**Lecture – 17**
**Matrix Representation of Linear Transformations - 2**

So, welcome to lecture series on advanced linear algebra. We have already understood the relation between the set of linear transformations which are defined over a finite number of spaces with the set of matrices, I mean to say the matrix representation of the linear transformations.

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In that context, let me also continue that one that I left as an example. Let T: $F^3 \rightarrow F^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_2 + x_3)$, suppose that I have taken this one. So, consider B = { $\alpha_1$ = (1, 0, 0), $\alpha_2$ = (0, 1, 0), $\alpha_3$ = (0, 0, 0)}. So, this is an ordered basis over 3-tuple space $F^3$ & B' = {$\beta_1$= (1, 0), $\beta_2$= (0, 1)}. We know how to obtain $[T]_{BB'} = A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. The question is suppose a matrix is given to me, how I will define the corresponding linear transformation, is it possible to do it? Let me talk about this specific example case. So, here we can find what is the matrix representation of A. A will be definitely again 2x3 matrix. So, it will be a 2x3 matrix. A= $[A_1 \ A_2 \ A_3]$.

So, $T(\alpha_1) = (1, 0) = c_1\beta_1 + c_2\beta_2 = c_1(1, 0) + c_2(0, 1)$ this implies, $c_1 = 1$ and $c_2 = 0$. $[T(\alpha_1)]_{B'} = A_1$.

S, so $(1, 0) = A_1$ will be the first column. And $[T(\alpha_2)]_{B'} = A_2$. So, $T(\alpha_2) = (1, 1) = (c_1, c_2)$.

$T(\alpha_3) = (1, 1)$. So, I have got 2x3 matrix A, fine. So then can I express with the help of the A the linear transformation T for this case? Now for this case to define T I have to take $T(X = (x_1, x_2, x_3))$

$$= (x_1, x_2, x_3) A^T = (x_1, x_2, x_3) \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = (x_1 + x_2 + x_3, x_2 + x_3).$$ So, in this example what I found

that if the T is a mapping from 3-tuple space to 2-tuple space and if I represent A as the matrix representation of T related to B and B' when B and B' is given to us, so in this case the linear transformation T can be defined as you know $X A^T$. Now, let me do the similar thing for this changing the space.

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Now, if I consider example to say $T: F^3 \rightarrow F^2$ defined by $(x_1, x_2, x_3)^T \rightarrow (x_1 + x_2 + x_3, x_2 + x_3)^T$. Now, I will also consider, $B = \{\alpha_1 = (1, 0, 0)^T, \alpha_2 = (0, 1, 0)^T, \alpha_3 = (0, 0, 1)^T\}$ and $B' = \{\beta_1 = (1, 0)^T, \beta_2 = (0, 1)^T\}$. Now, again if I go the matrix representation of T, $[T]_{BB'} = ?$

$T^*(\alpha_1) = (1, 0)^T = c_1\beta_1 + c_2\beta_2$ , $T^*(\alpha_2) = (1, 1)^T$, $T^*(\alpha_3) = (1, 1)^T$, $[T^*]_{BB'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = A$,

$T^*(X = (x_1, x_2, x_3)^T) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. With the help of the matrix representation of the linear transformation one can also go to the corresponding linear transformation, one can define linear transformation also.

Now, we are curious to know one thing that we have seen for any T: V → W where V and W are finite dimensional and then for any pair of ordered basis B and B′ where the B is the basis for V and B′ is ordered basis for W, we have a matrix representation of T. Now consider T is a linear operator on a finite dimensional vector space V of dimension n over field say F, now consider this one.

Let B = $(\alpha_1, \alpha_2, ... \alpha_n)$ be an ordered basis for V. Since my V and W are same, therefore I will also have matrix representation of T related to the ordered basis B and B because both are same, $[T]_{BB}$ = A where A is a nxn square matrix over F. For sake of simplicity, I will write down this as simply matrix representation of T related to ordered basis B that is, $[T]_B$ = A, I might write like this.

