

Advanced Linear Algebra
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Lecture – 20
Linear Functional - 3

So now I will discuss if I consider the subspace of a finite dimensional vector space having dimension of the subspace as k and dimension of space is suppose n , then we will show that the subspace of dimension k is basically intersection of $(n - k)$ hyperspace. I mean it is like this. As consequences of my previous result if W is a k dimensional subspace of n dimensional vector space, then W is the intersection of $(n - k)$ hyperspace in V . I will prove first through the examples and then theoretically also I will prove it.

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Corollary: If W is a k -dim subspace of an n -dim vector space V , then W is the intersection of $(n-k)$ hyperspaces in V .

Ex: W be the subspace of \mathbb{R}^5 spanned by
 $\alpha_1 = (2, -2, 3, 4, -1)$, $\alpha_2 = (0, 0, -1, -2, 3)$
 $\alpha_3 = (-1, 1, 2, 5, 2)$ $\alpha_4 = (1, -1, 2, 3, 0)$
 Find the required hyperspace which annihilate W .

Solⁿ: First we have to find the dim of W .

$$\begin{pmatrix} 2 & -2 & 3 & 4 & -1 \\ -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 3 & 0 \\ -1 & 1 & 2 & 5 & 2 \\ 2 & -2 & 3 & 4 & -1 \\ 0 & 0 & -1 & -2 & 3 \end{pmatrix}$$

So, I have taken a subspace W of \mathbb{R}^5 space, I mean space of 5-tuples. So, the dimension of the $V = n = 5$ and the subspace w which is basically spanned by the given four vectors say $\alpha_1 = (2, -2, 3, 4, -1)$, $\alpha_2 = (0, 0, -1, -2, 3)$, $\alpha_3 = (-1, 1, 2, 5, 2)$, $\alpha_4 = (1, -1, 2, 3, 0)$. I do not know dimension of this the subspace W . The dimension of the subspace maybe 4, maybe 3, maybe 2 or maybe 1. If it is say 4, then I have to prove that this subspace is intersection of $(5 - 4)$ hyperspace.

I mean one hyperspace. If it is 3, then I have to prove that it is basically intersection of 2

hyperspace means that there exist two linear functionals on \mathbb{R}^5 which are linearly independent to each other, linearly independent and these two linear functionals will annihilate this subspace W that is the meaning of the above corollary also. So, W be the subspace of \mathbb{R}^5 spanned by this $\alpha_1, \alpha_2, \alpha_3$ find required hyperspaces which annihilates W , the question.

So, let me go to the solution. First I have to identify what is the dimension of the subspace W . So, first we have to find the dimension of W . So for this I will take the help of elementary row operations constructing a 4×5 matrix with first row α_1 , second row α_2 , third row α_3 and fourth row

is α_4 . So, if I count like that then I have,
$$\begin{bmatrix} 2 & -2 & 3 & 4 & -1 \\ -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{bmatrix}$$
. There may be some mistake

in my algebra, so you should also cross check before accepting it,

Now let me go for the row-reduced echelon form. So, for this what I will do I will take this basically first row is something $1, -1, 2, 3, 0$ and then $-1, 1, 2, 5, 2$. Then I will go for $2, -2, 3, 4, -1$. Basically, I would have written the fourth row to the first row and your second row as it is, third row as the first row and the first row of the original matrix go to third row and then $0, 0, -1, -2$

and 3, this matrix
$$\begin{bmatrix} 2 & -2 & 3 & 4 & -1 \\ -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ -1 & 1 & 2 & 5 & 2 \\ 2 & -2 & 3 & 4 & -1 \\ 0 & 0 & -1 & -2 & 3 \end{bmatrix}$$
.

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$$\begin{pmatrix} 1 & -1 & 2 & 3 & 0 \\ -1 & 1 & 2 & 5 & 2 \\ 2 & -2 & 3 & 4 & -1 \\ 0 & 0 & -1 & -2 & 3 \end{pmatrix} \xrightarrow{R_2+R_1, R_3-2R_1} \begin{pmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 8 & 2 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 8 & 2 \\ 0 & 0 & -1 & -2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R$$

\Rightarrow Dim of Subspace W is 3.

