# Advanced Linear Algebra

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## Lecture – 25
## Eigenvalue and Eigenvector of Diagonalizable Linear Operator

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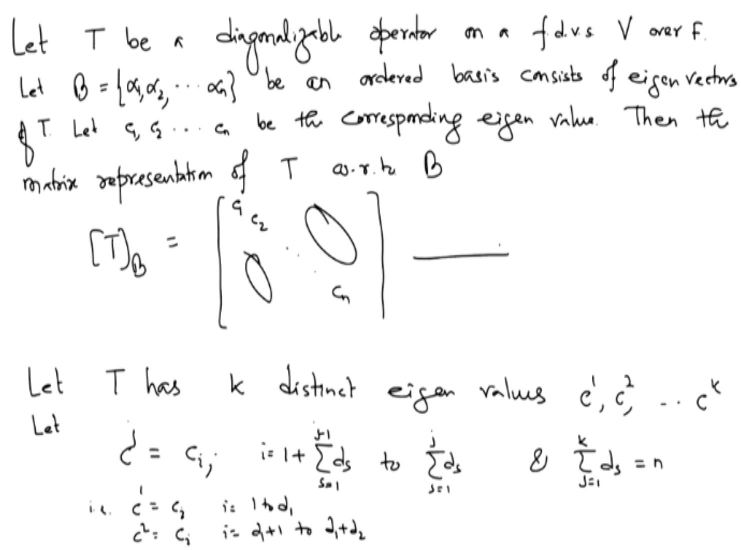

Welcome to lecture series on advanced linear algebra. Today, I will talk about the eigenvalue and eigenvector of diagonalizable linear operator. Already I have defined what is the meaning of or the definition of diagonalizable linear operator. So, if I consider T be a diagonalizable operator on a finite dimensional vector space V over the field F and let B = $\{\alpha_1, \alpha_2, \dots\ \alpha_n\}$ be the ordered basis consist of eigen vectors of T.

And say $c_1, c_2, \dots\ c_n$ be the corresponding eigenvalue, then the $[T]_B = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & c_{n-1} & 0 \\ 0 & 0 & 0 & 0 & c_n \end{bmatrix}$.

So, this means that if I consider any other ordered basis on vector space V, then T will also have a matrix representation with respect to that also. So, I can see that the matrix will be similar to this diagonal matrix.

Indirectly I will say that the matrix representation of T with respect to this ordered basis is basically a diagonal matrix, okay fine. Again it may happen that $c_1, c_2, \dots c_n$ all are not distinct. Suppose $c_1$ repeated $d_1$ times and similarly other eigenvector, there are suppose k distinct eigenvalues $c^1, c^2, \dots c^k$, these are the ones I have written with top suffix like this $c^1, c^2, \dots c^k$ and then maybe I can define the $c^j = c_i$, i = 1 +$\sum_{s=1}^{j-1} d_s$ to $\sum_{s=1}^{j} d_s$ & $\sum_{s=1}^{k} d_s = n$.

What I mean to say that $c^1 = c_i$, $i = 1$ to $d_1$ this is this one because all are same eigenvalues. Similarly, $c_2$ also defined say $c^2 = c_i$, i = $d_1 + 1$ to $d_1 + d_2$, similarly $c^3, \dots c^k$.

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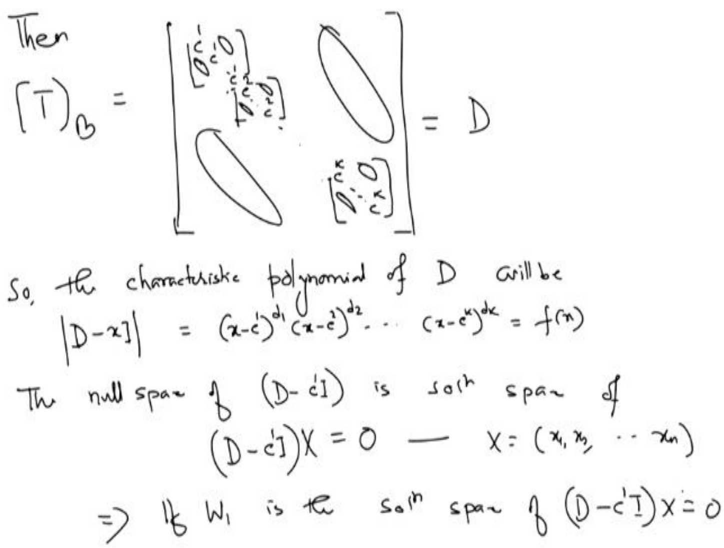


