Advanced Linear Algebra

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Lecture – 21
Linear Functional and the Transpose of Linear Transformation - 1

So, welcome to this lecture series. In our last lecture, we have seen that if we consider a vector space V of dimension say n and V^* with a corresponding dual space, then for any ordered basis $(\alpha_1, \alpha_2, \ldots \alpha_n)$ of V, there exist a dual basis $f_1, f_2, \ldots f_n$ for V^*. Of course, we have seen in last example when you consider a dual basis $f_1, f_2, \ldots f_n$ of vector space V^*, we have also found a basis $\alpha_1, \alpha_2, \ldots \alpha_n$ for the corresponding space V that we have checked through the numerical examples.

But in general, we have not answered the questions if V be a vector space defined over the filter F and V^* with the correspondent dual space and if f_1, f_2, f_3 be the basis for V^* the question is whether for each sub basis is there a dual of some basis for V. I mean this is f_1, f_2, f_3 this basis for V^* is a dual for some basis for the V that is the question. I mean does that exist $\alpha_1, \alpha_2, \ldots \alpha_n$ in V which would basis for the V. To answer this question, we can proceed like this.

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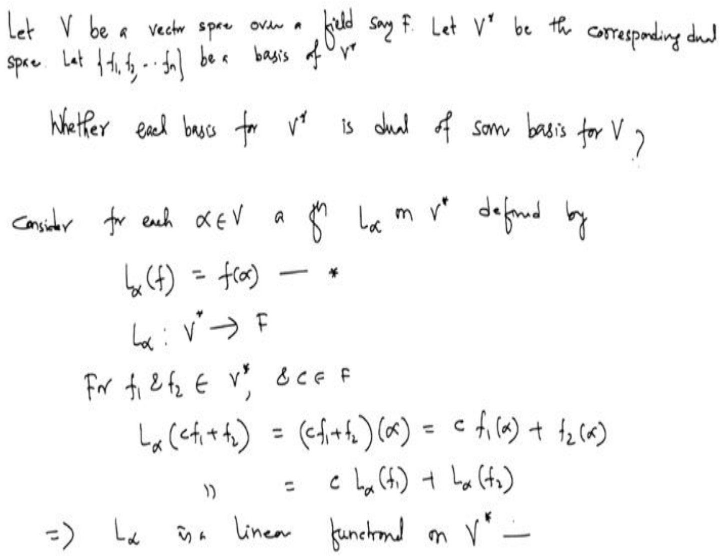

Consider for each $\alpha \in$ V a function L_α on V^* defined by L_α(f) = f(α). So, basically this implies it is a function, so L_α: $V^* \to F$ this one. This function we can check it is also linear function, I mean for f_1 & $f_2 \in V^*$ and c ∈ V, $L_\alpha(cf_1 + f_2) = (cf_1 + f_2)(\alpha) = cf_1(\alpha) + f_2(\alpha) = c\,L_\alpha(f_1) + L_\alpha(f_1)$L. So, this implies this L_α is a linear functional on V^*. So, for each alpha we have introduced a linear functional L alpha like this.

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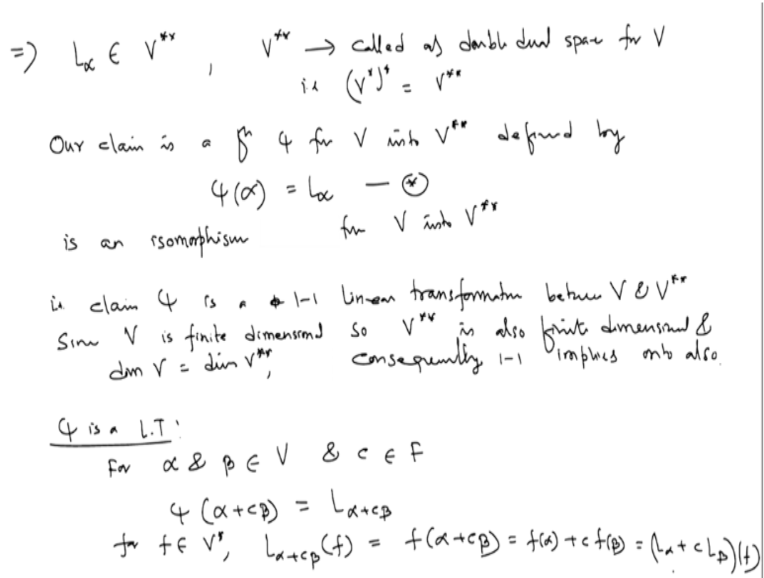

So, this implies that $L_\alpha \in V^{**}$, star, this is called the double dual space. V^{**}, star is called as double dual space for vector space V, i.e. $(V^*)^* = V^{**}$. So, we see that for each α one can introduce a linear functional L_α defined like what we have already done. Now, my claim is that a function φ from V into V^{**} defined by the $\varphi(\alpha) = L_\alpha$, is an isomorphism from V into V^{**}.

