Advanced Linear Algebra
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Lecture – 19
Linear Functional - 2

So, welcome to my lecture series. In my last class, we have seen that if V be a finite dimensional vector space over a field F and B $= (\alpha_1, \alpha_2, \ldots\ \alpha_n)$ be an ordered basis for B, then there exist a dual basis $\{f_1,\ f_2,\ \ldots f_n\}$ for V^* such that any linear functional f on V^* can be written as a linear combination of f_i where the coefficient of f_i the functional value at α_i and any vector alpha can be written as linear combinations of $(\alpha_1, \alpha_2, \ldots\ \alpha_n)$ where the coefficient of α_i is basically function value of $f_i(\alpha)$. So, let us give an example and see it.

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Let V be a finite dim vector space over a field F.
Let $B = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be an ordered basis for V. Then there exists a dual basis $\{f_1, f_2, \cdots f_n\}$ for V^* such that

$$f = \sum_{j=1}^{n} f(\alpha_j) f_j \quad \text{— (i)}$$

$$\alpha = \sum_{j=1}^{n} f_j(\alpha)\alpha_j \quad \text{— (ii)}$$

Ex: Let V be the space of all polynomial f^n s of degree ≤ 2, over F.
Consider t_1, t_2, t_3 be any three distinct constants from F.

$L_i(p) = p(t_i)$ $\quad i = 1, 2, 3$

$L_i : V \to F$

$L_1(p) = p(t_1),\quad L_2(p) = p(t_2),\quad L_3(p) = p(t_3)$ —

Claim $\{L_1, L_2, L_3\}$ is a basis of V^*

So, let me consider one example. Let V be the space of all polynomial function of degree ≤ 2 and over the F and polynomial functions are defined for F to F only. Consider $(t_1,\ t_2,\ t_3)$ be any three distinct constants from the field F that I consider. So, let me define linear functional on V. L is a linear functional on V, so L_i(p) $=$ p(t_i), i=1,2,3. L_i: V → F. $(L_1,\ L_2,\ L_3)$.

So L_i is basically map from your V to your field F. So, for i = 1, 2, 3 I have three functions. In my last lecture, we have already seen each L_i that we have defined is basically linear functional. So, I have three linear functionals. So, this means that $L_1(\mathrm{p}) = \mathrm{p}(t_1)$, $L_2(\mathrm{p}) = \mathrm{p}(t_2)$, $L_3(\mathrm{p}) = \mathrm{p}(t_2)$. Claim: (L_1, L_2, L_3) as a basis of V^*.

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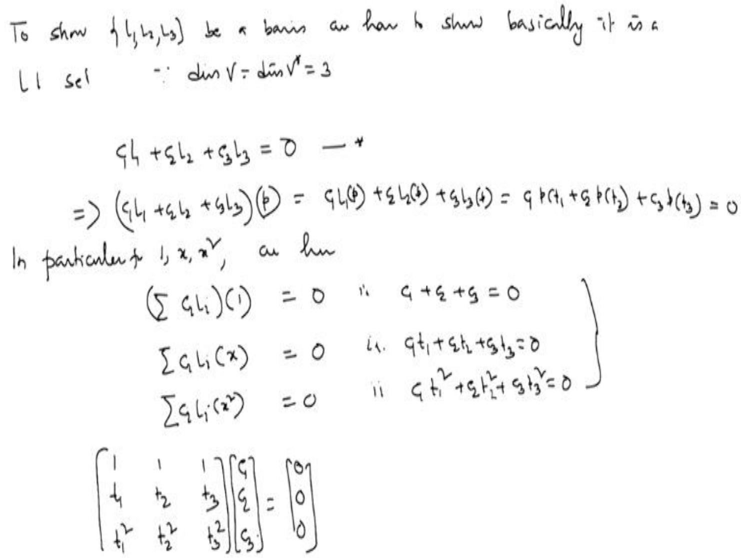

To show (L_1, L_2, L_3) be a basis I have to show basically it is a linearly independent set. Since dimension of V = dimension of V^* = 3, so if I have a three linearly independent element of V^*, certainly they will form a basis. Now to prove this is linearly independent, let be consider linear combination of L_1, L_2, L_3 . So, for any linear combination is given by $c_1L_1 + c_2L_2 + c_3L_3$, suppose this $c_1L_1 + c_2L_2 + c_3L_3 = 0$. I have to show that $c_1 = c_2 = c_3 = 0$ is only solution.

