Advanced Linear Algebra
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Lecture – 18
Linear Functional – 1

So, welcome to lecture series on advanced linear algebra. We have already discussed different types of linear transformations. Today, I will introduce another specific type of linear transformations which we are calling as linear functional. So, what is linear functional?

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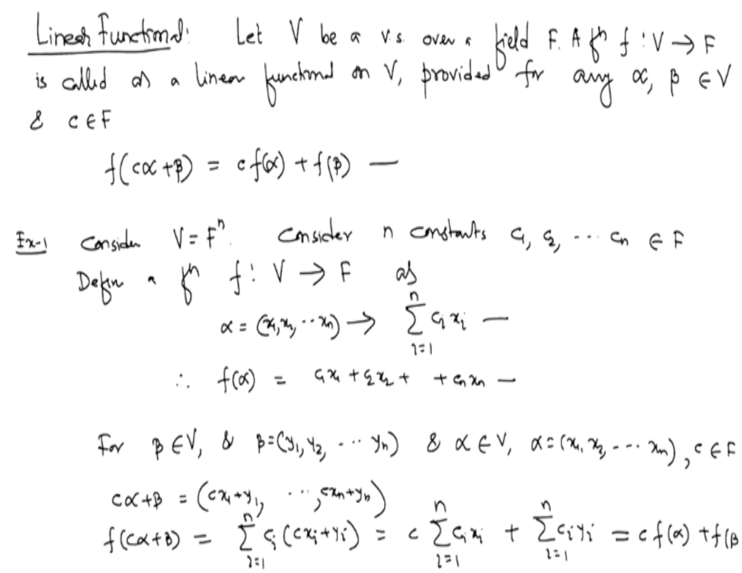

Let V be a finite dimensional vector space, not necessarily it has to be finite dimensional, let me check V be a vector space over the field F. A function f: V → F is called as a linear functional from V to F or linear functional on V, provided for any α, β ∈ V & c ∈ F, f(αc + β) = c f(α) + f(β). So, this is the definition of basically linear transformations.

So, f is a linear transformation V to F. Here this vector space is the field itself that is W in our usual definition of a linear transformation T from V to W, here W = F only. So, this is basically definition of linear functional. So let us consider some examples. Consider V =F^n over the field F. Consider n constants (c_1, c_1, c_3) ∈ F.

Define a function f: V → F as for any elements $\alpha = (x_1, x_2, x_3) \rightarrow \sum_{i=1}^{n} c_i\, x_i$. So, I have defined a function from V to F like this. So f(α) = $c_1\ x_1 + c_2 x_2 + \cdots + c_n x_n$. We can quickly check whether this function is a linear transformation or not or I mean to say the linear functional or not.

For any other elements , $\beta \in$ V and $\beta = (y_1, y_2, \ldots y_n)$ and $\alpha\ \in$ V, α =($x_1, x_2, \ldots x_n$), c ∈ F, (αc+β) = (c $x_1 +\ y_1$, c $x_2 +\ y_2, \ldots$ c $x_n + y_n$), so it will be like this thing. So, f(αc+β) = $c_i \sum_{i=1}^{n}(c\, x_i + y_i)$ = $c \sum_{i=1}^{n} c_i\, x_i$ + $\sum_{i=1}^{n} c_i\, y_i$ = cf(α)+f(β).

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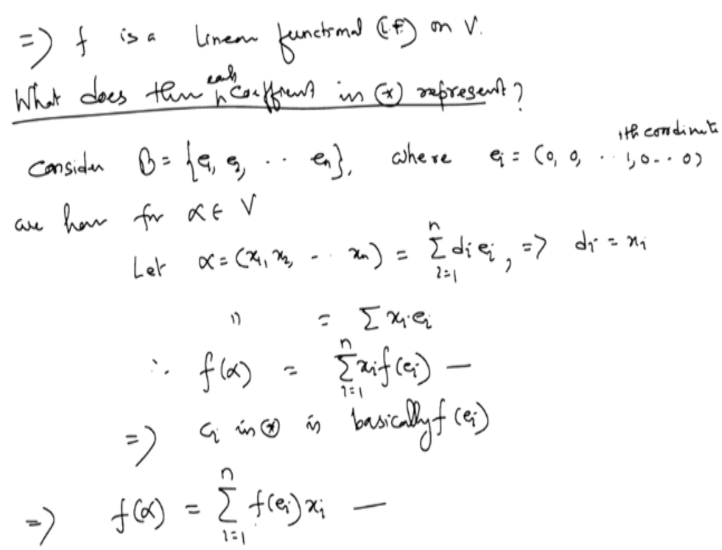

