

Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 2
Elementary Row Operations

So, we have discussed in our first lecture what is system of linear equations. We have also seen equivalent systems and we have also seen while discussing as a special case that solution of system equations as linear combinations of some column vector form type that is also we have seen in terms of as a geometrically, also in terms of equation of line or planes.

So, these things indicates that one has to proceed in a particular path to answer the question whether a system of equations, m equations in n unknowns have solution or not and if at all it has solution how to find. So, we need to represent that system of equations in terms of matrix form.

(Refer Slide Time: 01:46)

Matrix representation:

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = y_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = y_m \end{array} \right\} * \begin{array}{l} A_{ij} \in F \quad 1 \leq i \leq m \\ \quad \quad \quad \quad 1 \leq j \leq n \\ y_i \in F \quad \quad 1 \leq i \leq m \end{array}$$

$AX = Y$ —

where $A = (a_{ij})$ i.e. $m \times n$ matrix over F
 $X = (x_1, x_2, \dots, x_n)^T$, $n \times 1$ column matrix over F
 $Y = (y_1, y_2, \dots, y_m)^T$ $m \times 1$ " " "

Three basic elementary row operations:

(i) Multiplication of i th row of A by a non zero constant c
 Mathematically,

$$e(A_{ij}) = \begin{cases} a_{ij} & \text{if } i \neq r \text{ \& } 1 \leq i \leq m, 1 \leq j \leq n \\ ca_{ij} & \text{if } i = r \end{cases}$$

Recall we wrote the first system of m equation and n unknowns like $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = y_1; a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = y_2; a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = y_m$ where $a_{ij} \in F$ for $1 \leq i \leq m, 1 \leq j \leq n$ and of course $y_i \in F$ for again $1 \leq i \leq m$. So, this one I wrote as a system of m equation and n unknowns, I gave the name as star.

Now this system of equations can be written in a compact form like by introducing an $m \times n$ matrix say $AX=Y$

where $A = (a_{ij})$ that is $m \times n$ matrix over F and X is basically $(x_1, x_2, \dots, x_n)^T$ $n \times 1$ column matrix over F and $Y = (y_1, y_2, \dots, y_m)^T$ this $m \times 1$ column matrix over the surface. So, I can write down this star in a compact form $AX = Y$ as a matrix from this.

And again, then we said that $Y=0$ is my homogeneous systems and otherwise it is a non-homogenous system. We have discussed in our first class that to check whether a system has solution or not one has to follow elimination procedure. While doing the elimination procedures, one has to do basically three basic elementary row operations. I can do this elementary operation in terms of row or in terms of column.

So to check whether system has solution or not, we will consider the elementary row operation. So, there are three basic elementary row operations. The first one multiplication. This elementary row operation I will talk in terms of only matrix A even though I can do it for both sides I mean in the system $AX = Y$ elementary row operation for A as well as Y , but I will do it for sake of simplicity for A only and the concept can be extended easily for the Y also.

So, multiplication of r th row of A by a nonzero constant c . So this is one elementary row operation. Mathematically that is in function form I can write down $e(a_{ij}) = a_{ij}$ if $i \neq r$ and $1 \leq i \leq m, 1 \leq j \leq n$. And in the case of $i = r$, it is basically ca_{rj} if $i = r$. So, this is the first elementary row operation.

(Refer Slide Time: 07:14)

(i) Replacing r th row of A by c times r th row of A plus r th row

$$e(a_{ij}) = \begin{cases} a_{ij} & \text{if } i \neq r, 1 \leq i \leq m, 1 \leq j \leq n \\ ca_{rj} + a_{rj} & \text{if } i = r \end{cases}$$

(ii) Exchange of r th row by s th row of A

$$e(a_{ij}) = \begin{cases} a_{ij} & \text{if } i \neq r \text{ \& } i \neq s, 1 \leq j \leq n \\ a_{sj} & \text{if } i = r, 1 \leq j \leq n \\ a_{rj} & \text{if } i = s \end{cases}$$

Each elementary row operation is invertible & introduce an invertible matrix, called as elementary matrix.

