Advanced Linear Algebra

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Lecture – 23
Eigenvalue and Eigenvector of Linear Operator - 1

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Let V be a f.d. v. space & W also f.d. v.s. over a field F
Then Collection of all L.T. L(V,W) is also a v.s.

Let S denote set of all mxn matrix over F

$L(V,W) \xrightarrow{1-1} S$

$L(V,V)$

Eigen Value of a Linear operator: Let T be a linear operator on V & V is a vector space over F. Let $c \in F$. c is said to be an eigenvalue of T provided there exist $0 \neq \alpha \in V$ such that $T\alpha = c\alpha$ –

Here, α is called as eigenvector of T associated to eigenvalue c.

$\therefore$ $T\alpha = c\alpha \Rightarrow (T - cI)\alpha = 0$

So, welcome to lecture series on advanced linear algebra. We have studied linear transformations, linear operators, linear functionals. We have also seen that if a linear transformation T is defined over a finite dimensional vector space, I mean if V be a finite dimensional vector space(f.d.v.s.) and W also finite dimensional vector space over a field F, then we have seen that T is defined then collection of all linear transformations that is I have denoted as L(V,W) is also a vector space.

And we have seen if the dimension of V is n and dimension of W is m, then dimension of L(V,W) is (mn). Apart from that we have also seen that if I consider say let S denote set of all mxn matrix over F, then there is a 1-1 onto map between the set of L(V,W) and S, there is a 1-1 map between these two sets. And for each linear transformation in L(V,W) then we say one can associate mxn matrix about the same way.

Now, I will basically stick to instead of linear transformation, linear operators, I mean I will stick to this space L(V,V) set of all linear transformation from V into V. So, this is basically collection of all linear operator on V. So finite dimensional space for a given ordered basis one can have matrix representations of the linear operator also. So, one can associate each linear operator by in an nxn square matrix also.

Now, we want to introduce an interesting phenomenon for the linear operator, what is that? Eigenvalue of a linear operator. Let T be a linear operator on V & V is a vector space over the field F. Let c ∈ F, c said to be an eigenvalue of operator T provided there exist $0 \neq \alpha \in V$ such that $T(\alpha) = c\alpha$. Here alpha is called as eigenvector of the T associated to eigenvalue c. So this implies $T(\alpha) = c\alpha = (T\text{-}cI)\alpha = 0$.

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$(T - cI)\alpha = 0$ for some nonzero $\alpha \in V$

$\Rightarrow (T-cI)$ is singular operator.

If V be a f.d.v.s. say dim n. Then for a given ordered basis T can be represented by an nxn matrix say A.

Eigen value of a square matrix: Let A be an nxn matrix over a field F. Any scalar $c \in F$ is said to be eigenvalue of A provided there exist a nonzero column vector α such that

$$A\alpha = c\alpha$$

$$\Rightarrow (A - cI)\alpha = 0$$

$\therefore$ $(A - cI)$ is not invertible

$$\Rightarrow \det(A - cI) = 0$$

T is a linear operator on V and I is also linear operator on V, so (T – cI) is also linear operator on V. Now $(T\text{-}cI)\alpha = 0$, for some $0 \neq \alpha \in V$ implies (T – cI) this operator mapping nonzero vector to zero vector, so (T – cI) is singular operator, when the operator is singular. If V be a finite dimensional vector space say dimension n, then T can be associated with the nxn matrix.

Then for a given ordered basis T can be represented by a nxn matrix say A. So, if it is finite dimensional then we can associate this operator by nxn square matrix if the dimension of this vector V is n and this matrix is also defined over the same field F. So, one can also define the concept of eigenvalue for a square matrix also. So, let me define concept of eigenvalue of a metric space. So, let me define eigenvalue of a square matrix. Let A be a nxn matrix over field say F.