So, if this is the case then the matrix representation of T, this diagonal matrix what I have written here can again rewrite like this,

$$\begin{bmatrix} \begin{bmatrix} c^1 & 0 & 0 \\ 0 & c^1 & 0 \\ 0 & 0 & c^1 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} c^k & 0 & 0 \\ 0 & c^k & 0 \\ 0 & 0 & c^k \end{bmatrix} \end{bmatrix} = D.$$

I am basically writing some sort of block diagonal matrix type. Again, in this block this matrix is again a diagonal matrix with entries are here $c^1, c^1, c^1$ like that, then $c^2, c^2, c^2$ repeat, this is basically $d_1$, first one is $d_1$x$d_1$ matrix, second is $d_2$x$d_2$ matrix and last one $d_k$x$d_k$, so that your $\sum_{i=1}^{k} d_i = n$.

So, this is basically relating the matrix representation of T when some roots are repeated. So, in this case if you want to calculate the characteristic polynomial, so in this case the characteristic polynomial will be what? Suppose if I write this is equal to say D, then so the characteristic polynomial of D will be |D – xI| = $(x - c^1)^{d_1}(x - c^2)^{d_2} \ldots (x - c^k)^{d_k}$ = f(x). So, this is obviously the characteristic polynomial effects of this linear operator T. Interestingly, if I want to see the null space, so in then the null space of (D – $c^1$I), so this is equal to this will basically again a diagonal matrix the first block matrix that is of $d_1 \text{x} d_1$ all entries diagonal entries will be equal to 0.

But second, third, up to k-th one block matrix the diagonal entry will be $c^2 - c^1$, all the elements for second block element will be $c^2 - c^1$, and for last one $c^k - dc^1$. Since $c^1, c^2, \ldots c^k$ are distinct, so the difference is also nonzero. As a result, if I want to know that the null space of (D – $c^1$I) then this basically is solution space of (D – $c^1$I)X = 0. So, certainly I am getting because all except the first $d_1$, if I consider X = $(x_1, x_2, \ldots x_n)$

I will have only $d_1$ number of free variables. Apart from $d_1$, all other variables $x_i$= 0 because I am getting straight way the expressions from the (D – cI)X. If I write this equation, I am getting all the value of $x_i$ = 0 except first $d_1$ entries. So, $x_1$ to $xd_1$, I am not saying anything, but apart from $xd_1$+1 to $x_n$ all will be 0. So, this implies I am getting exactly d 1 number of free variables in the solution of (D – $c^1$I) X = 0.

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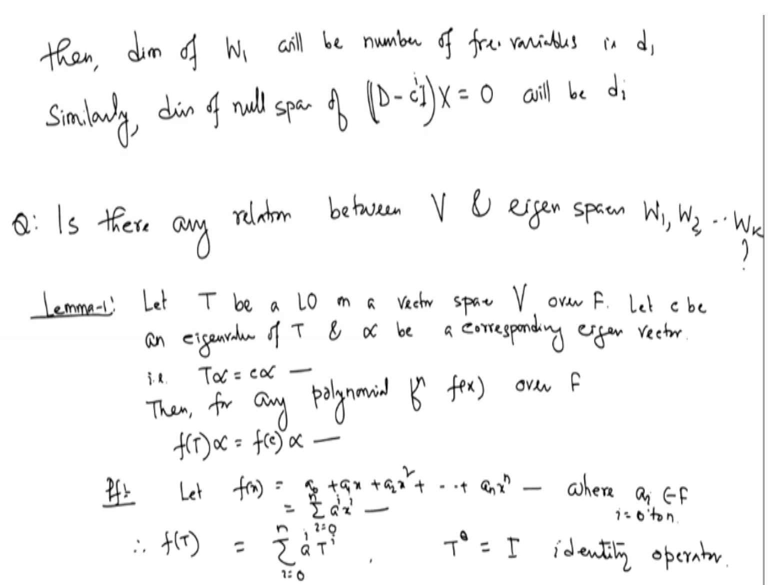

This implies if $W_1$ is the solution space of (D – $c^1$I) X = 0 then the dimension of $W_1$ will be number of free variables that is exactly $d_1$ because all other entries are 0. Similarly, the dimension of null space of (D – $c^i$I) = 0 will be $d_i$, so that is I am getting from the definition of the diagonalizability of the operator T, fine. Now, we are curious to know is there any relation between V over V is the operator defined and eigenspaces $W_1, W_2, \ldots W_k$.

