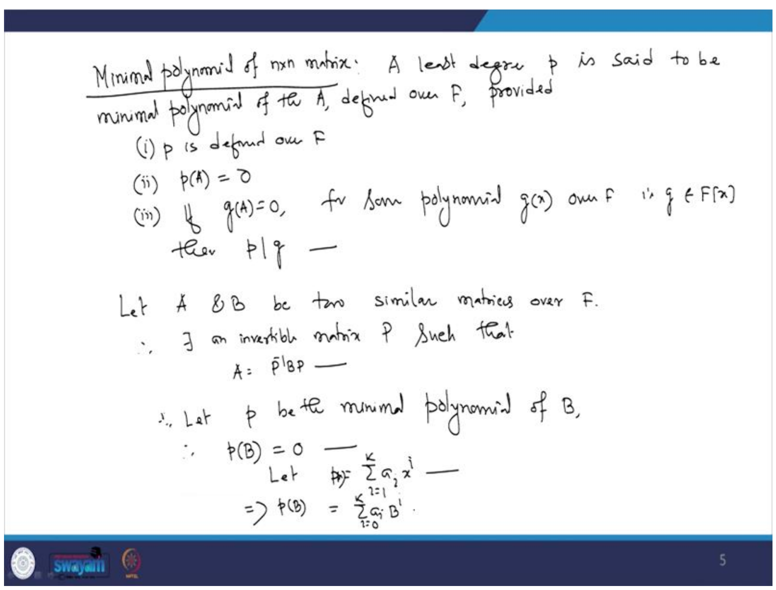
So, existence of finite degree polynomial for finite dimensional space is guaranteed. Whether this is the least degree polynomial which annihilate T or not that is next question please will soon prove that the characteristic polynomial of the operator T is defined on a finite dimensional Vector space then the characteristic polynomial also annihilates the operator T see that I will prove it please later on please n which is much smaller than $n^2 + 1$.

So, characteristic polynomials there but before going to characteristic polynomials then the question is does there exist if further minimum degree polynomials which will annihilate the operator T. Answer is maybe yes. So, what is that? That will be basically in terms of minimal polynomial. Minimal polynomial place of a linear operator on a finite dimensional vector space Place what is minimum polynomial of a linear operator on a finite dimensional vector space defined on a finance Vector space.

Let T be a linear operator on a finite dimensional Vector space say V over the field say F. Since if I go back to the our previous logic that in the polynomial field f(x) if some polynomial is say p(x) annihilate the operator T then any other polynomials multiple to the p in also annihilate V. So, it form an ideals in that polynomial field page now since the ideal in a polynomial field certainly there will be least degree monic polynomial which will act as a generator for the ideal place and that list degree monic polynomial we are defining as a minimal polynomial for the operator T.

So, least degree Monic polynomial p over F is called as minimal polynomial of operator T provide it, (i) p(T) = 0. (ii) q(x) be any other polynomial such that q(T) = 0, then $\frac{p}{q}$ So, this is the definition of the minimum polynomial of a linear operator.

(Refer Slide Time: 15:18)



