Advanced Linear Algebra
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Lecture – 22
Linear Functional and the Transpose of Linear Transformation - 2

So, we have seen in our last lecture for a given linear transformation from a vector space V into vector space W, it introduces another linear transformation the T^* from V^* to V^*. In fact, this is a unique element for each linear transformation T introduces a unique linear transformation T^* from W^* to V^* such that $T^t(g) = gT$, basically. I mean to say T^t operating on g, $g \in W^*$, $T^t g(\alpha) = g(T(\alpha))$ where any vector $\alpha \in V$.

I mean to say this $T^t g$ is a basically linear functional on V^* So, for a given linear transformation T, we have another linear transformation $T^t: W^* \rightarrow V^*$. As in the case of linear transformation T from V to W, we know when V is the finite dimensional vector space, then we know, rank of T + nullity of T = dimension of V.

Question is whether similar type of results hold good for the case of this transpose of linear transformation T, that is T^t? I mean to say if the rank of T is known to us what way this will be related to the rank of T transposes? How the range space of T transpose, how the null space of T transpose is related by the null space or range space of T? So, these things we have to study. Here I am claiming like this.

Say let V and W be vector spaces over the field F and T be linear transformation from V into W, then the null space of T transpose is the annihilator of the range space of T? If the range space of T = R_T, then ${R_T}^o$ basically the null space of T^t. So, this is true irrespective of V and W whether finite dimensional or infinite dimensional, it is true for both the cases. If V and W both are finite dimensional in that case, rank of T = rank of T^t.

And the range space of T, T transpose is the basically annihilator of the null space of T. I mean range space of T transpose is if we consider N is the null space of T, then range space of $T^t = N^o$ that is the annihilator of the null space of the N, null space of transpose of T.

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Theorem: Let V & W be vector spaces over a field F, and let T be a linear transformation from V into W. The null space of T^t is the annihilator of range of T. If V & W are finite dimensional then

(i) $\text{rank}(T^t) = \text{rank}(T)$

(ii) the range of T^t is the annihilator of the null space of T.

Pf! Let N_{T^t} denote null space of T^t

$T^t : W^* \to V^*$

for any $g \in N_{T^t}$,

$T^t(g) = gT = 0$ i.e. zero functional

$\Rightarrow (T^t g)(\alpha) = g(T(\alpha)) = 0$ for $\forall \alpha \in V$

$\Rightarrow g \in R_T^0$, where R_T denote range of T & R_T^0 is the annihilator of R_T

$\Rightarrow N_{T^t} \subset R_T^0$ —(1)

So, let us give a proof of this result, very nice interesting result so that later on we will correlate this concept with the matrix. Although, we have already used the concept of transpose of a matrix, all these things we have already used, but the relation between the transpose of the matrix and linear transformation is not studied so far, so that we are going to highlight and you will come to that part later on.

But before that let me; correlate the range pace of T^t and null space of T^t with some relation with the range space of T and null space of T. First claim is that the null space of T^t is the annihilator of the range of T. Let N_{T^t} denote null space of T transpose. We know T^t: $W^* \to V^*$. For any $g \in N_{T^t}$ we have from the definition, $T^t(g) = gT = 0$ i.e. zero functional.

This implies that $(T^t g)(\alpha) = g(T(\alpha)) = 0$, for $\forall\, \alpha \in V$. This implies, see since g is annihilating all elements in the range space of T, so $g \in {R_T}^o$ where R_T denotes range of T and ${R_T}^o$ is the annihilator of R_T. We said all the linear functional which annihilate R_T is called ${R_T}^o$. So, $g \in {R_T}^o$, this implies that $N_{T^t} \subset {R_T}^o$.

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If $g \in R_T^0 \Rightarrow g(T(v)) = 0$

i.e. $g(T(\alpha)) = 0 \ \forall \alpha \in V$

$\Rightarrow g \in N_{T^t}$

$\therefore R_T^0 \subset N_{T^t}$ — (2)

$\therefore$ (1) & (2) $\Rightarrow R_T^0 = N_{T^t}$ —

· We have for finite dimensional vector spaces V & W

$\dim R_T + \dim R_T^0 = \dim W$ -

If dim of W is m & dim of V is n

then $\dim R_T^0 = \dim W - \dim R_T = m - \dim R_T$

Let rank of T be r

$\Rightarrow \dim R_T^0 = m - r = \dim N_{T^t}$

Again if $g \in {R_T}^o$, this implies g(T(v)) = 0, i.e. g(T(α)) = 0 $\forall \alpha \in$ V. So, this implies, $g \in N_{T^t}$ because g annihilating entire range space meeting all our purpose. So, g belongs to this one, so we have ${R_T}^o \subset N_{T^t}$. So equation(1) & equation(2) implies ${R_T}^o = N_{T^t}$.

