Advanced Linear Algebra
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Lecture – 24
Eigenvalue and Eigenvector of Linear Operator- 2

In my last class, we have defied the concept of eigenvalue and eigenvector of a linear operator defined on a vector space V. If the dimension of the vector space is finite, then I also defined the concept of eigenvalue and eigenvector of square matrix associated to linear operator defined on finite dimensional space. We have taken some examples. We have seen in some case the eigenvalues of a matrix may not have eigenvalues and in some case it has eigenvalue. Then we have also seen how to find the eigenvectors associated to the corresponding eigenvalue.

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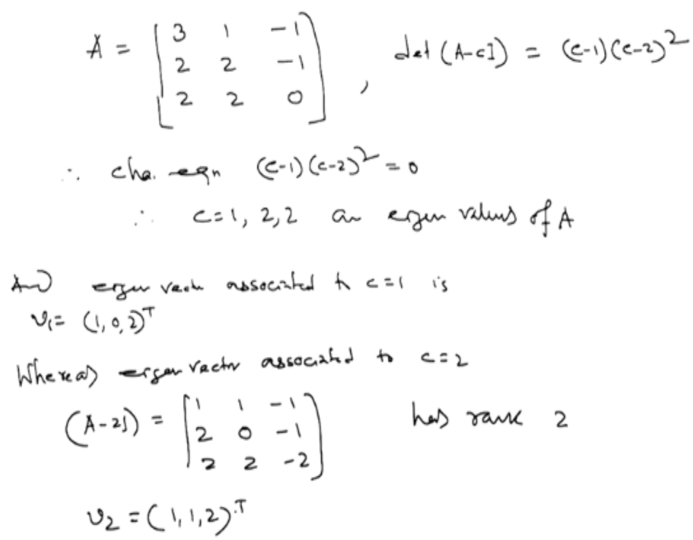

If you recall I took last time this is the matrix A = $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$. We have seen the characteristic polynomial of this matrix is (c-1) $(c-2)^2$. So, c = 1, 2, 2 are basically eigenvalues of the matrix A and the eigenvector associated to eigenvalue c = 1 we have calculated as $v_1 = (1, 0, 2)^T$ whereas for eigenvalue 2 that is $v_2 = (1, 1, 2)^T$.

Here (A–2I) if we consider this matrix then I have seen that rank of this matrix is again 2. So, it has only one linearly independent solution (A–2I) operating on X = 0. So, I have only one linearly independent eigenvector associated two eigenvalue c = 2.

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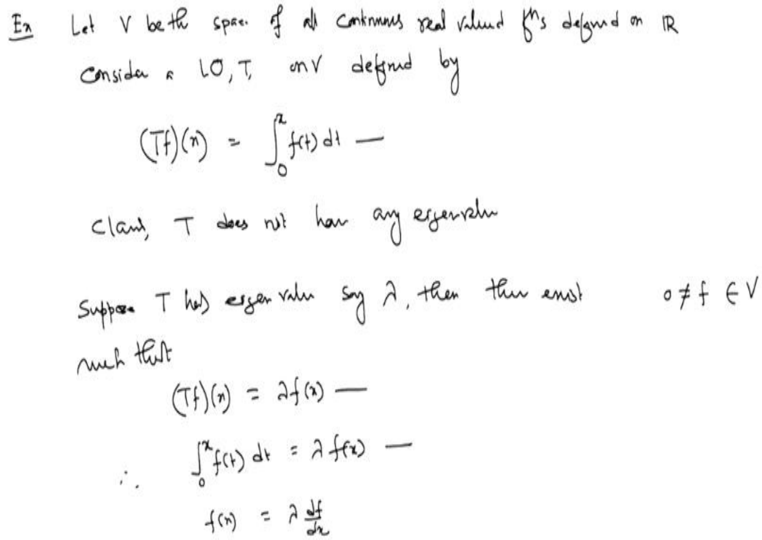

Now, let me consider another example. Let V be the space of all continuous real valued functions defined on real line R. So, V be vector space or dimension infinite. Consider a linear operator(L.O.) T on V defined by (Tf)(x) = $\int_0^x f(t)dt$. So, this is the definition of the operator T on V. We already seen this function T is a linear operator on V. Claim: T does not have any eigenvalue. So, how to prove it? Suppose T has eigenvalue say λ.