Ex

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= 0 \end{aligned} \right\} \Rightarrow AX = 0$$

Then the next one I am saying that replacing r th row of A by c times s th row of A plus r th row. So, mathematically it is basically $e(a_{ij}) = a_{ij}$ if $i \neq r$ and $1 \leq i \leq m, 1 \leq j \leq n$ and this is $a_{rj} + ca_{sj}$ if $i = r$ and this one. Certainly, this will lie in between 1 to n . The third one exchange of r th row by s th row of A .

Now, these elementary row operations again can be written in functional form. It is like this $e(a_{ij}) = a_{ij}$ if $i \neq r$ and if $i \neq s, 1 \leq j \leq n$ and this is a_{sj} if $i = r$ and $1 \leq j \leq n$ and this is a_{rj} if $i = s$. So, this is my third elementary row operation. The total elementary row operation is nothing, I can say as some sort of combinations of these three elementary row operations.

Interestingly each elementary row operation is invertible and introduces an invertible matrix called an elementary matrix. For example, system is $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0$, I have taken homogeneous case, but this can be done for the non-homogeneous case also. And $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = 0$, and $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = 0$. This goes to I can say $AX = 0$ where A is a 3×4 matrix and X is a 4×1 column matrix space.

(Refer Slide Time: 11:29)

Multiplying 2nd eqn by $0 \neq c$, we will have new system

$$BX = 0$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here, $E_1 A = B$

$$E_2 E_1 A = A$$

Using graph of elementary row operation one can have a new system say $BX = 0$

Equivalent matrices: Two matrices A & B of order $m \times n$ are said to be equivalent provided one can be obtained from other by considering a finite number of elementary row operations

Multiplying second equation by $0 \neq c$, so in this case we will have new system say $BX = 0$ where the B is the first row of A , third row of B is also third row of A but the second row of B is c times second row of A . Now, this one I can write down if I multiply by $1/c$ to the second equation of $BX = 0$, certainly I will get back to my first system $AX = 0$. So, here say to obtain B , I have introduced an elementary matrix E_1 which is equal to like this $1 \ 0 \ 0 \ 0 \ c \ 0 \ 0 \ 0 \ 1$.

Here, $E_1 A = B$, you can immediately check this is true. Now, if I consider E_2 , E_2 is another matrix which I have obtained by multiplying second equation of $BX=0$ by 1 by c . Then I will also introduce another invertible matrix C . So that will be some form, it will be again $1\ 0\ 0\ 0\ 1$ by $C\ 0\ 0\ 0\ 1$. So, we say that $E_1 E_2 A = A$. So, each though I have shown it for the multiplication by constant term, nonzero constant, but it can be also shown for any other two operations what I introduced here.

For that also you will get elementary row matrices which are invertible. So, when you are doing some elementary row operations using these three basic elementary row operations, one may have new systems. So, now using couple of elementary row operations one can have a new system say $B^* X=0$ where this $B^* X=0$ is a new system obtained from my suppose given system $A X=0$, then you see that both these systems are equivalent.

But before that let me introduce one more terminology called equivalent matrices. Two matrices A and B of order $m \times n$ are said to be equivalent provided one can be obtained from other by considering finite number of elementary row operations. So, the way we used to get equivalent matrices from this we can also talk about the solution of the system $A X=0$ and $B X=0$, where A and B are equivalent matrices also. So, it is like this.

(Refer Slide Time: 16:35)

Theorem: Let A & B be two $m \times n$ matrices over the field F & are also row-equivalent. Then the system $A X=0$ & $B X=0$ have the same solutions

Pf: Let B can be obtained for A by a finite number of elementary row operation

$$A \rightarrow A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_k = B$$

Let us consider B can be obtained for A by 1 elementary row operation.

While doing this, we have obtained the system $B X=0$ where each eqn is linear combination of eqns of $A X=0$

\Rightarrow The system $A X=0$ & $B X=0$ are equivalent

\therefore They have same sol^{ns}.