So, a basis $\{P_1, P_2, P_3\}$ of W
 where $P_1 = (1, -1, 0, -1, 0)$, $P_2 = (0, 0, 1, 2, 0)$, $P_3 = (0, 0, 0, 0, 1)$

So let me do this one, goes to again, this can be written as,

$$\begin{aligned}
 & \begin{bmatrix} 2 & -2 & 3 & 4 & -1 \\ -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ -1 & 1 & 2 & 5 & 2 \\ 2 & -2 & 3 & 4 & -1 \\ 0 & 0 & -1 & -2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 8 & 2 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2 & 3 \end{bmatrix} \rightarrow \\
 & \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 8 & 2 \\ 0 & 0 & -1 & -2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \\
 & \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R
 \end{aligned}$$

The calculation of above elementary row operation as follow like this, 1, -1, 2, 3, 0. Just adding 0, 0, 4, 8, 2. And second one multiplying first row by 2 and subtracting the third row I will have 0. So basically ($R_3 \rightarrow R_3 - 2R_1$). So that is equal to 0, -2 R_1 means the plus this will also 0, -1, 4 and then -6 that is -2 and -1. So I will have like this. Then next one is 0, 0, I have -1 and -2, 3. So if I go the next step this is going to be again 1, -1, 2, 3, 0 and so I can make it here.

So I can exchange the third row to this one, so 0 0 1 2. And then I can have the third one if I do this one 0, 0, 4, 8, 2. So let me write down 0, 0, -1, -2. So one more step let me go, it will be 1, -1, 2, 3, 0; 0, 0, 1, 2, 1 and this one multiplying with 4 and subtract, so it will be 0, 0, 0 and this will be also 0, So 2 - 4 is equal to -2 will come into the picture and this will be again 0, 0, 0, 0 and I will simply add it so it will be basically 4.

So this means that I am getting one more step. So 1, -1, 2, 3, 0; 0, 0, 1, 2, 1 and next one will be 0, 0, 1 and this is also 0, 0, 0, 0, 0. So though it is not a row-reduced echelon form, to make it a row-reduced echelon form I have to do one more step, let me go ahead. This is equal to 1, then I have -1, and I have to multiply two and subtract this once, so this will be 0, so multiplying second row by 2 and subtracting from first row, so then a 3 - 4 that is equal to -1.

Basically -2 will come into the picture, 0 - 2 yeah fine and then I have 0, 0, 1. and here I am

getting 2 and 1, 0, 0, 0, 0, 1; 0, 0, 0, 0, 0. So, again I can make it this is equal to simply 0, so this is also possible. So, this implies the dimension of subspace W is 3. So we have a basis say consisting of $\{\beta_1, \beta_2, \beta_3\}$ of W where $\beta_1=(1, -1, 0, -1, 0)$, $\beta_2=(0, 0, 1, 2, 0)$ and $\beta_3=(0, 0, 0, 0, 1)$. So, I have basically a basis for the subspace W. Now, question is I have to find linear functionals which will annihilate W.

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Let f be a L.F on \mathbb{R}^5

$$f(x_1, x_2, x_3, x_4, x_5) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5$$

f it annihilate W

$$\Rightarrow \begin{aligned} f(\beta_1) = 0 &\Rightarrow c_1 - c_2 - c_4 = 0, \quad c_1 = c_2 + c_4 \\ f(\beta_2) = 0 &\Rightarrow c_3 + 2c_4 = 0 \Rightarrow c_3 = -2c_4 \\ f(\beta_3) = 0 &\Rightarrow c_5 = 0 \end{aligned}$$

Put $c_2 = 1, c_4 = 0$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} f_1 &= x_1 + x_2 \\ f_2 &= x_3 - 2x_4 + x_5 \end{aligned}$$

Put $c_2 = 0, c_4 = 1$, \therefore Null span of f_1 & f_2 includes W, $\Rightarrow W = \text{Null span of } f_1 \text{ & } f_2$

So, let f be a linear functional on \mathbb{R}^5 . Let me define, it is defined by $f(x_1, x_2, x_3, x_4, x_5) = (c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5)$ and it annihilates W. So, this implies $f(\beta_1) = 0$, $f(\beta_2) = 0$, $f(\beta_3) = 0$, where $\beta_1=(1, -1, 0, -1, 0)$, $\beta_2=(0, 0, 1, 2, 0)$ and $\beta_3=(0, 0, 0, 0, 1)$. So, $f(\beta_1) = 0$ this implies $c_1 - c_2 - c_4 = 0$, $c_1 = c_2 + c_4$.