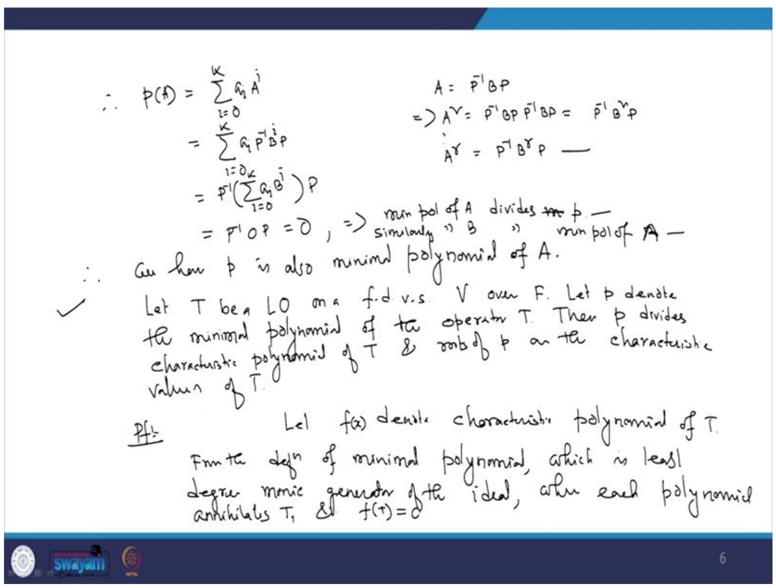
In the same Spirit I can talk about the minimal polynomial of a square Matrix. Also of a nxn or square matrix a least degree polynomial p is said to be minimal polynomial of the matrix a defined over F, provided (i) p is defined over F. (ii) p(A) = 0. (iii) if g be another polynomial, g(A) =

0, for some polynomial g(x) over F, i.e. $g(x) \in$ f(x) basically then $\frac{p}{g}$ so, the same definition as in the case of the linear operator please.

So, from this definition it is clear that similar matrices have same minimal polynomial. Let A & B two similar matrices right matrices over F. So, there exists and invertible matrix P such that your $A = P^{-1}BP$, because A and B are similar. So, that is the definition from this definition of similarity I have; there existing invertible vertex P such that $A = P^{-1}BP$.

So, let p be the minimal polynomial of B. So, p(B) = 0, let $p(x) = \sum_{i=1}^{k} a_i x^i$. So, let $p(B) = \sum_{i=1}^{k} a_i B^i$.

(Refer Slide Time: 19:36)



So, what can you say about the p(A). So, $p(A) = \sum_{i=0}^{k} a_i A^i = \sum_{i=1}^{k} a_i P^{-1}B^iP = P^{-1}(\sum_{i=1}^{k} a_i B^i)P = P^{-1}0\,P = 0$, where $A = P^{-1}BP$, $A^2 = P^{-1}BP\,P^{-1}BP = P^{-1}B^2P$, $A^r = P^{-1}B^rP$. So, we have p is also minimal polynomial of A. Actually p(A) = 0 means if the minimal polynomial of A equal to skew then certainly $\frac{p}{q}$ this implies minimal polynomial of A divides p.

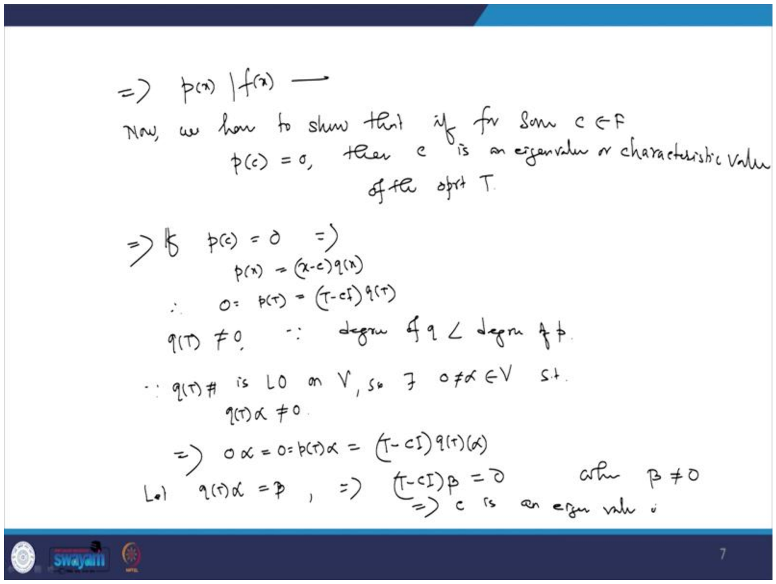
In same way minimal polynomial B and divides minimal polynomial of A. So, from these two only you can conclude that p is also minimal polynomial of A. So, we have that minimum polynomial of a similar matrices are basically same. Another interesting results the relation between the characteristic polynomial and minimum polynomials. Let T be a linear operator on a finite dimensional vector space V over the field F.