Then there exist 0 ≠ f ∈ V such that (Tf)(x) = λf(x) $\implies \int_0^x f(t)dt = \lambda$f(x) $\implies f(x) = \lambda \frac{df}{dx}$. It is given to us f is continuous, since f is a continuous so the $\int_0^x f(t)dt$ is differentiable from the fundamental theory with calculus, we can say it is differentiable. So, differentiating both the sides we have $f(x) = \lambda \frac{df}{dx}$. I can also add if it is confusing, I can say that it is all continuous real valued differentiable function also so that $f(x) = \lambda \frac{df}{dx}$.

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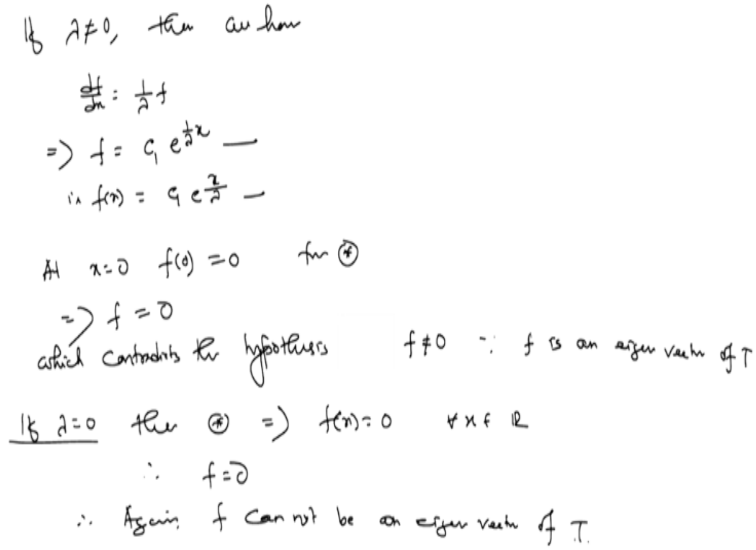

So there are two possibilities, the $\lambda = 0$, or $\lambda \neq 0$. If $\lambda \neq 0$, then we have, $\frac{df}{dx} = \frac{1}{\lambda} f \implies$ f = $c_1 e^{\left(\frac{1}{\lambda}\right)x}$. I mean i.e. f(x) = $c_1 e^{\frac{x}{\lambda}}$. Now, we see that at x = 0 if I use the initial condition at x = 0, f(x) = 0 because f(x) = $\frac{1}{\lambda}\int_0^x f(t)dt$.

So, if x = 0, certainly f(0) = 0 from this definition, f(0) = 0 $\implies$ f = 0, whose contradict the hypothesis f ≠ 0 since f is an eigenvector of T. So, this is not possible. Now, if λ = 0 this case, for λ = 0 what we have? Then * $\implies$ f(x) = 0 $\forall x \in$ R. So, this implies f = 0 because if λ = 0, $\int_0^x f(t)dt = 0$, so in differentiating I will have f(x) = 0. So, I will have f = 0.

So, again f cannot be an eigenvector of T. So, we see that both the case when $\lambda = 0$ as well as $\lambda \neq 0$, the operator T does not have eigenvalue.

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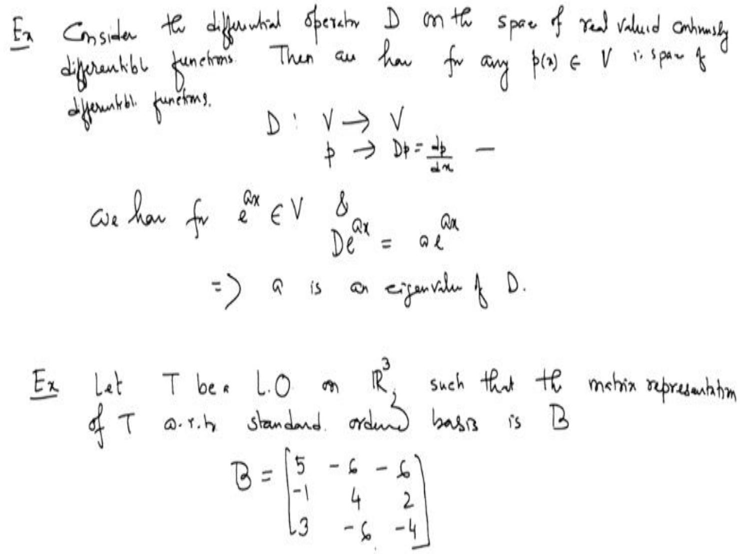