Now, consider B′ = $\left(\alpha_1{}', \alpha_2{}', ... \alpha_n{}'\right)$ be another ordered basis this for V. $[T]_{B'}$ = B which is also nxn squared matrix over F. Now, we are curious to know whether any relation between A and B exist or not. Does there exist a relation between A and B? This is the question. So, answer is yes. Now, how to obtain that relation, what is that relation, so let us analyze that one.
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We have for any $\alpha \in V$, $[\alpha]_B = P[\alpha]_{B'}$ where $P$ is nonsingular. Since $\alpha_j' = \sum_{i=1}^n P_{ij}\alpha_i$ and if I say that $[\alpha]_B = X$, $[\alpha]_{B'} = X'$. then we know $\alpha = \sum_{j=1}^n x_j'\,\alpha_j' = \sum_{j=1}^n x_j'\sum_{i=1}^n P_{ij}\alpha_i = \sum_{j=1}^n(\sum_{i=1}^n P_{ij}x_j')\alpha_i$, This implies $X = PX'$. If you recall this relation that $[\alpha]_B = P[\alpha]_{B'}$. So $[T\alpha]_B = [T]_B[\alpha]_B$. Then $[T\alpha]_{B'} = [T]_{B'}[\alpha]_{B'}$. This implies $[T\alpha]_B = [T]_B P[\alpha]_{B'}$ because $[\alpha]_B = P[\alpha]_{B'}$. This implies from 3, we have $[T\alpha]_{B'} = [T]_{B'}[\alpha]_{B'}$ this is $P^{-1}[T\alpha]_B = [T]_{B'}[\alpha]_{B'}$.

This implies that $P^{-1}[T]_B[\alpha]_B = [T]_{B'}P^{-1}[\alpha]_B$ this implies $(P^{-1}[T]_B - [T]_{B'}P^{-1})[\alpha]_B = 0$ and this is true for $\alpha$ this implies $P^{-1}[T]_B = [T]_{B'}P^{-1}$ this implies $P^{-1}[T]_B P = [T]_B$ this implies $P^{-1}[T]_B P = [T]_{B'}$.

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$$P^{-1}[T]_B P = [T]_{B'}$$

$$\Rightarrow P^{-1}AP = B$$

Theorem: Let $V$ be a f.d.v.s of dim $n$ over $F$. Let $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ &
$B' = \{\alpha_1', \alpha_2', \ldots \alpha_n'\}$ be two ordered bases of $V$. Let $T$ be a linear
operator on $V$. Then

$$[T]_{B'} = P^{-1}[T]_B P$$

Where $P$ is an $n \times n$ invertible matrix over $F$ & $P = [P_1, P_2 \ldots P_n]$
the $j$th column $P_j$ denote $[\alpha_j']_B$

In fact, if $P$ is any $n \times n$ invertible matrix over $F$ & $T$ is a LO on
$V$ of dim $n$. Let $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be an ordered basis of $V$, then

So, $P^{-1}[T]_B P = [T]_{B'}$ so, this is the relation. So, this implies if it is given to us if the matrix representation of the operator T related to ordered basis B equal to say A, then $P^{-1}A P = B$. I can conclude like this in terms of theorems like this. Let V be a finite dimensional vector space of dimension n over the field say F.

Let $B = (\alpha_1, \alpha_2, \ldots \alpha_n)$ & $B' = (\alpha_1', \alpha_2', \ldots \alpha_n')$ be two ordered bases of V. Let T be a linear operator on V. Then $[T]_{B'} = P^{-1}[T]_B P$ where P is an nxn invertible matrix over F and if P = $[P_1, P_2, \ldots P_n]$, the j-th column $P_j$ denote $[\alpha_j']_B$. In fact, if P is any nxn invertible matrix over F and T is a linear operator on V of dimension n and let $B = (\alpha_1, \alpha_2, \ldots \alpha_n)$ be an ordered basis of V.

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then exist another ordered basis $B' = \{\alpha_1', \alpha_2', \ldots \alpha_n'\}$ such that
$[\alpha_j']_B = P_j$ (i.e. $j$th column of $P$) &

$$[T]_B = P[T]_{B'} P^{-1} \quad \underline{\qquad}$$

Ex $T: V = F^{3 \times 1} \to V$

$T\alpha = A\alpha$, where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

Let $\alpha = (x_1, x_2, x_3)^T$

$T\alpha = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} (x_1, x_2, x_3)^T$

Consider $B = \{e_1 = (1,0,0)^T, e_2 = (0,1,0)^T, e_3 = (0,0,1)^T\}$

$B' = \{\alpha_1' = (1,1,1)^T, \alpha_2' = (1,1,0)^T, \alpha_3' = (1,0,0)^T\}$

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Then there exist another ordered basis $B' = \left(\alpha_1', \alpha_2', \dots \alpha_n'\right)$ such that $[\alpha_j']_B = P_j$ (i.e j-th column of $P$ ). In fact, there is an ordered basis B' such that like this and $[T]_B = P[T]_{B'}P^{-1}$.So, we have seen the existence of that type of relation already we have proved. Now, if one gives an invertible matrix $P$ and suppose one ordered basis say of B is given to us, then we can immediately find another ordered basis B' consisting of $\left(\alpha_1', \alpha_2', \dots \alpha_n'\right)$ where $[\alpha_j']$ is basically the j-th column of the matrix $P$ and then this relation also holds good.