A null space of (D – $c^i$I) is basically eigenspace of the eigenvalue $c_i$, I mean eigenspace related to the eigenvalue $c_i$. So, this is the question is there any relation between V and eigenspace is $W_1, W_2, \ldots W_k$ that is the question. Before going to answer this these questions, let us talk about some standard results. So, let me write one result in terms of lemma 1 like this. Let T be a linear operator on a vector space V over F.

Let c be an eigenvalue of T and $\alpha$ be a corresponding eigenvector, i.e. T($\alpha$) = c$\alpha$, then for any polynomial function f(x) over F, f(T($\alpha$)) = f(c)$\alpha$, this is very simple result. If c is an eigenvalue of the operator T, $\alpha$ is the corresponding eigenvector that is T($\alpha$) = c$\alpha$, then for any polynomial function f(x) over the field F, f(T($\alpha$)) = f(c)$\alpha$.

Proof is very straightforward. Let $f(x) = a_o + a_1x^1 + a_2x^2 + \cdots + a_nx^n$ where $a_i \in F$, i = 0 to n. So $f(x) = \sum_{i=0}^{n} a_i x^i$. So $f(T) = \sum_{i=o}^{n} a_i T^i$, where $T^o = I$.

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$$\Rightarrow f(T)\alpha = \left(\sum_{i=0}^{n} a_i T^i\right)(\alpha) = \sum_{i=0}^{n} a_i T^i(\alpha)$$
$$= (a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n)(\alpha)$$
$$= a_0 I\alpha + a_1 T\alpha + a_2 T^2\alpha + \cdots + a_n T^n(\alpha)$$
$$= a_0\alpha + a_1 c\alpha + a_2 c^2\alpha + \cdots + a_n c^n\alpha$$
$$= (a_0 + a_1 c + a_2 c^2 + \cdots + a_n c^n)\alpha$$
$$= f(c)\alpha$$

$T^2\alpha = ?$

$T(T(\alpha)) = T(c\alpha) = c\,T(\alpha) = c\,c\alpha = c^2\alpha$

$T^r\alpha = c^r\alpha$

Lemma-II: Let T be a L.O. on a f.d.v.s. V over F. Let $c_1, c_2 \ldots c_k$ be the only distinct eigenvalues of T. Let $W_i$ be the eigenspace associated to eigenvalue $c_i$, i = 1 to k. Let $W = W_1 + W_2 + \cdots + W_k$. Then $\dim W = \sum_{i=1}^{k} \dim W_i$ —

Infact, if $B_1, B_2, \ldots B_k$ be the ordered basis of $W_1, W_2, \ldots W_k$, respectively, then $B = \{B_1, B_2, \ldots B_k\}$ will be an ordered basis of W.

So, this implies f(T)$\alpha$ = $(\sum_{i=0}^{n} a_i \mathrm{T}^i)(\alpha)$ = $\sum_{i=0}^{n} a_i \mathrm{T}^i(\alpha)$ = $(a_o \mathrm{I} + a_1\mathrm{T} + a_2\mathrm{T}^2 + \cdots + a_n\mathrm{T}^n)(\alpha) = a_o\mathrm{I}\alpha + a_1\mathrm{T}\alpha + a_2\mathrm{T}^2\alpha + \cdots + a_n\mathrm{T}^n\alpha = a_o\alpha + a_1\mathrm{c}\alpha + a_2\mathrm{c}^2\alpha + \cdots + a_n\mathrm{c}^n\alpha = (a_o\mathrm{I} + a_1\mathrm{c} + a_2\mathrm{c}^2 + \cdots + a_n\mathrm{c}^n)\alpha$ = f(c)$\alpha$, where $\mathrm{T}^2$ = T(T($\alpha$)) = T(c $\alpha$) = cT($\alpha$) = $c(c\alpha)$ = $\mathrm{c}^2\alpha$ and similarly $\mathrm{T}^r(\alpha) = \mathrm{c}^r\alpha$.