This implies that $(c_1L_1 + c_2L_2 + c_3L_3)(\mathrm{p}) = c_1L_1(\mathrm{p}) + c_2L_2(\mathrm{p}) + c_3L_3(\mathrm{p}) = c_1\mathrm{p}(t_1) + c_2\mathrm{p}(t_2) + c_3\mathrm{p}(t_3) = 0$, then definitely for any element p ∈ V. So, in particular for the set $\{1, \mathrm{x}, \mathrm{x}^2\}$ we have, $(\sum_{i=1}^{k} c_i L_i)(1) = 0$, i.e $(c_1 + c_2 + c_3) = 0$ and

$(\sum_{i=1}^{k} c_i L_i)(x) = 0$, i.e. $(c_1t_1 + c_2t_2 + c_3t_3) = 0$ and $(\sum_{i=1}^{k} c_i L_i)(\mathrm{x}^2) = 0$ i.e $(c_1{t_1}^2 + c_2{t_2}^2 + c_3{t_3}^2) = 0$. So this implies that I will have, $\begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ {t_1}^2 & {t_2}^2 & {t_3}^2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. My claim is that the coefficient matrix is basically nonsingular.

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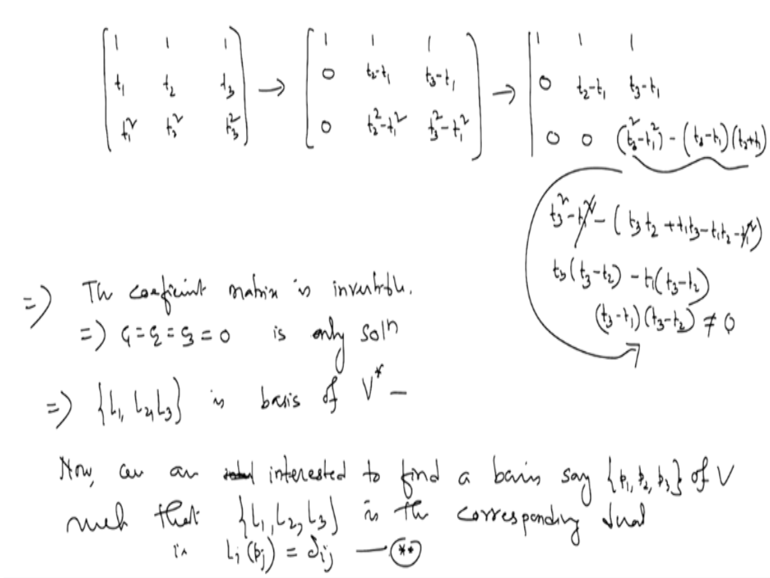

So for that I have to check, if I do the elementary row operations, I am getting what? I am getting $\begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ {t_1}^2 & {t_2}^2 & {t_3}^2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & t_2 - t_1 & t_3 - t_1 \\ 0 & {t_2}^2 - {t_1}^2 & {t_3}^2 - {t_1}^2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & t_2 - t_1 & t_3 - t_1 \\ 0 & 0 & ({t_3}^2 - {t_1}^2) - (t_3 - t_1)(t_2 + t_1) \end{bmatrix}$. So if I simplify this $({t_3}^2 - {t_1}^2) - (t_3 - t_1)(t_2 + t_1)$ = $t_3(t_3 - t_2)$- $t_1(t_3 - t_2)$ = $(t_3 - t_1)(t_3 - t_2) \neq 0$. So, this implies that the coefficient matrix is invertible. So, this implies that $c_1 = c_2 = c_3 = 0$ is only solution. So, this implies that your L_1, L_2, L_3 is a basis of V^*, fine.

Now, I have one question here, I want to find a basis for the space B for which this $\{L_1, L_2, L_3\}$ will be dual. Now, we are to find a basis say $\{p_1, p_2, p_3\}$ of V such that $\{L_1, L_2, L_3\}$ is the corresponding dual. That is $L_i\left(p_j\right) = \delta_{ij}$. So, I want to find that $\{p_1, p_2, p_3\}$ for which this relation, so it is suppose (**), so this relation will hold good, how to find it?

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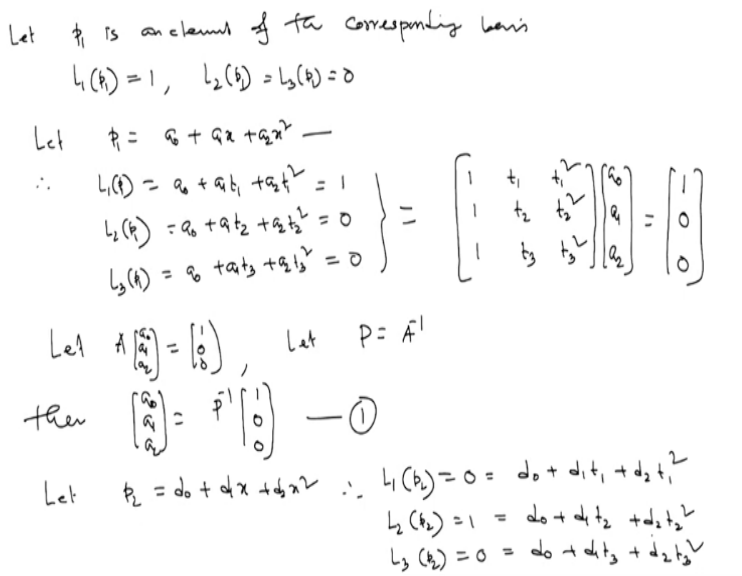