So, this implies that f is a linear functional(L.F.) on V. So, I can raise the question what does this coefficient in s(*)? If I define here this is my say star, so what does the coefficient in star, each coefficient in star represent? This is the question we can quickly answer by this thing. So, consider B = {$e_1, e_2, \ldots e_n$} where e_i = (0, 0, 1, 0, 0) where the i-th coordinate. Then we have for $\alpha \in$ V, let $\alpha = (x_1, x_2, \ldots x_n) = \sum_{i=1}^{n} d_i\, e_i$ t his implies that $d_i = x_i$. So, $\alpha = \sum_{i=1}^{n} x_i\, e_i$. So, f($\alpha$) = $\sum_{i=1}^{n} x_i\, f(e_i)$. So, this implies $c_i(*)$is basically $f(e_i)$.

So, $f(\alpha) = \sum_{i=1}^{n} c_i\, x_i$ that show this c_i is basically the image of e_i. So, this implies we are getting f(α) = $\sum_{i=1}^{n} f(e_i)\, x_i$. The question is, is it possible to generalize this concept for any finite dimensional vector space? I mean one can write down if ordered basis is given to you, then is it possible that you can define the functions f(α) = $\sum_{i=1}^{n} f(e_i)\, x_i$?

Now, we will definitely answer this question, but before going to answer this question let me give some more examples of some more linear functionals and then I will come to this question.

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<u>Ex</u> Let V be all nxn matrix over F

Defn a f^n

$$f: V \to F$$
$$A \to \text{trace of } A$$
$$\Rightarrow f(A) = \sum_{i=1}^{n} A_{ii} \quad — (*)$$

Then, we have for any A & B $\in$ V & c $\in$ F

$$f(cA+B) = \sum_{i=1}^{n}(cA_{ii}+B_{ii}) —$$
$$= c\sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$$
$$= cf(A) + f(B)$$

$\therefore$ f is also a L.F.

Let me consider second examples like this. Let V be all nxn matrix over a field F. So, define a function f: V → F as show this any element is matrix A → trace of A. This means that I am defining f (A) = $\sum_{i=1}^{n} A_{ii}$. If I consider this is the definition of the function then we can immediately check this function is also a linear transformation from V to F, I mean this function f is a linear functional.

Here we have for any A & B ∈ V and c ∈ F, f(cA+B) = $\sum_{i=1}^{n}(cA_{ii}+B_{ii}) = c\sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = cf(A) + f(B)$, so f is also a linear functional.

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Ex Let V be the space of all polynomial fn's from F into F

$\Rightarrow$ for $p \in V \Rightarrow \exists\, n \in \mathbb{N}$, s.t.

$$p = \sum_{i=0}^{n} p_i x^i = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n \text{ —}$$

Consider a constant $t \in F$

Define a fn $f: V \rightarrow F$

$$p \rightarrow p(t)$$

$$\Rightarrow \quad f(p) = p(t) \text{ —}$$

f is also a linear functional from V into F (H.W.)

Ex: Let V be the space of all continuous & real valued fn on $[a, b]$

i.e. $V = C[a,b]$ —

Define a fn $L: V \rightarrow F$.

So, I have taken example over the finite dimensional space, now let me consider a couple of examples into the infinite dimensional vector space. Let V be the of all polynomial function from field say F into f. So, this implies for p ∈ V implies ∃ n(positive integer) ∈ N(natural number) such that, $p = \sum_{i=0}^{n} p_i x^i = p_o + p_1 x + p_2 x^2 + \cdots + p_n x^n$.

Consider a constant t ∈ F. Define a function f: V → F as p → p(t). So, this implies that f(p) = p(t). So, if I define like this, then again you can cross check that f is also a linear transformation from V to F or is a linear functional from V into F, you can check it. So, this you can consider as a homework to check it whether f is a linear functional or not. In mathematics, most of the times you use standard examples.

So let me consider that one, say let V be the space of all real valued continuous functions, continuous and real valued function on [a, b], i.e. V = c[a, b]. So, suppose this is my space. Define a function from V to F L: V → F.

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$L(g) = \int_a^b g(t)dt$ —

is also a L.F on V.

Consider $L(V,F)$ i.e. collection of all LF on V.
which is also a vector space & if dim of V is finite say n
then the dim $L(V,F) = n \times 1 = n$

$L(V,F) = V^*$, which is called as Dual space of Vector space V.