This is small you can say theorems. Let A and B be two $m \times n$ matrices over the field F and are also row equivalent. Then the system $A X=0$ and $B X =0$ have the same solutions. See the proof is

very straightforward, it can be done like this. Let B can be obtained from A by a finite number of elementary row operations, $A \rightarrow A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_k = B$.

So, if I say that the two matrices A and B are row equivalent means I can obtain one from others by finite number of elementary row operations. Let me consider A can be obtained from B by exactly one elementary row operation, then I will have again, suppose you have done for that one that system $AX=0$ and $BX=0$ have same solution, then again applying B one more elementary row operation I will have another new system then again so that B and suppose D they have the same solution.

So, I will basically give the proof for these results considering that B can be obtained from A by a single elementary row operation. Let us consider B can be obtained from A by 1 elementary row operation. So, when you are doing the elementary row operation that may be elementary row operation one, it may be elementary row operation two or elementary row operation three the way I have defined it.

I mean multiplying by constant to some row or maybe extending some row by multiplying with another row or constant plus adding to it or extending the rows also. So, I have taken only one elementary row operation has been used for A and obtain B. So, while doing these things we have obtained the system $BX=0$, where each equation is a linear combination of equations of $AX=0$. This implies the system $AX=0$ and $BX=0$ are equivalent, they have same sort of solutions.

So, if I consider two matrices which are equivalent and row equivalent to each other, then their corresponding system $AX=0$ and the system $BX=0$, also have the same solution. Since our objective is to answer the questions what we are raised that how to check whether system has solution or not and if the solution exists how to find the solutions, so we have to basically follow a particular path through which we can take the help of computer and answer our question.

(Refer Slide Time: 22:32)

Row-reduced matrix: An $m \times n$ matrix A , defined over a field F , is said to be a row-reduced matrix, provided

- (i) The first nonzero entry of each nonzero row must be equal to 1. This entry is also called as leading entry.
- (ii) Any column containing the leading entry of some nonzero row, then all other entries of that column are zero.

Ex $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$ Row-reduced matrix

Ex-2 $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$

Ex-3 $B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore B$ is also a row-reduced matrix.

So, in these directions let me introduce one terminology it is called row-reduced matrix. An $m \times n$ matrix A , defined over a field F is said to be a row-reduced matrix provided first condition the first nonzero entry of each nonzero row must be equal to 1. So let me also add one more thing in the first condition, this entry is also called as leading entry, so leading entry for that corresponding row.

Any column containing the leading entry of some nonzero row, then all other entries of that column are zero. So, for example if I consider A equal to zero matrix, say $0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$, can I say that it is the row-reduced matrix? Certainly answer is yes, why? Does it satisfy condition 1 or condition 2? Answer is again yes. Condition one says that the first nonzero entry of each nonzero row, since there are no nonzero rows, so there is no question of the first condition and since there is no leading entry, so therefore second criteria is also by default satisfied.

So, it is a row-reduced matrix. Let me consider another one. Say our trivial identity matrix say $1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0$ and $0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1$. Does it satisfy again the criteria 1 and 2? Answer is yes because first nonzero entry of first row is 1 and all the entries of the first column which contain the leading entry of the first row are zero also and it is also valid for the second column which contains the leading entry of the second row and similarly for the third column and fourth column.

So, it satisfied all the required criteria of the definition of row-reduced matrix. But these are all

trivial examples, this is example 2. Let me take some nontrivial examples. Let me take say B is equal to something like this $0\ 0\ 0\ 0\ 1$ and $0\ 1\ 0\ 0\ 0$ and $0\ 0\ 1\ 0\ 0$ and $0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0$. Now, again see this is a 5×5 matrix and actually the first row is also nonzero and the first nonzero entry is 1 which is repeating in the fifth column and fifth column all other entries are 0.