This is another lemma, I can say lemma 2. Let T be a linear operator on a finite dimensional vector space V over the field F. Let $c_1, c_2, \ldots c_k$ be the only distinct eigenvalues of T. Let $W_i$ be the eigenspace associated to eigenvalue $c_i$, i = 1 to k. Since V is a finite dimensional vector space this $W_i$ will be also finite dimensional. Let $W = W_1 + W_2 + \cdots + W_k$. Then dim(W) = $\sum_{i=1}^{k} \dim(W_i)$.

In fact, if $B_1, B_2, \ldots B_k$ be the ordered basis of $W_1, W_2, \ldots W_k$ respectively then B = $\{B_1, B_2, \ldots B_k\}$ will be an ordered basis of W. So, this is a small result. So, let us try to give the proof of this result.

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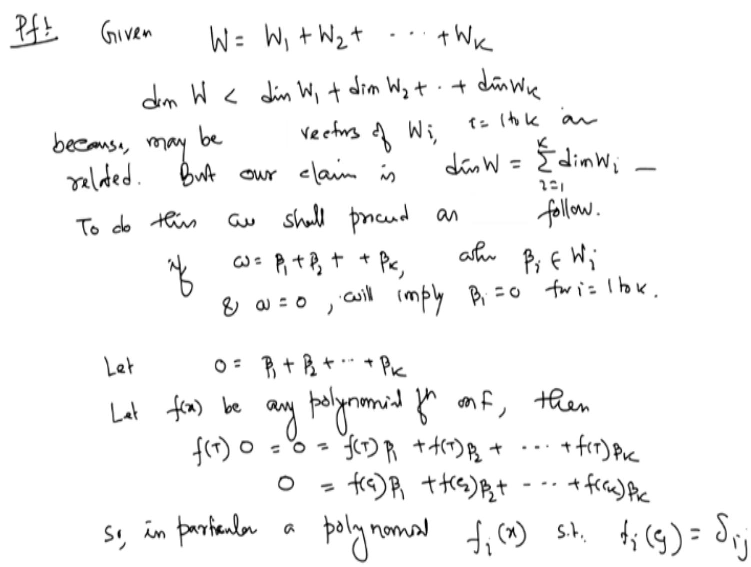


So, given $W = W_1 + W_2 + \cdots + W_k$. Since each of $W_i$ is finite dimensional, so W has to be finite dimensional, there is no question or doubt. And $W_1, W_2, \ldots W_k$ are the eigenspaces associated to eigenvalues $c_1, c_2, \ldots c_k$ respectively. Now, by showing that dim(W) $<$ dim($W_1$) + dim($W_2$) +…+ dim($W_k$) we are basically showing that eigenspaces associated to distinct eigenvalues are linearly independent.

Apparently, we will see that dim(W) $<$ dim($W_1$) + dim($W_2$) +…+ dim($W_k$) because maybe some vectors of $W_i$, i = 1 to k are related, so I mean to say they are linearly dependent, they are interrelated type. So, in that case and since the W is basically sum of $W_1, W_2, \ldots W_k$, so certainly the dimension of W is expected that it will be less than or equal dim($W_1$) + dim($W_2$) +…+ dim($W_k$) On the basis of the hypothesis that eigenvectors of $W_i$ may be related to the vector of the $W_j$. But, our claim is that it is not less than, it will be exactly equal to basically, I mean dim(W) =dim($W_1$) + dim($W_2$) +…+ dim($W_k$), this is our claim. But our claim is dim(W) = $\sum_{i=1}^{k}$ dim($W_i$). . So, to do this what I will do, we will will proceed as below.

So, we should show that if w = $\beta_1$+$\beta_2$…+$\beta_k$ where $\beta_i \in W_i$ and w = 0 $\Longrightarrow$ $\beta_i$ = 0 for i = 1 to k, we want to show this one. So, first I will show this one that if the sum of these element w = 0 means each of the $\beta_i$ = 0 that I have to show first, then we will go for the next step. So, let 0 = $\beta_1$+$\beta_2$…+$\beta_k$. let f(x) be any polynomial function on F.