<u>Theorem</u>: Let V be a f.d. v.s. over a field say F. Let $B = \{\alpha_1, \cdots \alpha_n\}$
be an ordered basis of V. Then there exist an ordered basis
$B^* = \{f_1, f_2, \cdots f_n\}$ of V^* such that
$f_i(\alpha_j) = \delta_{ij} \cdot \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$, i.e. Kroneckar Delta

As L(g) = $\int_a^b g(t)dt$. So, if I define this type of function L from V to F, so this is again you can check it is a linear functional and satisfy the definition of linear transformations. Because this integral is a linear operator, so we can quickly check L($g_1 + g_2$) = L(g_1)+L(g_2) , L(cg) = c L(g) is linear functional on V, you can check it. So, we have seen couple of examples of linear functionals.

Now, consider L(V,F) means that is collection of all linear functional on V. So, we know this is also a vector space because it is a special case of L(V,W) which is also a vector space and if dimension of V is finite say n, then the dimension of L(V,F) = nx1 = n. In general L(V,F) we are denoting as V^*, which is called as dual space of vector space capital V, I mean V^* is the dual of V.

We can also cross check the concept of basis as you have seen in the case of L(V,W) here also suppose a basis on V is given and also basis in capital V is given, then also one can introduce a basis on L V, F also. As we have seen that if you recall while talking about the vector space L(V,W) and V is a vector space of dimension n and W is a vector space dimension m, we have shown it that there are exactly (mn) number of linear transformations E_{pq} which are satisfying that.

Suppose if you consider the basis of the V is say B consisting of $(\alpha_1, \alpha_2, \ldots \alpha_n)$ and if I consider B' the basis for the W consisting of say $(\beta_1, \beta_2, \ldots \beta_m)$.then we have defined E_{pq}. So, if you recall correctly $E_{pq}(\alpha_j) = \beta_p$ if j = q, otherwise equal to 0. So, in that way we introduced (mn) number of linear transformations. The same concept also holds good here.

The only thing is that we have to find out what would be the corresponding appropriate linear functionals when an ordered basis in B is given to us. So, for this let us write down one small theorem. Let V be a finite dimensional vector space over a field say F. Let B = $(\alpha_1, \alpha_2, \ldots \alpha_n)$ be an ordered basis of V, then there exist an ordered basis B^* = $\{f_1,\ f_2,\ \ldots\ f_n\}$ the set of linear functionals of V^* such that $f_i(\alpha_j) = \delta_{ij}$, i.e. Kronecker delta. So, this is δ_{ij} =1 when i = j, δ_{ij} =0 when i $\neq$ j.

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Also, for any $f \in V^*$

$$f = \sum_{j=1}^{n} f(\alpha_j) f_j \quad \text{— (i)}$$

For any $\alpha \in V$, $\alpha = \sum_{j=1}^{n} f_j(\alpha)\, \alpha_j$ — (ii)

Pf: Given $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ as an ordered basis of V.

Consider $B' = \{1\}$ as basis of F.

$\therefore$ for i, i = 1 to n, one can define a unique LF

f_i on V such that

$f_i(\alpha_j) = \delta_{ij}$ —

$\therefore$ there are n L.F.

claim, $\{f_1, f_2, \ldots f_n\}$ is L.I

For any linear combination

$f = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = \Sigma c_i f_i$

Also for any f $\in$ V^*, f = $\sum_{j=1}^{n} f(\alpha_j)\, f_j$. So, any linear functional can be written as a linear combination of f_j with the coefficients as function value at α_j and for any α $\in$ V, α $=\sum_{j=1}^{n} f(\alpha_j)\, f_j$. So, let us give a proof. Given B = $(\alpha_1, \alpha_2, \ldots \alpha_n)$ as an ordered basis of V. Consider B^* = {1} as basis of F.

So, for each i, i = 1 to n, one can define a unique linear functional f_i on V such that $f_i(\alpha_j) = \delta_{ij}$. One can define or I can say that just a unique linear functional f_i such that $f_i(\alpha_j) = \delta_{ij}$, I meant to say $f_i(\alpha_i)$ = 1when i=j and $f_i(\alpha_j)$ = 0, when j$\neq$i. So, this type of existence of linear transformation already we have proved it. So, I can give a guarantee that this type of linear functional exists.

So, existence is guaranteed from the previous lecture. So, one can say there are n such linear functionals on V satisfying this type of condition. It is basically saying that E_{pq} recall here p = 1, q varying from 1 to n. So, my E_{1q} or E_{p1} is my f_i. So, I hope this will be clear how one can say there exists a unique linear transformation or linear functional f_i on V which will map α_i I to 1, all other elements to 0. So, there are n linear functionals.

Claim: This $\{f_1, f_2, \ldots f_n\}$ is linearly independent. For any linear combinations say $f = (c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = \sum_{i=1}^{n} c_i f_i$.