So, let me take another linear operator on the same space. Consider the differential operator D on the space of real valued continuously differentiable functions. So, the space is again infinite dimensional space and we have considered the operator as D that is differential operator. Then we have for any p, I can say p(x) ∈ V that is a space of differentiable functions, so D: V → V defined as p → Dp = $\frac{df}{dx}$.

So if I consider this differential operator on this space we have for e^{ax} ∈ V & De^{ax} = ae^{ax} $\Longrightarrow$ a is an eigenvalue of D. So, in this case we have the operator has eigenvalue even the space having dimension infinite. Let me take another example. Let T be a linear operator on R^n, I mean space of n-tuple.

Such that the matrix representation of T with respect to standard ordered basis B which is given by say, so let me take here n = 3 the space of 3-tuple, B = $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$.

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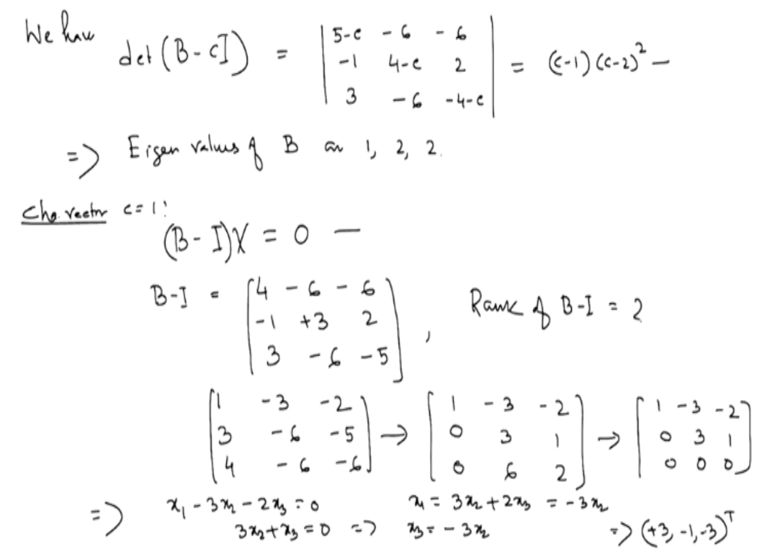

We have, det(B–cI) = $\begin{vmatrix} 5-c & -6 & -6 \\ -1 & 4-c & 2 \\ 3 & -6 & -4-6 \end{vmatrix}$ = (c-1) $(\text{c}-2)^2$. So for this matrix the characteristic polynomial is exactly same what I considered that first matrix A, 3x3 matrix that is a matrix representation operator. So, both A and this present matrix B have the same characteristic polynomial.

So, this implies eigenvalues of B = 1, 2 and 2. Let us quickly see what is the characteristic vector associated to 1 and 2 and 2. So characteristic vector associated to c =1 should be calculated by (B – I)X = 0. Now, the matrix (B – I) equal to what? We will have (B-I) = $\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}$. So B – I equal to this matrix. We can quickly check the Rank of this (B – I) = 2, how to show it? So that we can do through elementary row operations, anyhow you have to find the eigenvector.

So for this using elementary row operation then I am getting like this $\begin{bmatrix} 1 & -3 & -2 \\ 3 & -6 & -5 \\ 4 & -6 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -2 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So, rank of this matrix (B – I) = 2.