Any scalar $c \in F$ is said to be an eigenvalue of A provided there exist a nonzero column vector α such that $A\alpha = c\alpha$. So, this matrix this is an nxn matrix A and I is the nxn identity matrix. So $(A - cI)\alpha = 0$. So, since there exists $0 \neq \alpha$ such that $(A - cI)\,\alpha = 0$, so $(A - cI)$ is not invertible. So, this implies that $\det(A - cI) = 0$.

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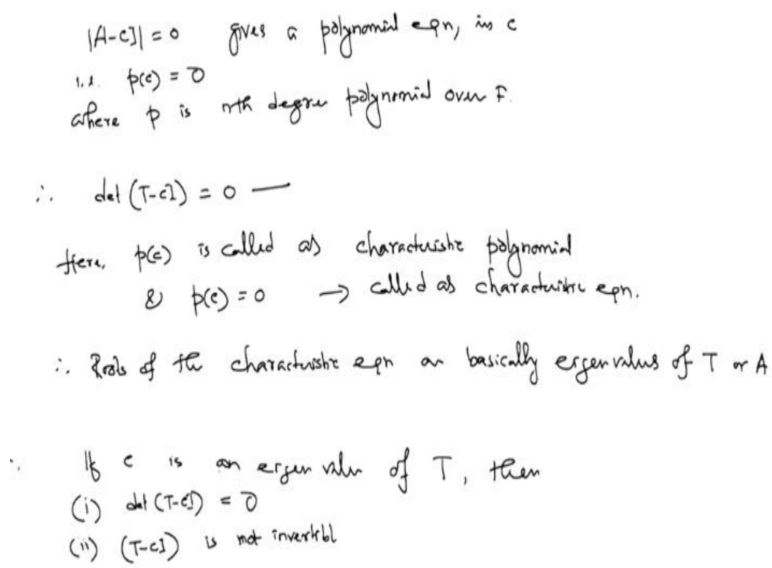

So, this gives $|(A–cI)| = 0$ gives a polynomial equation in c and degree of the polynomial that is of the form I can say something like $p(c) = 0$ where p is an n-th degree polynomial over the field F. So, I can see that when T is a linear operator on a finite dimensional space and c is an eigenvalue of the operator T, then $(T–cI)$ is singular so in that case I can say that determinant of $(T–cI) = 0$.

I mean when T is basically represented by some matrix A, then $\det(A–cI) = 0$ which is equivalent to saying that $\det(T–cI) = 0$. Here this p© is called as characteristic polynomial and $p(c)= 0$ called as characteristic equation. Roots of the characteristic equation basically eigenvalues of T or A. So, if c is an eigenvalue of T, then

(i) $\det(T–cI) = 0$.

(ii) $(T–cI)$ is not invertible.

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Let c be an eigenvalue of T. Let W be the collection of all $\alpha \in V$ such that $T\alpha = c\alpha$. W is called as eigenspace of T associated to eigenvalue c.

Claim W is a subspace of the vector space V.

For α & $\beta \in W$ we have

$$T\alpha = c\alpha \quad \& \quad T\beta = c\beta$$

$$\therefore \quad T\alpha + T\beta = c\alpha + c\beta$$

$$\therefore \quad T(\alpha+\beta) = c(\alpha+\beta) \qquad [\because T \text{ is a L.T}$$

$\therefore \alpha+\beta \in W$.

Again for $d \in F$ & $\alpha \in W$

we have $T(d\alpha) = dT(\alpha) = dc\alpha$

$$\Rightarrow T(d\alpha) = c(d\alpha)$$

$\therefore d\alpha \in W$.

Another important terminology is like this let c be an eigenvalue of operator T. Let W be the collection of all $\alpha \in$V such that the relation Tα = cα is satisfied then this W is called as eigenspace of T associated to eigenvalue c. So, we have defined what do you mean by eigenspace of a linear operator associated to an eigenvalue c. Claim: W is a subspace of the vector space V, how?