$f(\beta_2) = 0$, $c_3 + 2c_4 = 0$ this implies $c_3 = -2c_4$. and if $f(\beta_3) = 0$ this implies $c_5 = 0$. So now I have to solve this system to get the solution. This implies that your $c_3 = -2c_4$ and $c_1 = c_2 + c_4$, $c_5 = 0$. So this means I have two free variables, c_2 and c_4 . So put $c_2 = 1, c_4 = 0$, I will have $c_1 = 1$ and $c_3 = 0$,

$c_5 = 0$. So this implies I will have one set of $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. This is one and other one put $c_2 = 0$ and $c_4 = 1$

$=1$, if I do like this $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. So I will have this one. So, this implies that I have two linear

functionals I can say, $f_1 = x_1 + x_2$ and $f_2 = x_1 - 2x_3 + x_4$. So these are linear functionals which will basically annihilate your W.

Annihilate means the null space of f_1 and null space of f_2 contain the set W. So the null space of f_1 and f_2 include W. So, this implies that $W = (\text{null space of } f_1) \cap (\text{null space of } f_2)$, so intersection of two hyperspaces. So when the dimension of W is 3, then we see that W is basically the intersection of two hyperspaces. So, in similar way one can also suppose subspace is given of dimension k.

Then we see that there exists exactly $(n - k)$ hyperspaces, which will basically intersection of the hyperspaces is your W. Now, let me twist the problems, now it is like this. Over the vector space V, of course finite dimensional, suppose some linear functional is given to me f_1, f_2, f_3 like that. Now, I want to know what is that set W of vector space V which will be annihilated by this f_1, f_2, f_3 like whatever linear functional given to me, how to do it? So, for that one can proceed like this.

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Consider a vector $V = F^n$ of dim n.
 Consider linear fnd
 $f_1(x_1, x_2, \dots, x_n) = A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n$
 $f_2(x_1, x_2, \dots, x_n) = A_{21}x_1 + \dots + A_{2n}x_n$
 \vdots
 $f_m(x_1, x_2, \dots, x_n) = A_{m1}x_1 + \dots + A_{mn}x_n$

Now, we are interested to find a subspace W of V which will be annihilated by above m-L.F.

Now, consider a vector space V of dimension say n. Consider linear functionals $f_1(x_1, x_2, x_3)$. So here I have taken here $V = F^n$, suppose now, $f_1(x_1, x_2, x_3) = A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n$, $f_2(x_1, x_2, x_3) = A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n$, $f_m(x_1, x_2, x_3) = A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n$. So, n linear functional is given to us which are defined over the vector space V.

Now we are interested to find a subspace W which will be annihilated by m linear functionals, how to find it? So, here what you have to do is first we have to find the set of linearly independent linear functionals. This has to do with the help of elementary row operations, get the corresponding, this is basically equivalent to say AX , now this AX you have to reduce to RX , where R is a row-reduced equivalent form of the matrix A .

Then you will exactly come to know out of m linear functionals how many linear functionals you are getting which are linearly independent. So, suppose you get thus r number of linearly independent linear functionals. Once r is obtained, then we have to find what is the subspace of V which will be annihilated by this r number of linearly independent linear functionals that is basically solution of $RX=0$.

So, from that we will get a solution space W . I will get a basis for the W and the linear combination of all the basis element will talk about the subspace W which will be annihilated by this f_1, f_2, \dots, f_m linear functional. So, you can consider as a homework and if we have difficulties, we will discuss later. Please Thank you.