We have for finite dimensional vector space V and W, for finite dimensional vector space V & W, we have dimension of R_T + dimension of ${R_T}^o$ = dimension of W. So, if dimension of W is m and dimension of V is n, then dimension of ${R_T}^o$ = dimension of W – dimension of R_T = m – dimension of R_T. Let rank of T be r. So, this implies dimension of range space of T = r.

So, this implies, dim of ${R_T}^o$ = (m – r) = N_{T^t}, because we have seen that null space of T^t, I mean transpose of T is basically annihilator of the range space of T. So, that is why dimension of N_{T^t} = (m – r).

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∴ From rank-nullity theorem

dim N_{T^t} + rank(T^t) = dim W^* = m

$\Rightarrow$ rank(T^t) = m − (m−r) = r = rank (T)

Let N denote null space of T

claim $N^0 = R_{T^t}$

We have for any $f \in R_{T^t}$, then exist $g \in W^*$

such that $f = gT = T^t(g)$

$\Rightarrow$ for $\alpha \in N$, $f(\alpha) = 0$ ∵

$f(\alpha) = g(T(\alpha)) = 0$

$\Rightarrow$ $f \in N^0$

$\Rightarrow$ $R_{T^t} \subset N^0$.

So, if dimension of N_{T^t} = (m – r), then from rank-nullity theorem we have, dim (N_{T^t}) + rank(T^t) = dim of W^* = m. So, this implies, rank(T^t) = m – (m – r) = rank(T). So, rank of T^t and rank of T are same, this is a very fundamental result and classical also. So, we see the rank of linear transformation T and its transpose are same if these are defined over the finite dimensional vector spaces.

Now, we want to show other result that is the range of T^t is the annihilator of the null space of T. Let N denote null space of T. Claim: $N^0 = R_{T^t}$. We have for any f ∈ R_{T^t}, there exist g ∈ W^* such that, f = gT = T^t(g).

So, this implies for $\alpha \in$ N, f(α) = 0 because f(α) = g(T(α)) = 0, because α is null space of T, so it is 0. So, this implies f ∈ N^0 , I mean f annihilates your N. So any element f ∈ R_{T^t} annihilate N. So, this implies $R_{T^t} \subset N^0$.

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dim N^0 = ?

We have dim N = $n -$ dim $R_T = n - r$

$\therefore$ dim N + dim N^0 = dim $V = n$

$\Rightarrow$ dim $N^0 = n - (n-r) = r =$ dim R_{T^t}

$\therefore$ $R_{T^t} = N^0$

<u>Theorem</u>: Let V & W be two finite dimensional vector spaces over a field F. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$ & $\mathcal{B}' = \{\beta_1, \beta_2, \cdots \beta_m\}$ be ordered bases of V & W respectively. Let $\mathcal{B}^* = \{f_1, f_2, \cdots f_n\}$ & $\mathcal{B}'^* = \{g_1, g_2, \cdots g_m\}$ be the dual for $\mathcal{B}$ & $\mathcal{B}'$ respectively. Let T be a linear transformation from V into W, with $[T]_{\mathcal{B},\mathcal{B}'} = A$ & $[T^t]_{\mathcal{B}'^*,\mathcal{B}^*} = B$. Then

$A_{ij} = B_{ji}$ — $1 \leq i \leq m$ & $1 \leq j \leq n$.

We know the dim(R_{T^t})= r. Now, let me see what is the dim(N^o) = ? We have dim(N) = n – dim(R_T) = (n – r). So, dim(N) = (n – r). So, dim(N) + dim(N^o) = dim(V) = n. So, this implies your dim(N^*) = n – (n – r) = r = dim(R_{T^t}). So $R_{\mathrm{T}^t} = \mathrm{N}^0$.

Since both are having same dimension and range space of T is a subspace of N^*, so this implies $R_{\mathrm{T}^t} = \mathrm{N}^*$. So, range space of T transpose equal to the annihilator of the null space of linear transformation T. So, this nice relation we have established, now let us go to the matrix representation of the transpose of the linear transformation. So, for this let us write down this result.

When I am talking about the matrix representations, certainly I am saying that V and W are both finite dimensional vector spaces. So, let V and W be finite dimensional vector spaces over a field say capital F. Let B = $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ and $\mathrm{B}' = (\beta_1,\ \beta_2,\ \dots\ \beta_m)$ be ordered bases of V and W respectively. Let $\mathrm{B}^* = (f_1, f_2, \dots f_m)$ and $\mathrm{B}'^* = (g_1, g_2, \dots g_m)$ be the dual for B and B' respectively.