So, construction of the $(\alpha_1', \alpha_2', \dots \alpha_n')$ basically already we have seen it, if an invertible matrix is given to us then $\alpha_j'$ can be constructed like this. Whether this $(\alpha_1', \alpha_2', \dots \alpha_n')$ will be linearly independent or not that is the question. We can immediately check, this will be also linearly independent. I want to give you as home assignment to cross check that the $(\alpha_1', \alpha_2', \dots \alpha_n')$ constructed by this method.

I mean where $\alpha_j' = \sum_{i=1}^{n} P_{ij}\, \alpha_j$ is given to us okay. So, we have to prove that $(\alpha_1', \alpha_2', \dots \alpha_n')$ is ordered basis for the matrix p and with respect to that basis these types of relations also hold good that we have to prove it. So, now let me consider some examples and see. So, let me consider $T: V = F^{3x1} \rightarrow F^2$, Now, this to V, this is linear operator on V. Say $T\alpha = A\alpha$, where A $= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, suppose A is this one. So, here alpha means basically 3x1 column vector, so I can say let $\alpha = (x_1, x_2, x_3)^T$. So, $T(\alpha) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, Consider t wo ordered bases B equal to our standard ordered basis for $F^{3x1}$ which is similar to $F^3$, only instead of row it will be column just taking the transpose.

So, my first one let me consider B $= \{e_1 = (1,0,0)^T,\ e_2 = (0,1,0)^T,\ e_3 = (0,0,1)^T$. This is the standard basis on $F^{3x1}$. And another one say $B' = \{\alpha_1' = (1,1,1)^T,\ \alpha_2' = (1,1,0)^T,\ \alpha_3' = (1,0,0)^T$. So, we can immediately check that B and $B'$ are the ordered bases for the space $F^{3x1}$. So, that you can check, it is not a difficult task. Now, let us see the matrix representation of T related to ordered basis B and matrix representation of T related to ordered $B'$.

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$T((x_1,x_2,x_3)^T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Let $[T]_B = G$

$T((1,0,0)^T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = c_1 e_1 + c_2 e_2 + c_3 e_3$

$T((0,1,0)^T) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$\therefore [T]_B = G = A$

$[T]_{B'}$

$T((1,1,1)^T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1+c_2+c_3 \\ c_1+c_2 \\ c_1 \end{bmatrix}$

$G = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

$H = [T]_{B'} = \begin{bmatrix} 3 & 3 & 2 \\ -1 & -2 & -1 \\ 0 & 2 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1+c_2+c_3 \\ c_1+c_2 \\ c_1 \end{bmatrix}$

It is given to us $T((x_1, x_2, x_3)^T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. So, this means that let $[T]_B = G$, some

name, then what is G? So certainly, G will be 3x3 matrix. Now, $T((1,0,0)^T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = c_1 e_1 + c_2 e_2 + c_3 e_3$, if I consider that you will see that $c_1 = 1$, $c_2 = 1$, $c_3 = 2$. Similarly

$T((0,1,0)^T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. And similar for 3${}^{rd}$ column. So, $G = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. So, the

$[T]_B = G = A$. Now, what can you say about the $[T]_{B'}$? So, for that again I have to say

$T((1,1,1)^T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1+c_2+c_3 \\ c_1+c_2 \\ c_1 \end{bmatrix}$

Now the next is again for the $T((1,1,0)^T) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, I will have basically last one

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1+c_2+c_3 \\ c_1+c_2 \\ c_1 \end{bmatrix}$. So, $H = [T]_{B'} = \begin{bmatrix} 3 & 3 & 2 \\ -1 & -2 & -1 \\ 0 & 2 & 0 \end{bmatrix}$ So, I will have this one. So, I will have

the matrix representation of p related to ordered basis $B'$equal to H equal to like this vector. Now,

our aim is to find a nonsingular matrix $P$ such that $H = P^{-1}G\,P$.

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We want fund an invertible 3x3 matrix $P$ s.t

$$H = P^{-1} G P \quad \text{—}$$

then $P_j = [\alpha_j']_B$

We want to find an invertible 3x3 matrix $P$ such that $H = P^{-1}G\,P$ that we have to find. Here $\alpha_j = [\alpha_j']_B$. So, I hope you will be able to calculate this and cross check. So please complete this work and if you have any difficulties, you can raise your questions. I hope this definitely would help to understand our next lecture. Thank you.