For α & $\beta \in$ W, we have Tα = cα and Tβ = cβ. Tα + Tβ = cα+cβ. So, T($\alpha + \beta$) = c(α+β). So, it is a linear transformation so this assumes this property, I can have T($\alpha + \beta$) = c(α+β). So, this implies if α & β in W, I mean if it is eigenvector associated to the eigenvalue c(α+β) is also eigenvector of T associated with eigenvalue c.

In fact, your (α+β) $\in$ W. Again, for any d $\in$ F and $\alpha \in$ W, I have to show that (dα) $\in$ W. We have T(dα) = dT(α) = dcα. So, this implies that T(dα) = c(dα). So this implies (dα) $\in$ W. So, W is a subspace of the vector space V.

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Let A & B be any two similar matrices over F.
claim characteristic polynomial of A = characteristic polynomial of B.

Sol^n Let $p(c)$ denote characteristic polynomial of A

$\therefore$ $p(c) = |A-cI|$ — (*)

Given A is similar to B

$\therefore$ there exist an nxn invertible matrix P such that

$A = P^{-1}BP$ — (1)

$\therefore$ $p(c) = |A-cI| = |P^{-1}BP - cP^{-1}P|$

$\therefore$ $p(c) = |P^{-1}|\,|B-cI|\,|P|$

$= |B-cI|$

$\therefore$ characteristic polynomial of B = $p(c)$.

Another interesting thing is let A & B be any two similar matrices over the field F. Claim: Characteristic polynomial of A = characteristic polynomial of B, how? Let p(c) denote characteristic polynomial of A. So p(c) = |A–cI|. Given A similar to B, so according to definition of similar matrices there exist an nxn invertible matrix P such that, $A = P^{-1}BP$. So this relation holds good.

So, one can write down that $p(c) = |A-cI| = |P^{-1}BP - cP^{-1}P| = |P^{-1}|\,|B-cI|\,|P| = |B-cI|$. So, characteristic polynomial of B = p(c). So, both A & B have the same characteristic polynomials. This means what? This means both of them have same characteristic equations, means both of them have same characteristic values. So, if A and B are two similar matrices, then the characteristic values of both will be same or in terms of linear operator one can also say similar way.

Suppose T_1 & T_2 are two linear operators defined on a finite dimensional vector space V where T_1 & T_2 both are similar, I mean there exist a linear operator say U such that $T_1 = U^{-1}T_2U$. So, in this case the characteristic polynomial of T_1 and characteristic polynomial of T_2 will be same. So, we have seen the definition of characteristic values of a linear operator or a square matrix. So, let us consider some examples.

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Ex Let T be a L.O. on $\mathbb{R}^2$, such that the matrix representation of T w.r.t. standard ordered basis B is

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We have $\det(A - cI) = \begin{vmatrix} -c & -1 \\ 1 & -c \end{vmatrix} = c^2+1$

$\therefore$ c^2+1 is not factorizable over $\mathbb{R}$

i.e. $c^2+1 = 0$, $c = \pm i$

$\therefore$ A does not have eigenvalue over $\mathbb{R}$

But it has eigenvalue over $\mathbb{C}$.

First, I will read one small question that is it necessary that every linear operator will have eigenvalue? Let us see through this example. Let T be a linear operator(L.O.) on finite dimensional space R^2 such that the matrix representation of T with respect to standard ordered basis B is say A = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Now, question is let us see whether this operator has eigenvalue over the real line or not.

We have det(A – cI) = $\begin{bmatrix} -c & -1 \\ 1 & -c \end{bmatrix}$ = c^2+1. So, c^2+1 is not factorizable over real number. That is c^2+1= 0 if you consider this quadratic equation, we have c $=$ $\pm$i the only solution. So, A does not have eigenvalue over real line, but it has eigenvalue over complex number C.