So, to answer this question what I have to do I will take let p_1 is an element of the corresponding basis such that $L_1(p_1) = 1$, $L_2(p_1) = L_3(p_1) = 0$. Since p is a polynomial function of degree ≤ 2, so I can consider say $p_1 = a_o + a_1\mathrm{x} + a_2\mathrm{x}^2$. So, this implies that $L_1(p_1) = a_o + a_1 t_1 + a_2 {t_1}^2 = 1$.

$L_2(p_1) = a_o + a_1 t_2 + a_2 {t_2}^2 = 0$ and $L_3(p_1) = a_o + a_1 t_3 + a_2 {t_3}^2 = 0$. So, this implies that I will have system of equations like we will have, $\begin{bmatrix} 1 & t_1 & {t_1}^2 \\ 1 & t_2 & {t_2}^2 \\ 1 & t_3 & {t_3}^2 \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Let it is basically your A $\begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So, already we have seen if t_1, t_2, t_3 are distinct then the scope is the matrix is invertible.

So, if I find the inverse of p, let p = A^{-1} . Then the coefficient $\begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} = \mathrm{p}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So, we have to find the inverse of this matrix so that once this a_o, a_1, a_2 is known from this equation say equation 1, then I will be able to find the corresponding polynomial, so p_1 will be known to me. Exactly same way, I can also proceed for p_2. Let $p_2 = d_o + d_1\mathrm{x} + d_2\mathrm{x}^2$ so that $L_1(p_2) = 0 = d_o + d_1 t_1 + d_2 {t_1}^2$.

and $L_2(p_2) = 1 = d_o + d_1 t_2 + d_2 {t_2}^2$ and $L_3(p_3) = 0 = d_o + d_1 t_3 + d_2 {t_3}^2$.

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$$\begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix}\begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = P^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Similarly, for $P_3 = g_0 + g_1 x + g_2 x^2$

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} = P^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ —}$$

$\therefore$ $\{p_1, p_2, p_3\}$ will be known.
& Any polynomial fn of degree ≤ 2, p,

Basically, I am getting what? I am getting again same, $\begin{bmatrix} 1 & t_1 & {t_1}^2 \\ 1 & t_2 & {t_2}^2 \\ 1 & t_3 & {t_3}^2 \end{bmatrix}\begin{bmatrix} d_o \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. So, this implies $\begin{bmatrix} d_o \\ d_1 \\ d_2 \end{bmatrix} = \mathrm{p}^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. So, I can immediately calculate the coefficient of p_2. Similarly for p_3, I will have $\begin{bmatrix} g_o \\ g_1 \\ g_2 \end{bmatrix} = \mathrm{p}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. So, in this way we can immediately calculate corresponding polynomial for $p_3 = g_o + g_1\mathrm{x} + g_2\mathrm{x}^2$. So, $\{p_1, p_3, p_3\}$ will be known. And any polynomial function of degree ≤ 2, say it is p.

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$p = c_1 p_1 + c_2 p_2 + c_3 p_3$
$\quad = p(t_1)p_1 + p(t_2)p_2 + p(t_3)p_3$ —

Let V be vector space over a field say F.
Let f be a linear fnal on V.
$f : V \rightarrow F$
Let dim of V is n, then dim $V^* = n$.
$\Rightarrow$ dim of range space of f will be at most 1
If f is a zero L.F. then the range space of f is zero subspace of F
$\Rightarrow$ Null space of zero LF is V.

If f is a nonzero L.F. then rank of f is 1
& Nullity of f is $n-1$, if dim of V is n.

I can write p = $c_1\, p_1 + c_2 p_2 + c_3 p_3 = p(t_1)p_1 + p(t_2)p_2 + p(t_3)p_3$. So, any polynomial can be written like this thing. I hope you have understood how to find the corresponding basis for which $\{L_1, L_2, L_3\}$ will be dual basis. So, now let me consider the linear functional and subspace. Let V be a vector space over a field say F. Let f be a linear functional on V.

So, f: V → F. Let dimension of V is n. Then dimension of $V^* = n$, because the dimension of f =1, So, this implies that dimension of range space have, f will be at most, how, if f is zero linear functional(L.F.) then the range space of f is zero subspace of f. So, this implies null space of zero linear functional is your entire space V.