So, this implies I can find the solution will be your equation $x_1 - 3x_2 - 2x_3 = 0$ and $3x_2 + x_3 = 0$ So, this implies that $x_3 = -3x_2$ and here I am getting $x_1 = 3x_2 + 2x_3$, this implies $x_1 = -3x_2$. So,

$x_3 = x_1$. So this implies if I substitute x_2 = 1, I am getting x_1 = –3, x_2 = 1, and x_3 = –3. So, this transpose will be eigenvector of the matrix B, so I am getting or I can write out a plus, there is minus, this is a plus so this is (3,-1,3)

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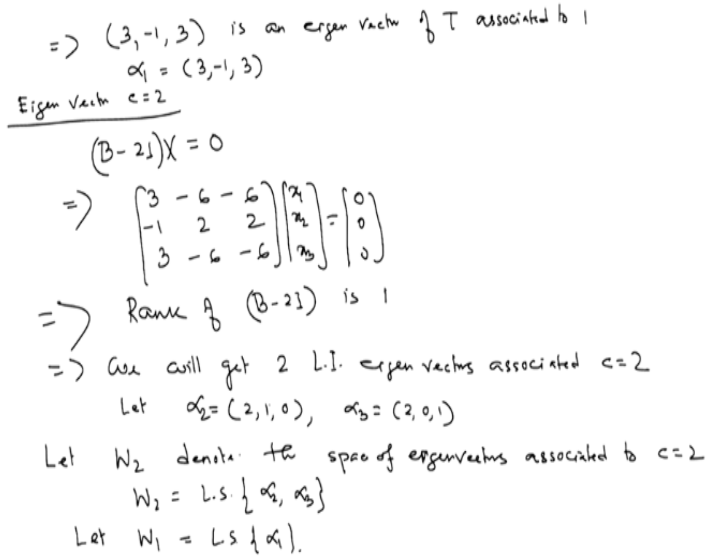

So, this implies (3,-1,3) is an eigenvector of T associated to 1. Let me represent that α_1= (3, –1, 3). Eigenvector associated to c = 2 that we can calculate the same way and this is equal to again (B – 2I)X = 0 $\Rightarrow = \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So, this implies immediately I can say the rank of (B – 2I) is 1 because all rows are basically scalar multiples of each other.

So, this implies I will have 2 free variables, so I will have two linearly independent eigenvectors. So, this implies we will get two linearly independent eigenvectors associated to c = 2. Let me consider α_2, I am sure you can find out about the two linearly independent eigenvector exactly same way what I did for the for c =1. So, let me write down the final solutions, say $\alpha_2 = (2, 1, 0)$ and α_3 = (2, 0, 1).

So, I will get two linearly independent eigenvectors like this. Let W_2 denote the space of eigenvectors associated to c = 2. So, that is W_2 = L.S.{α_2, α_3}. Let W_1 = L.S.{α_1}. So, then $W_1 \cap W_2$ of these two subspaces we already know that for different eigenvalues, the eigenspaces if we consider intersection of the two eigenspaces W_1, W_2 will be zero subspace.

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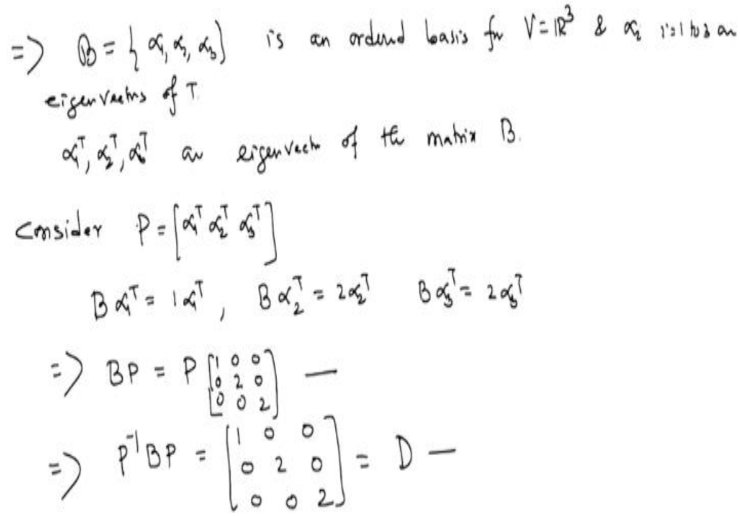

This implies if I consider $B = \{\alpha_1, \alpha_2, \alpha_3\}$ is an ordered basis for the space V = R^3 & α_i where i = 1, 2, 3 are eigenvectors of operator T. I can also say that here $\{{\alpha_1}^T, {\alpha_2}^T, {\alpha_3}^T\}$ are eigenvectors of the matrix B associated to eigenvalues 1, 2 and 2. Now consider a matrix P = $[{\alpha_1}^T, {\alpha_2}^T, {\alpha_3}^T]$. So, I have considered 3x3 matrix which columns are ${\alpha_1}^T, {\alpha_2}^T, {\alpha_3}^T$.