So the way we have introduced L_α and then I am defining a function φ from V into V^{**} defined by $\varphi(\alpha) = L_\alpha$. Now, my claim is that this φ is an isomorphism from V into V^* that is claim φ is a 1-1 linear transformation between V and V^{**}. Since, V is finite dimensional so V^{**} is also finite dimensional and, dimension of V = dimension of V^{**}.

Consequently, if I somehow prove that φ is a 1-1 linear transformation from V into V^{**} , then φ is also 1-1 onto linear transformation from V into V^{**} that is φ will be an isomorphism from V onto V^{**}. Consequently 1-1 implies onto also. So, first I have to prove that φ is a linear transformation. So, let us see how φ is a linear transformation. φ is a linear transformation(LT).

For α & β ∈ V and c ∈ F, we have $\varphi(\alpha + c\beta) = L_{\alpha+c\beta}$. Claim: $L_{\alpha+c\beta} = L_\alpha + c\,L_\beta$. For f ∈ V^*,

$L_{\alpha+c\beta}$ (f)= f(α+cβ) = f(α)+cf(β) = (L_α+ c L_β)(f).

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$\Rightarrow L_{\alpha+c\beta} = L_\alpha + c L_\beta$

$\Rightarrow \varphi(\alpha+c\beta) = L_{\alpha+c\beta} = L_\alpha + c L_\beta = \varphi(\alpha) + c\,\varphi(\beta)$

$\therefore$ φ is a L.T.

For $\alpha \neq 0$, if $\varphi(\alpha) = 0$

$\Rightarrow L_\alpha = 0 \Rightarrow L_\alpha(f) = 0 = f(\alpha) \quad \forall f \in V^*$

But we know, for $\alpha \neq 0$, one will get at least one f in V^*

Such that $f(\alpha) \neq 0$

$\Rightarrow \exists f$ for which $L_\alpha(f) = f(\alpha) \neq 0$

$\therefore$ For $\alpha \neq 0$, $L_\alpha \neq 0$

$\Rightarrow \varphi$ is 1-1

So, this implies that $L_{\alpha+c\beta}$ = L_α+ c L_β. So, this implies, $\varphi(\alpha + c\beta) = L_{\alpha+c\beta} =$ L_α+ c L_β $= \varphi(\alpha) +$ c$\varphi(\beta)$. So, φ is a linear transformation(L.T.). For $\alpha \neq 0$, if $\varphi(\alpha) = 0$, implies that L_α= 0. So, L_α is a linear functionals on V^*. So, implies that L_α(f) = 0 = f(α), $\forall$ f $\in$ V^*.

But we know for $\alpha \neq 0$, one will get at least one such f on V^* such that f(α) ≠ 0, already we have explained in our previous lectures if you consider a nonzero element in V one can immediately find out a nonzero linear functional f on V^* such that f(α) ≠ 0. So, based on that we have a nonzero linear functionals, so this implies ∃ f for which, L_α(f) = f(α) ≠ 0.

This implies that so for $\alpha \neq 0$, $L_\alpha \neq 0$, I mean 0 linear functional. So, this implies that your φ is 1-1 because the null space of φ is only zero subspace, φ is 1-1.

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$\therefore$ φ is a 1-1 L.T. for V into V^{**}

$\therefore$ φ is an isomorphism for V into V^{**}

We have for any L on V^*, $\exists$ an unique $\alpha \in V$ s.t.

$L = L_\alpha$ i.e. $L(f) = L_\alpha(f) = f(\alpha)$ —

$\therefore$ If $\{f_1, f_2, \dots f_n\}$ is a basis of V^*, then, there exist a dual basis $\{L_1, L_2, \dots L_n\}$ for V^{**} such that

$L_i(f_j) = \delta_{ij}$ — $1 \le i \le n$, $1 \le j \le n$

Again, for each L_i $\exists$ α_i in V s.t.