If f is a nonzero linear functional, then rank of f is 1 and nullity of f is (n – 1), if dimension of V is n. If V is an infinite dimensional vector space, then I will say that the null space of nonzero linear functionals will be 1 less than the dimension of the vector space V. But since in the case of infinite, then there is no question of that one, so I will simply say that for finite dimensional vector space the dimension of the null space will be 1 less than dimension of V.

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Hyperspace: Any subspace of dim (n-1) of a vector space V of dim n is called a Hyperspace.

Somtime, it is also Known as Hyper plane.

$\therefore$ For a nonzero L.F, f, on a finite dim vector space V of dim n the null space of f is a Hyper space.

Annihilator: Let V be a f.d.v. space Let S be a subset of V. Then the set of linear functionals which annihilate S is called annihilator of S

Let $S^0 = \{f \in V^* : f(\alpha) = 0 \quad \forall \alpha \in S\}$

So, now let me introduce one terminology called hyperspace. Any subspace of dimension (n – 1) of a vector space V of dimension n is called a hyperspace, some time it is also known as hyperplane. So, this implies for the nonzero linear functional(L.F.) on a finite dimensional vector space V of dimension n, the null space of f is a hyperspace. Let me introduce one more terminology that is annihilation. Let V be a finite dimensional vector space.

Let S be a subset of V, then the set of linear functionals which annihilate S is called annihilator of s. Note that this set of linear functional which annihilate S is basically a subspace of vector space V star. Let S^o = { f ∈ V^* : f(α) = 0 ∀ α ∈ S}. So, this is the definition of S^o that is annihilator of the set S.

So, if the set S is subspace, then also S^o will be annihilator of the subspace S. So, S no need to be a subspace, even if it is a subset then also one can have annihilator and the annihilator will be same.

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<u>Theorem</u>: Let V be a f.d.v.s. of dim n, over a field F. Let S be a subspace of V of dim k, Then

$$\dim S + \dim S^o = \dim V.$$

<u>Pf</u>: Let $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ be a basis of S

Let $B' = \{\alpha_1, \alpha_2, \ldots \alpha_k, \alpha_{k+1}, \ldots \alpha_n\}$ be the basis of V

$\Rightarrow$ then exist a dual basis $B^* = \{f_1, f_2, \ldots f_n\}$ such that

$$f_i(\alpha_j) = \delta_{ij}$$

$\Rightarrow$ for any $f \in V^*$

$$f = \sum_{j=1}^{n} f(\alpha_j) f_j$$

If f annihilate S $\Rightarrow$ f annihilate all α_i, i = 1 to k.

$$\Rightarrow \quad f = \sum_{j=k+1}^{n} f(\alpha_j) f_j$$

Small result like this. Let V be a finite dimensional vector space(f.d.v.s.) of dimension n over a field F. Let S be a subspace of V of dimension k, then, dim of S + dim S^o = dim of V. How to prove it? Proof is also very easy. So, let me consider this. Let $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ be a basis of S. Then it can be extended by adding (n – k) extra linearly independent element to a basis for the B.

So, let $B' = \{\alpha_1, \alpha_2, \ldots \alpha_k, \alpha_{k+1}, \ldots \alpha_n\}$ be the basis of V. So, this implies there exist a dual basis $B^* = \{f_1, f_2, \ldots f_n\}$ such that $f_i(\alpha_j) = \delta_{ij}$. So, according to our last results, I can say there is a dual basis B^* consisting of $f_1, f_2, \ldots f_n$ such that $f_i(\alpha_j) = \delta_{ij}$ = 1 when i=j and $f_i(\alpha_j) = \delta_{ij}$ = 0 when i ≠ j.

So, this implies for any f ∈ V^*, f ∈ V^*, f = $\sum_{j=1}^{n} f(\alpha_j)\, f_j$ that is what we have already learnt from last results. So, I can express any linear functionals of V^*, as a linear combination of this thing. Now, if f annihilates let S implies f annihilate all α_i, i = 1 to k, so this implies that f can be written as a linear combination of $\sum_{j=k+1}^{n} f(\alpha_j)\, f_j$. This $\{f_1, f_2, \ldots f_n\}$ is a linearly independent set of V^*.

And so now this f_{k+1} to f_n will be again linearly independent elements and any linear functional f which annihilate S can be written as a linear combination of this (n – k) linear functional.

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$\Rightarrow$ dim of $S^0 = n-k$

& dim of $S = k$

$\Rightarrow$ dim S + dim $S^0 = k + (n-k) = n$ —

This implies that the dim of S^o = (n – k) and dim of S = k. So, this implies that dim of S + dim of S^o = k + (n – k) = n. So, I see that dimension of any subspace S of a vector space V plus dimension of its corresponding annihilator is equal to dimension of V.

In fact, if I consider any k dimensional subspace of the vector space V of dimension n or any finite dimensional vector space if its dimension is n, then I will say that the subspace S can be considered as intersection of (n – k) hyperspace also, which I am going to prove in my next class. Thank you.