Then we have $B{\alpha_1}^T = 1{\alpha_1}^T$, $B{\alpha_2}^T = 2{\alpha_2}^T$, $B{\alpha_3}^T = 2{\alpha_3}^T$. So, $\Longrightarrow$ BP = P$\begin{bmatrix}1&0&0\\0&2&0\\0&0&2\end{bmatrix}$. Here this P matrix is invertible because there the $\alpha_1, \alpha_2, \alpha_3$ are basically linearly independent, so the corresponding columns will be also linearly independent.

So, $\Longrightarrow$ $P^{-1}BP = \begin{bmatrix}1&0&0\\0&2&0\\0&0&2\end{bmatrix} = D$. So, what I find here when two operators defined over the same spaces that is R^3 and both the operators which are having suppose they have matrix representation with respect to standard ordered basis say A and B.

Even though they have same characteristic values, but the nature of the operator or nature of the corresponding matrix representation are different. First matrix what I considered in the very beginning A we have seen it has only two eigenvectors associated to eigenvalues 1, 2, 2. So, we did not have any scope to write the first matrix A as some, we could not find any invertible matrix P such that $P^{-1}AP$ equal to some diagonal matrix.

But per second operator which has matrix representation with respect to standard ordered basis of B, we see that this B is similar to a diagonal matrix even though both the operators have same characteristic values. So, having same characteristic values does not mean that the operator or metrics will have same nature. Another important thing what we found it here that if an operator is defined over a finite dimensional space.

And the field is complex number we have the characteristic polynomials which is factorizable over the complex **(()) (26:33)** therefore the linear operator will have eigenvalues. But if I consider infinite dimensional space, then even this space is defined over the complex field then also the linear operator may or may not have eigenvalues that already you have seen through these two examples.

So, now let us try to understand what is the use of studying eigenvalues, eigenvector of a given linear operator R or T. Introduction of this eigenvalue or eigenvector is basically we are looking for the answer of a couple of questions, let me write down the questions.

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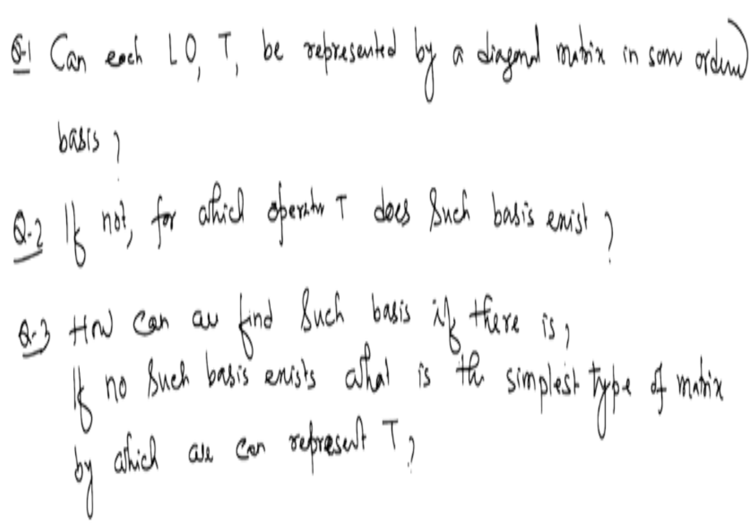

So, there are a couple of questions that can each linear operator(L.O.) T be represented by a diagonal matrix in some ordered basis? Because we have seen in our last example matrix A is not diagonalizable, I can say that it is not similar to a diagonal matrix, but B is similar to a diagonal matrix. Second question is if not for which operator T does such basis exist? Third is how can you find such basis if at all exist? If not, then what is the simplest type of matrix representation of T has and how to check it?