I mean they are the bases for the V^* & W^* respectively. Let T be a linear transformation from V into W with, $[T]_{B,B'}$ = A, where A is nxn matrix & $[\mathrm{T}^t]_{B^*,B^*}$ = B, where B is mxm matrix. Then $A_{IJ} = B_{IJ}$ 1≤i≤m,1≤j≤n. So, if B stands for the transpose matrix, I mean matric representation of the transpose linear transformation of T and if A denotes the matrix representation of T, then A and B are related like this, basically A = transpose of B or B = transpose of A.

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Pf: It is given $T: V \to W$

$$[T]_{B,B'} = A \quad m \times n \text{ matrix over } F$$

$$\& \quad [T^t]_{B'^*,B^*} = B \quad n \times m \quad " \quad "$$

$$\therefore \quad [T(\alpha_j)]_{B'} = A_j \quad \text{i.e. } j\text{th column of } A$$

$$\therefore \quad T(\alpha_j) = \sum_{k=1}^{m} A_{kj} \beta_k \; — \qquad \& \quad T^t(g_j) = \sum_{k=1}^{n} B_{kj} f_k \; —$$

$$\text{Now} \quad \left(T^t(g_i)(\alpha_j)\right) = g_i(T(\alpha_j))$$

$$= g_i\left(\sum_{k=1}^{n} A_{kj} \beta_k\right) \qquad i = 1 \text{ to } m$$

$$= \sum_{k=1}^{n} A_{kj}\, g_i(\beta_k) = A_{ij} \quad \because \; g_i(\beta_k) = \delta_{ik}$$

So, let us give the proof. It is given T: V → W and $[T]_{B,B'}$ = A, where A is mxn matrix over F and $[T^t]_{B'^*,B^*}$ = B, where B is nxm matrix over F. So, if we consider, $\left[T(\alpha_j)\right]_{B'} = A_j$. i.e. it is basically j-th column of matrix A. So $T(\alpha_j) = \sum_{k=1}^{m} A_{k_j} \beta_k$ & $\mathrm{T}^t(g_j) = \sum_{k=1}^{n} B_{k_j} f_k$. $\left(\mathrm{T}^t(g_j)(\alpha_j)\right) = g_i\left(T(\alpha_j)\right) = g_i(\sum_{k=1}^{n} A_{k_j} B_k) = \sum_{k=1}^{n} A_{k_j} g_i(B_k) = A_{IJ}$, Where i=1 to m, beacuse $g_i(\beta_k) = \delta_{ik}$.

This implies like this thing, so $(\mathrm{T}^t(g_i))(\alpha_j) = A_{ij}$.

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$$T^t(g_i) = \sum_{k=1}^{n} B_{ki} f_k \qquad \text{from (ii)}$$

$$\Rightarrow \left(T^t(g_i)\right)(\alpha_j) = \left(\sum_{k=1}^{n} B_{ki} f_k\right)(\alpha_j)$$

$$= \sum B_{ki} f_k(\alpha_j)$$

$$= B_{ji} \; — \; \text{(iv)}$$

$$\therefore \text{(iii)} \; \& \; \text{(iv)}$$

$$\Rightarrow \quad A_{ij} = B_{ji} \qquad \text{for} \quad 1 \le j \le n \; \& \; 1 \le i \le m$$

$\therefore$ B is the transpose of A.

At the same time according to our given conditions, $\mathrm{T}^t(g_i) = \sum_{k=1}^{n} B_{ki} f_k$. So, this implies if this information given this is 1, this is 2, so from 2 we have like this thing. So, $\left(\mathrm{T}^t\left(g_i\right)\right)\left(\alpha_j\right) = \left(\sum_{k=1}^{n} B_{ki}\, f_k\right)(\alpha_j) = \sum_{k=1}^{n} B_{ki}\, f_k(\alpha_j) = B_{ji}$. So, this is equal to when k = j then only I will have 1, otherwise 0.

This is suppose 4. So, 3 and 4 implies that $A_{ij} = B_{ji}$ for $1 \le i \le m, 1 \le j \le n$. So, B is the transpose of a matrix A, so that is all the proof. So, we see that the matrix representation of T transpose(T^t) is basically transpose of the given matrix representation of the linear transformation T.

So, we see that the image of $(\mathrm{T}^t(g_i))(\alpha_j)$, so in this case it is basically B_{ji} that is the g_i, i-th entry of the matrix B whereas previously we have already obtained the $(\mathrm{T}^t(g_i))(\alpha_j)$ is basically the i-jth entry of the matrix A that is A_{ij}. So, since both are same scalar quantity, certainly $A_{ij} = B_{ji}$. So, this implies matrix representation of the transpose of transformation T is basically the transpose of the matrix representation of the linear transformation T.

So, B is the transpose of A. I hope this is clear to all of you and you can utilize this concept for proving some other nice results in the linear transformation. Thank you.