Then fT(0) = 0 = f(T)$\beta_1$ + f(T)$\beta_2 + \cdots +$ f(T)$\beta_k$ = f($c_1$)$\beta_1$ + f($c_2$)$\beta_2 + \cdots +$ f($c_k$)$\beta_k$, this is true for any polynomial, in particular a polynomial $f_i(x)$ s.t. $f_i(c_j)$= $\delta_{ij}$. I mean to say suppose this polynomial is a polynomial of degree n so one can consider Lagrange interpolant polynomial of degree n and we can have this type of criteria satisfied, $f_i(c_j)$= $\delta_{ij}$.

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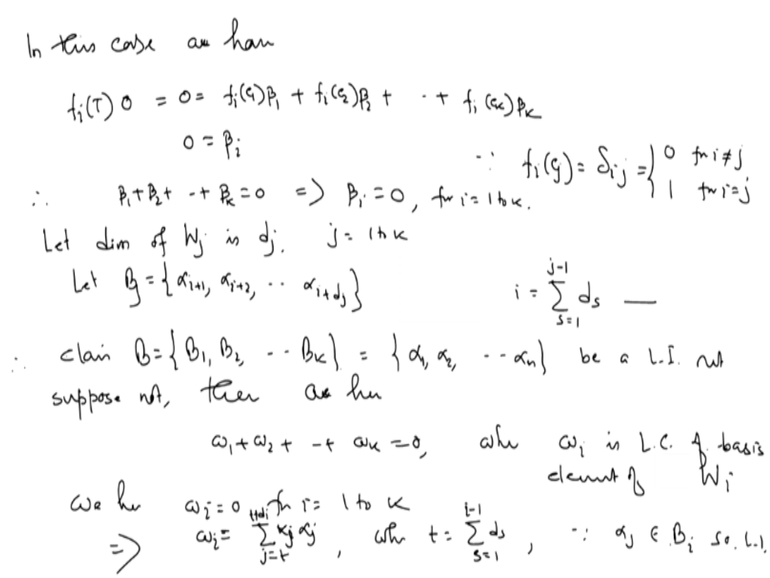


In this case, we have $f_i(T)0 = 0 = f_i(c_1)\beta_1 + f_i(c_2)\beta_2 + \cdots + f_i(c_k)\beta_k$. Since, $f_i(c_j)$= $\delta_{ij}$ =

$\begin{cases} 0 & for\ i \neq j \\ 1 & for\ i = j \end{cases}$. This implies 0 = $\beta_i$, so this is true. See this I have taken polynomial $f_i$, here I can say i = 1 to k. You can take $f_1, f_2, \ldots f_k$ and applying each of them we will see that $\beta_1$= 0, $\beta_2$ = 0, and $\beta_i$ = 0 So, this implies that $\beta_1$+$\beta_2$…+$\beta_k$ = 0 implies $\beta_i$ = 0 for i = 1 to k.

Let dimension of $W_j$ is $d_j$ for j = 1 to k. So, let $B_j$ ordered basis this is consisting of I can say okay, so let me consider the ordered versus for the $W_j$, $B_j = \alpha_i + 1, \alpha_i + 2, \ldots \alpha_i + d_j$ where i = $\sum_{s=1}^{j-1} d_s$. So, I have $B_1, B_2, \ldots B_k$ they are the ordered basis for $W_1, W_2, \ldots W_k$. Now claim, B = $\{B_1, B_2, \ldots B_k\}$ = $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a linearly independent set, suppose not.

Then we have something like $w_1 + w_2 + \cdots + w_k = 0$ where $w_i$ is linear combination of basis element of $W_i$. Now, if $w_1 + w_2 + \cdots + w_k$= 0 according to our last result, we have $w_i$ = 0 for i = 1 to k. So, this implies that $w_i$ , $\sum_{j=t}^{t+d_i} k_j \alpha_j$, where t = $\sum_{s=1}^{i-1} d_s$, so this one like this thing.

So, this implies that since they are all linearly independent elements, therefore all the coefficients have to be equal to 0. Since, $\alpha_j \in B_i$ , so linearly independent.