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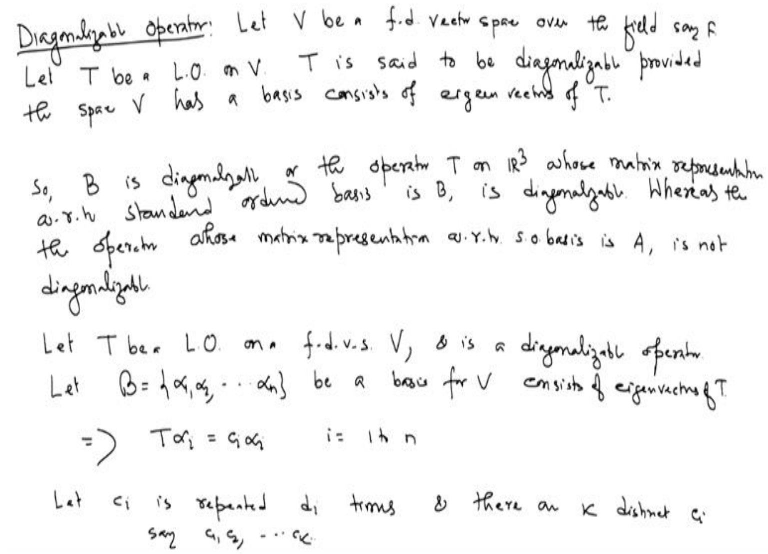

To answer the set of questions that we have raised let me initiate by introducing a terminology called diagonalizable operator. What is diagonalizable operator? Let V be finite dimensional vector space over the field say F, it may be a real number or a complex number. Let T be a linear operator on V. T is said to be diagonalizable provided the space V has a basis consists of eigenvector of T.

So, according to the definition of the diagonalizable operator if we recall last two operators whose matrix representation was A and B, so I can say the matrix B is diagonalizable or the operator T on R^3 whose matrix representation with respect to standard ordered basis B is diagonalizable whereas the operator whose matrix representation is not diagonalizable. Now, let me raise the question if the operator is diagonalizable what sort of benefit we are getting?

Let T be a linear operator on a finite dimensional vector space V and is a diagonalizable operator. So, let B = $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ be a basis for V consists of eigenvectors of T. So, this implies there exists n scalars $\{c_1, c_2, \dots c_n\}$ such that $T(\alpha_i) = c_i\alpha_i$, i = 1 to n. Here c_i may not be distinct, c_i may be repeated. Let c_i is repeated d_i times and there are k distinct c_i that is say $\{c_1, c_2, \dots c_k\}$ not to confuse, let me consider say c_1 is repeated to d_1 times, c_2 is repeated d_2 times and c_k is repeated d_k.

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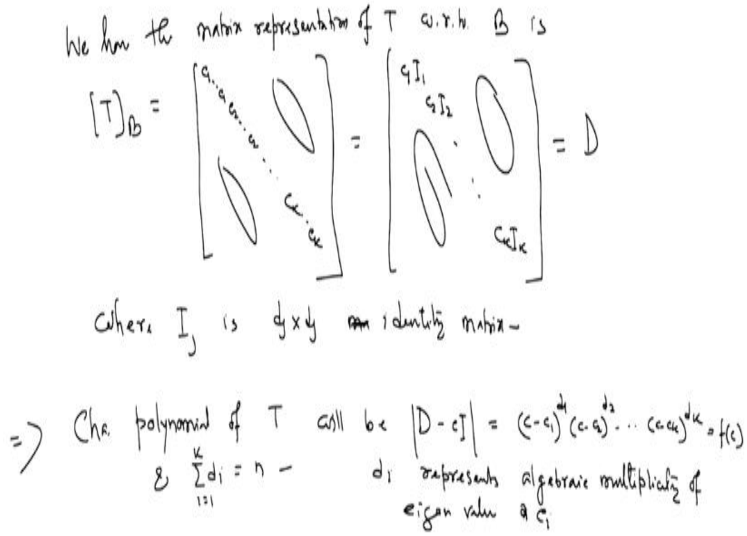