Then second row which is also nonzero and the leading entry appearing in second column and all other entries on that corresponding column are zero also. Similarly for the third row, leading entry appears in the third column and all other entries 0. So, it has satisfied 1 as well as 2. So, B is also row-reduced matrix. So, B is also a row-reduced matrix. Now the question is, is it possible that if one picked off any $m \times n$ matrix defined over the field say F, one can also have corresponding row-reduced matrix? Answer is yes.

(Refer Slide Time: 29:11)

Theorem: Each $m \times n$ matrix A is row-equivalent to an $m \times n$ row-reduced matrix R .

Pf: $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}$

First, we have to check, whether first row of A is a nonzero row or not.
 If it is zero row, then condⁿ (1) of the defⁿ of row-reduced matrix is satisfied. Else, let the first nonzero entry of the 1st row appear in k th column. Then, divide 1st row by a_{1k} , so condⁿ (2) is satisfied. Add (a_{ik}) times 1st row to all rows $i=2$ to m . So that condⁿ (3) will be also satisfied.

Go to 2nd row.
 If 2nd row is a zero row then nothing has to be done for this row. Else, find out the 1st nonzero entry of this row. Let it appear in k_1 column. Then divide 2nd row by a_{2k_1} .

So, this is small result in terms of theorem. Each m cross n matrix A is row equivalent to a row-reduced matrix R , again $m \times n$ row-reduced matrix R . How to prove it? Given matrix A is your $a_{11} \dots a_{1n}$ and then $a_{m1} \dots a_{mn}$ this matrix. To check whether the first row is nonzero or not, whether first row of A is a nonzero row or not. If it is zero row then condition 1 of the definition of a row-reduced matrix is satisfied by default, else let the first nonzero entry of the first row appear in k th column.

Then divide first row by a_{1k} or that is multiplying 1 upon a_{1k} to the first row, then the condition 1

will be satisfied because the first nonzero entry will be 1. So, condition 1 is satisfied. To satisfy the corresponding condition 2 the leading entry appearing in the k th column all other entry has to be equal to 0. Then add $(-a_{ik})$ times first row to all rows $i=2$ to m , so that condition 2 will be also satisfied. Go to second row.

Again if second row is zero rows then nothing has to be done, else find out the first nonzero entry of the second row and then divide that corresponding number to the second row so that leading entry will be equal to 1. Add third row, fourth row and all other rows by multiplying second row by corresponding number k , suppose is k_1 , suppose leading entry appear in k_1 column, say then a_{ik_1} , i will basically from your 3 to m so that again second condition will be satisfied.

And if one do it like this way, certainly after a finite step we will come to the row-reduced matrix form. If second row is zero row, then nothing has to be done for this row. Else find out the first nonzero entry of this row, let it appear in k_1 column. Then divide second row by a_{2k_1} . So, the first criteria will be again satisfied.

(Refer Slide Time: 35:36)

Also add $(-a_{ik_1})$ times 2nd row to all rows for $i=3$ to m .
 \therefore 2nd criteria will also satisfied
 Continue in this way.
 After a finite step, we will get corresponding row-reduced matrix R .
 \therefore The system $AX=0$ & $RX=0$, after R is row-reduced matrix of A , have same solⁿs.
 $\therefore A$ & R are row equivalent

Now also add $(-a_{ik_1})$, I have given the name k_1 right, First add $(-a_{ik_1})$ times second row to all rows for $i = 3$ to m , second criteria will be also satisfied. Continue in this way. After a finite step, one will get corresponding row-reduced matrix R . So, for each $m \times n$ matrix one can have a corresponded row-reduced matrix also.

Again similarly I can say that the system $AX=0$ and $RX=0$ where R is a row-reduced matrix of A have same solution since A and R are row equivalent because R is obtained from A by finite number of elementary row operations. So, therefore A and R are row equivalent, as a result the system $AX=0$ and $RX=0$ have the same solution. So, we have seen that for any $m \times n$ matrix A one can have corresponding a row-reduced matrix R and the corresponding system $AX=0$ as well as $RX=0$ they have same solutions. Thank you.