Let p denote the minimal polynomial of the operator T then p divides characteristic polynomial of T and roots of p are the characteristic values or eigen values of T. So, if T be a linear operator defined on a finite dimensional vectors with V over the field F and if p denote the minimal polynomial of the operator T then p must divides the characteristic polynomial of T and roots of p are the characteristic values of T.

p divides characteristic polynomial T that is coming as a consequence of our definition of the ideal and definition of the minimal polynomial. Since we know the set of polynomial which annihilate the operator T is an idea and minimum polynomial is basically least degree monic generator of that idea. So, certainly if q is the characteristic polynomial of the operator T means I will show later on that characteristic polynomial also annihilates the operator T.

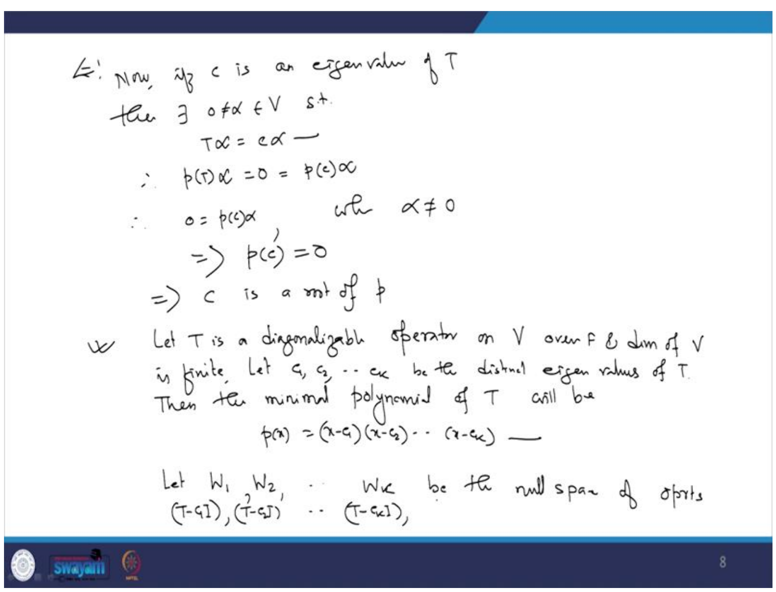
So, definitely then p must divide your q also because degree of q will be higher than the degree of p. So, p divides characteristic polynomial is coming as a consequence of the definition of the minimum polynomial. Let f denote characteristic polynomial of T this is you can say f(x) this is a polynomial functions. From the definition of minimal polynomial which is least degree monic generator of the ideal each polynomial and inlets the operator f(T) = 0.

(Refer Slide Time: 26:35)



$\Longrightarrow \frac{p(x)}{f(x)}$ so, this part is coming as a consequence of the definition. Now I have to show that if for some c ∈ F, p(c) = 0, then c is an eigen value eigen value or characteristic value of the operator p(c) = 0. So, let me do I can say this one is if and only if, I mean if p(c) = 0 if and only if an eigen value of the operator T, so, if p(c) = 0 $\Longrightarrow$ p(x) = (x-c) q(x). So, 0 = p(T) = (T-cI) q(T), q(T) $\neq$ 0, since degree of q $<$ degree of p. Since q(T) $\neq$ 0 is a linear operator on V. So, ∃ 0 $\neq$ α ∈ V s.t. q(T)α $\neq$ 0. So, $\Longrightarrow$ 0α = 0 = p(T)α = (T-cI)q(T)(α) = 0. So, let q(T)α = β, $\Longrightarrow$ (T-cI) β = 0, where $\beta \neq 0$ $\Longrightarrow$ c is an eigen value in value of T.