So, in this case we have the matrix representation of the operator T with respect to ordered basis B should be like this $[T]_B = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & c_{k-1} & 0 \\ 0 & 0 & 0 & 0 & c_k \end{bmatrix} =$ $\begin{bmatrix} c_1 I_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 I_2 & 0 & 0 & 0 \\ 0 & 0 & c_3 I_3 & 0 & 0 \\ 0 & 0 & 0 & c_{k-1} I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & c_k I_k \end{bmatrix} = D$ where I_j is d_jxd_j identity matrix. So, this implies the characteristic polynomial of T will be |D – cI| = $(c-c_1)^{d_1}(c-c_2)^{d_2} \dots .(c-c_k)^{d_k}$ = f(c) so you see that it is a characteristic polynomial of the operator T and here, $\sum_{i=1}^{k} d_i$ = n, , d_i represents algebraic multiplicity of eigenvalue c_i.

Power of the (c-c_i) in the characteristic polynomial will denote the algebraic multiplicity of the corresponding eigenvalue. Now, suppose we are interested to know what about the eigenspace associated to the eigenvalue c_1?

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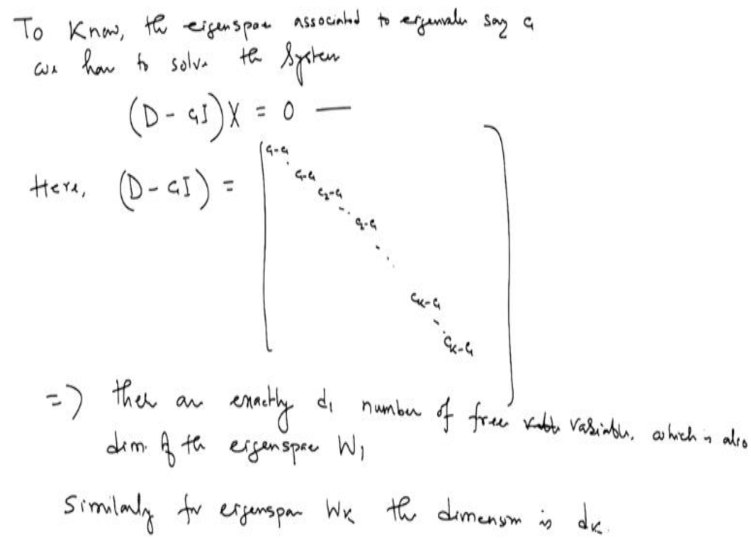

To know the eigenspace associated to eigenvalue say for sake of simplicity I take c_1 only, we have to solve the system (D – c_1X) = 0, we have to solve this system. Here if I consider, (D – c_1I)=

$$\begin{bmatrix} c_1 - c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 - c_1 & 0 & 0 & 0 \\ 0 & 0 & c_3 - c_1 & 0 & 0 \\ 0 & 0 & 0 & c_{k-1} - c_1 & 0 \\ 0 & 0 & 0 & 0 & c_k - c_1 \end{bmatrix}$$

. Since c_1 is different from c_2, c_3, … c_k therefore this shows that the first row, second row up to d_1 rows all will be zero rows.

Next after d_1 rows all other rows are nonzero rows. So, this implies that I have only exactly d_1 number of free variables. So, this implies there are exactly d_1 number of free variables that is exactly d_1 number of zeros are on the principal diagonal of the matrix (D –c_1I) and that is the basically number of free variables, which is also dimension of the eigenspace W_1. Similarly, if you go for c_2 or c_3 or c_k, the number of zeros in (D –c_kI) will give you the dimension of the eigenspace W_k.

Similarly, for eigenspace W_k, the dimension will be d_k. So, if the matrix operator is diagonalizable, then we are able to say what is the eigenspaces, immediately you can calculate what is the dimension of the corresponding eigenspaces, we can see most of the characteristic of the operator immediately. So, that is why the diagonalizability is I can say most simplified structure when representation of an operator.

If you say that if it is diagonalizable, then operator is very simple. However, we have also seen that there are such situations even the operator is defined over finite dimensional space, but it is not diagonalizable. What should be the next step to understand the corresponding operator which is not diagonalizable, to understand what will be the most simplified structures, we will continue more in my next class.