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Ex-2

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

we have $\det(A - cI) = (c-1)(c-2)^2$

$\therefore$ cha. eqn: $(c-1)(c-2)^2 = 0$

$\therefore c = 1, 2, 2$

$\therefore$ A has eigenvalues 1, 2, & 2.

Eigenvector associated to eigenvalue 1:

Let $\alpha = (x_1, x_2, x_3)^T$

$(A - I)\alpha = \bar{0}$

$$\therefore \left(\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let me take another example. Say let me consider A = $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$, we have det(A – cI) = (c-1) $(c-2)^2$. So characteristic equation that is (c-1) $(c-2)^2 = 0$ for c = 1, 2, 2 are the eigenvalues of the matrix A.

So, A has eigenvalues 1, 2, and 2. To say that 1 is the eigenvalue of the matrix A means I have to show that there exists a nonzero α such that A$\alpha = 1\alpha$. Similarly, to claim that 2 is an eigenvalue of the matrix A means I have to show there exist some nonzero β such that A β = 2 β . So, for this now question is how to find the eigenvector associated to the eigenvalue c, so eigenvector associated to eigenvalue 1?

So let me show how to calculate this one. Let $\alpha = (x_1, x_2, x_3)^T$ so that (A –I)α = 0 is satisfied. So this implies that, $\left\{ \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, I will have to solve this system basically.

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$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad -$$

$2x_1 + x_2 - x_3 = 0 \quad 2x_1 = x_3 - x_2$

$2x_1 + 2x_2 - x_3 = 0$

$x_2 = 0 \quad \therefore x_1 = x_3/2$

$\therefore$ $v_1 = (1/2, 0, 1)^T$ as sol^n of

$(A-I)\alpha = 0$

$\therefore$ v_1 is eigen vector associated to eigen value $c = 1$

Similarly, we can find out the eigen vector associated to eigen value $c = 2$

$v_2 = (1, 1, 2)^T$

So, this means that I have to solve basically what? $\begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So I have to solve basically this system. If you see A – I this matrix which is basically giving as this 2, 1, –1; 2, 1, –1; 2, 2, –1 certainly det(A – I) = 0. I mean the rank of this matrix will be certainly less than 3, we can immediately see that two rows are same and you can quickly check that this is having rank equal to basically 2.

So, rank equal to 2 means this system has solution space having dimension is 1 because rank + nullity = dimension of the space. Here dimension of space equal to 3, so rank is equal to we can quickly check this matrix is 2, so the nullity will be 3 – 2 = 1. So, let us calculate that solution space. So, this means that I have to solve $2x_1 + x_2 - x_3 = 0,\ 2x_1 + 2x_2 - x_3 = 0$. So, if I solve it, I will have basically $2x_1 = x_3 - x_2$ and $x_2 = 0$.

So, I will have $x_1 = \frac{x_1}{2}$. So, put $x_3 = 1$, I will have $v_1 = (\frac{1}{2}, 0, 1)^T$ as solution of (A –I)α = 0. So, v_1 is eigenvector associated to eigenvalue c = 1. Similarly, we can find out the eigenvector associated to eigenvalue c = 2. If I do the same way, similarly we can find out the eigenvector associated to eigenvalue c = 2 as $v_2 = (1, 1, 2)^T$.

So, we see that how $v_1 = (\frac{1}{2}, 0, 1)^T$ or we can multiply with 2 which is scalar quantity then it is basically $v_1 = (1, 0, 2)^T$ is the eigenvector associated to the eigenvalue 1 for the given matrix A. Exactly same way one can also find out the eigenvector associated to eigenvalue c = 2 and that has

$v_1 = (1, 1, 2)^T$. Note that this v_1 and v_2 are linearly independent.

I meant to say if the eigenvalues of the matrix A are distinct, then these corresponding eigenvectors are also linearly independent. You can consider it as a homework and check yourself. We will continue more about this eigenvalue, eigenvectors in the next class. Thank you.