$L_i = L_{\alpha_i}$

$\Rightarrow$ $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ related to $\{L_1, L_2, \dots L_n\}$

So, we have φ is a 1-1 linear transformation from V into V^{**}. Since already we explained the dimension of V is finite and that is equal to dimension of V^{**}, so certainly if it is 1-1 implies that it is also onto. So, φ is an isomorphism from V into V^{**}. So, we have seen there exist 1-1 onto map linear transformation from V and V^{**}. So that is α goes to L_α we have this one.

So, what do we have achieved from this? So, let me clarify this once. See we have for any L on V^* any linear functional on V^*, $\exists$ a unique $\alpha \in V$, s.t. $L = L_\alpha$, because since we have seen for the 1-1 map between V and V^{**}, so whenever you consider element in double star, I mean L is basically an element in the V^{**} so definitely an element in V.

So, for any L in V^{**} there will be an element $\alpha \in V$, s.t. $L = L_\alpha$. i.e. $L(f) = L_\alpha(f) = f(\alpha)$. So, this implies if $\{f_1, f_2, \dots f_n\}$ is a basis of V^*, then we know if this is the basis for a V^* then there exist a dual basis $\{L_1, L_2, \dots L_n\}$ for V^{**}, such that $L_i(f_j) = \delta_{ij}$ where $1 \le i \le n$, $1 \le j \le n$.

This already we know from previous studies that if we consider a basis for this space V^* then certainly there is a dual basis $\{L_1, L_2, \dots L_n\}$ for V^{**}. Again, for each L_i $\exists$ α_i in V such that $L_i = L_{\alpha_i}$. Previous result says that for each L_i $\exists$ α_i in V s.t. $L_i = L_{\alpha_i}$. So, this implies one will get $(\alpha_1, \alpha_2, \dots \alpha_n)$ related to $\{L_1, L_2, \dots L_n\}$.

Now if $\{L_1, L_2, \dots L_n\}$ is linearly independent certainly $(\alpha_1, \alpha_2, \dots \alpha_n)$ will be linearly independent and that is coming, I just forgot, because φ is 1-1. So, if α_1 and if suppose L_1, L_2 are linearly independent and α_1, α_2 not linearly independent they are dependent you will get a contradiction that φ carrying nonzero to zero vector.

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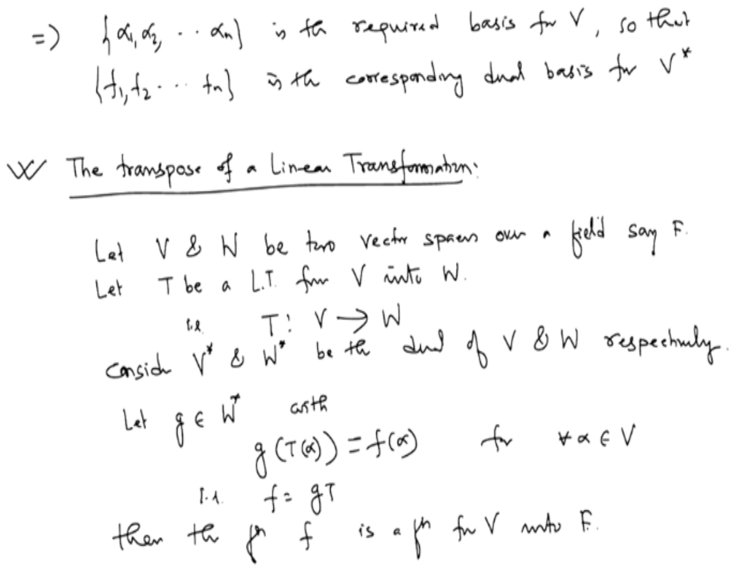

So, this implies that $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is the required basis for V so that $\{f_1, f_2, \ldots f_n\}$ is the corresponding dual basis for V^*. So, we see that if you consider a basis $\{f_1, f_2, \ldots f_n\}$ on V^* certainly there exist a basis $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ in V such that $\{f_1, f_2, \ldots f_n\}$ the dual of $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ basis. So, we have basically studied what is linear transformation, what is linear operator, linear functionals.