(Refer Slide Time: 29:33)

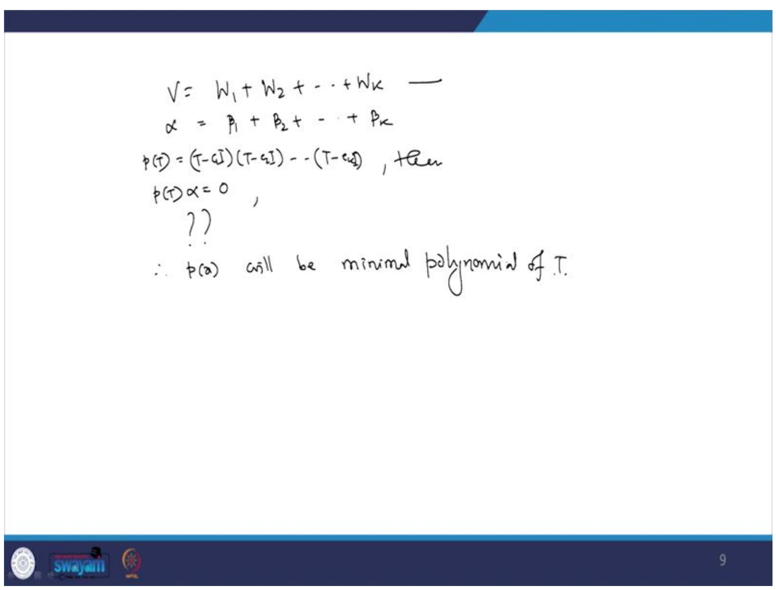


If c is an eigen value now it is other replays if and only if; if c is an eigen value of T then, $\exists\, 0 \neq \alpha \in$ V s.t. Tα = cα. p(T)α = 0 = p(c)α. So, 0 = p(c)α, where $0 \neq \alpha \implies$ p(c) = 0. So, this implies c is a root of p.

So, you see that the roots of minimal polynomial and roots of characteristic polynomial are same but the multiplicities may be different please. So, we see that c is also root of the minimum polynomial P. Another interesting results if T is a on vector space B over F and dimension of V is finite let T is a diagonal operator on V over a field F and dimension of V is finite let $c_1, c_2, \ldots c_k$ be the distinct eigen values of T then the minimal polynomial of T will be p(x) = (x-$c_1)(\mathrm{x} - c_2) \ldots . (\mathrm{x} - c_k)$.

So, this is the product of all (x-c_i) factor of linear product basically this result is coming as a consequence of if we consider let let $W_1, W_2, \ldots W_k$ be the null space of operator (T-$c_1 I)(\mathrm{T} - c_2 I) \ldots . (\mathrm{T} - c_k I)$.

(Refer Slide Time: 33:21)



Then V = W_1+W_2+ … +W_k and $\alpha \in$ V, $\alpha = \beta_1$+β_2+ …+ β_k. So, now p(T) = (T-$c_1 I)(\mathrm{T} - c_2 I) \ldots . (\mathrm{T} - c_k I)$, then if I cut this operator then you see that p(T) annihilates α, i.e. p(T)$\alpha = 0$ How? Say $\alpha = \beta_1$+β_2+ …+ β_k, see β_1 will be annihilated by (T-$c_1 I$), β_2 will be animated by (T-$c_2 I$) , and similarly β_k will eliminated by (T-$c_k I$). So, when p(T)α this Factor $\alpha = \beta_1$+β_2+ …+ β_k will be annihilated by some of the factors in p. So, therefore p(x) will be the minimal polynomial of T. So, I hope you have understood the definition of the annihilating polynomial and the concept of minimal polynomials relation between the roots of the minimal polynomial and characteristic polynomials.

If a operator define a very finite dimensional vector space and having $c_1, c_2, \dots c_k$ a distinct eigen values then its minimal polynomial is basically product of linear factors (x-c_1)$(\mathrm{x} - c_2) \dots . (\mathrm{x} - c_k)$, thank you.