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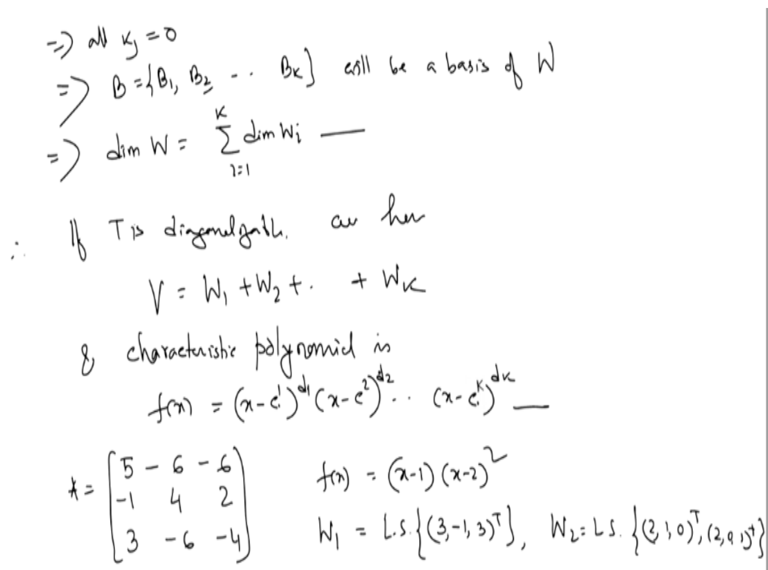

As the results $\Rightarrow$ all $k_j$ = 0. So, $\Rightarrow$ B = $\{B_1, B_2, \ldots B_k\}$ will be a basis of W. So, $\Rightarrow$ dim(W) = $\sum_{i=1}^{k} \dim(W_i)$. Now, I shall utilize and summarize the definition of diagonalizability of the operator. So, if T is diagonalizable then we have V = $W_1 + W_2 + \cdots + W_k$, & f(x) =

$(x - c^1)^{d_1}(x - c^2)^{d_2} \ldots (x - c^k)^{d_k}$, characteristic polynomial this one.

So, this is all about this diagonalizability of the operator. So, we see that if T is diagonalizable, then T can be written as a sum of subspaces $W_1, W_2, \ldots W_k$ where $W_i$ of the eigenspaces associated to the eigenvalues $c_1, c_2, \ldots c_k$ and the characteristic polynomial is f(x) = $(x - c^1)^{d_1}(x - c^2)^{d_2} \ldots (x - c^k)^{d_k}$. We can cross check these result by considering our previous examples where I considered a matrix that A equal to.

Now based on this we see if you recall our last example where A = $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$, we have corresponding characteristic polynomial is like f(x) = (x – 1) $(x - 2)^2$ and we have two distinct eigenvalues 1 and 2. So, the eigen spaces associated with eigenvalue 1 is $W_1$ = L.S.{ $(3, -1, 3)^T$}, $W_2 =$ L.S.{ $(2,1,0)^T$} is the eigenspace associated to eigenvalue 2. So, linear span of these two linearly independent vectors will give me the eigenspace $W_2$. And we see that your V = $W_1$ + $W_2$.

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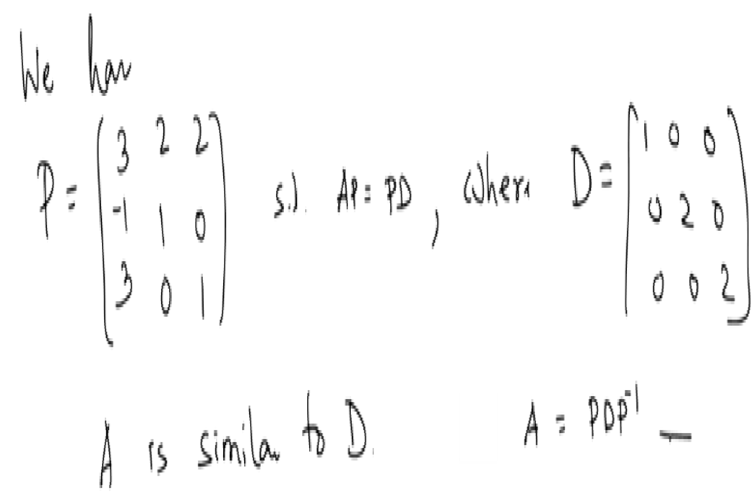


We have, P = $\begin{bmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ . we considered having nonsingular matrix P such that AP = PD, where D = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, the P consists of the eigenvectors only that is first column is eigenvector associated to eigenvalue 1 and other two columns are eigenvector associated to 2 only. So, we see that AP = PD. So, this implies that A is similar to D that is , I have A = $PDP^{-1}$ that is basically the beauty of diagonalizable operator or matrix. We will continue in next class. Thank you.