Now we want to see suppose a linear transformation is given to us whether these linear transformations introduce another linear transformation on some spaces, so we are interested for that, so basically I am talking about the transpose of a linear transformation. So, before introducing the transpose of a linear transformations, first let us see how to define, how they give you linear transformation introduce another linear transformation. Let V & W be two vector spaces over a field say F.

Let T be a linear transformation from V into W. So, i.e. $T: V \to W$. Consider V^* & W^* be the dual of V and W respectively. Let $g \in W^*$ means g be a linear functional on W with $g(T(\alpha)) = f(\alpha)$ $\forall \alpha \in V$, i.e. $f = gT$. So, I have considered the function which is composite of g and T. So, T is linear transformation from V into W and g is a linear functional on W^*.

So, take the (gT), composition of the g and T then I see that (gT) is basically functional or you can say function here from V to F. So, I have considered $g(T(\alpha)) = f(\alpha)\ \forall\ \alpha \in V$. Suppose is like these things, then the function f is a function from V into F. Since T is linear transformation and g is also linear transformation, linear functional linear transformation, so it is a product of two linear functions, so f has to be also linear.

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$$f(c\alpha+\beta) = (gT)(c\alpha+\beta) = g(T(c\alpha+\beta))$$
$$= g(cT(\alpha)+T(\beta))$$
$$= c(gT)(\alpha) + (gT)(\beta)$$

$\therefore$ f is a linear functional on V^*

Defn $T^t : W^* \to V^*$
$g \to T^t g$

Claim, T^t is a linear transformation fr W^* into V^*

For g_1 & $g_2 \in W^*$ & $c \in F$

$$T^t(cg_1+g_2) = (cg_1+g_2)T$$
$$\Rightarrow (T^t(cg_1+g_2))(\alpha) = ((cg_1+g_2)T)(\alpha)$$
$$= (cg_1)(T(\alpha)) + g_2(T(\alpha))$$
$$= cg_1(T(\alpha)) + g_2(T(\alpha))$$

So, if I consider $f(\alpha c + \beta) = (gT)(\alpha c + \beta) = g(T(\alpha c + \beta)) = g(c(T(\alpha)+T(\beta)) = (gT(\alpha)+(gT)(\beta)$. So, this is linear function, so f is a linear functional on V^*. Now, let me define $T^t: W^* \to V^*$. See from the linear transformation T, I am able to introduce a linear functional f like what we have done it here.

And that linear functional f is defined on basically V. So, I can define another function T^t which is mapping for $g \to T^t g$, where $g \in W^*$. I am defining say $T^t g$ and that $T^t g$ basically gT. Claim: $T^t: W^* \to V^*$. Because for g_1 & $g_2 \in W^*$ & $c \in F$, $T^t(cg_1 + g_1) = (cg_1 + g_1)T$

So, this implies that $(T^t(cg_1 + g_2))(\alpha) = ((cg_1 + g_2)T)(\alpha) = (cg_1)(T(\alpha)) + g_2(T(\alpha) = cg_1(T(\alpha)) + g_2(T(\alpha)$. So, this implies what?

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$\Rightarrow T^t(cg_1+g_2) = cT^t(g_1) + T^t(g_2)$ —

$\therefore$ T^t is a L.T.

T^t is called as the transpose of the linear transformation T

Ex Let $T: R^2 \to R^2$

$\alpha = (x_1, x_2) \to (x_1, 0)$

i.e. $T(x_1, x_2) = (x_1, 0)$

& $g \in R^{2^*}$ & $g(x_1, x_2) = ax_1 + bx_2$

$\Rightarrow (T^t(g))(\alpha) = g(T(\alpha)) = g(x_1, 0) = ax_1$ —

This implies, $\mathrm{T}^t(\mathrm{c}g_1 + g_2) = c\mathrm{T}^t(g_1) + \mathrm{T}^t(g_2)$. So, T^t is a linear transformation. T^t is called as the transpose of the linear transformation T. So T^t is the transpose of T. One can take an example say let T: $\mathrm{R}^2 \to \mathrm{R}^2$ and defined by say $\alpha = (x_1, x_2) \to (x_1, 0)$ i.e. $\mathrm{T}(x_1, x_2) = (x_1, 0)$. Let g $\in$ R^{2^*} & $g(x_1, x_2) = ax_1 + bx_2$.

So, this implies your $(\mathrm{T}^t(g))(\alpha) = g(T(\alpha)) = g(x_1, 0) = ax_1$. So, we will continue with this in the next lecture.