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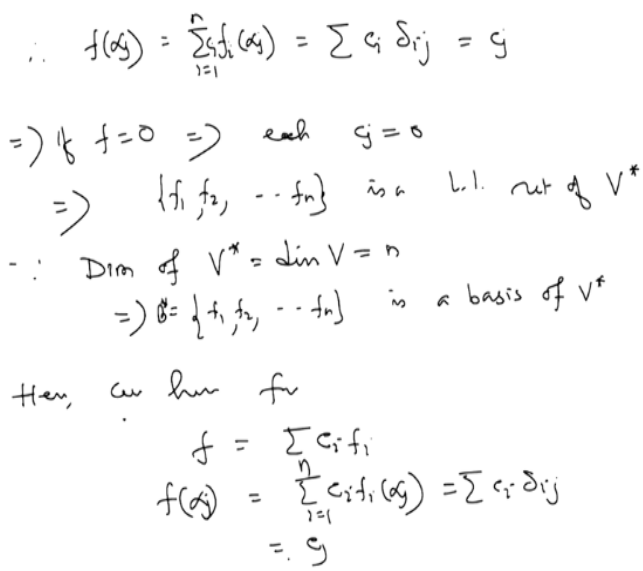

I mean if you take like this which is equal to f(α_j) = $\sum_{i=1}^{n} c_i\, f_i(\alpha_j)$ = $\sum_{i=1}^{n} c_i\, \delta_{ij}$ = c_j. So, this implies if f = 0 implies that each c_j = 0. So, this implies that $\{f_1, f_2, \ldots f_n\}$ is a linearly independent set of V^*.

And since dimension of V^* = dimension of V = n, implies the set if I denote B^*= $\{f_1, f_2, \ldots f_n\}$, then this is a basis of V^*. So, here we have for f =$\sum_{i=1}^{n} c_i\, f_i$, and $f(\alpha_j)$= $\sum_{i=1}^{n} c_i\, f_i(\alpha_j)$ = $\sum_{i=1}^{n} c_i\, \delta_{ij}$ = c_j. So, each coefficient $(c_1, c_2, \ldots c_n)$ are nothing $f(\alpha_1), f(\alpha_2), \ldots f(\alpha_n)$.

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$$\Rightarrow \quad f = \sum_{j=1}^{n} f(\alpha_j) f_j \quad — (*) \quad \Rightarrow \quad [f]_{B^*} = [f(\alpha_1), f(\alpha_2), \cdot\cdot\ f(\alpha_n)]^T$$

Again, for any $\alpha \in V$, we have

$$\alpha = \sum_{j=1}^{n} x_j \alpha_j \quad —$$

$$\Rightarrow \quad f_i(\alpha) = \sum_{j=1}^{n} x_j f_i(\alpha_j)$$

$$= \sum_{j=1}^{n} x_j \delta_{ij} = x_i$$

$$\Rightarrow \quad \alpha = \sum_{j} f_j(\alpha) \alpha_j$$

$$\Rightarrow \quad [\alpha]_B = [f_1(\alpha), f_2(\alpha), \cdot\cdot\ f_n(\alpha)]^T$$

This implies the function f = $\sum_{j=1}^{n} f(\alpha_j) f_j$. Again for any $\alpha \in$ V, we can write down α_j linear combination basis element of p that is $(\alpha_1, \alpha_2, \ldots\ \alpha_n)$. So let $\alpha = \sum_{j=1}^{n} x_j\, \alpha_j$. So, this implies that $f_i(\alpha_j) = \sum_{i=1}^{n} x_i\, f_i(\alpha_j) = \sum_{i=1}^{n} x_i\, \delta_{ij} = x_i$, this implies $\alpha = \sum_{j=1}^{n} f_j(\alpha)\, \alpha_j$, this implies $[\alpha]_B$ = $[\, f_1(\alpha), f_2(\alpha), \ldots f_n(\alpha)]^{\,T}$ and $[f]_{B^*}$ = $[\, f_1(\alpha), f_2(\alpha), \ldots f_n(\alpha)]^{\,T}$.

So, we have seen that any linear functional can be written as a linear combination of this $\{f_1, f_2, \ldots f_n\}$ where $\{f_1, f_2, \ldots f_n\}$ has been obtained based on the concept that given information that B = $(\alpha_1, \alpha_2, \ldots\ \alpha_n)$as ordered basis for the V. We have seen here that if you consider ith coordinate of that coordinate matrix that f(α_i), I mean functional value at α_i is basically ith coordinate of the coordinate matrix of f with respect to ordered basis B^*.

Exactly same thing we have seen for any vector $\alpha \in$ V, then the ith coordinate of the coordinated matrix of alpha with respect to ordered basis B is basically functional value of linear function $f_i(\alpha_i)$, which we will make it more clear when we will consider more examples, which I am going to do in my next class. Thank you.