Lecture – 16
Matrix Representation of Linear Transformations - 1

Welcome to lecture series. We have seen if V and W be two vector spaces of dimension say n and m respectively over the field F, then the collection of all linear transformations from V to W is a vector space. We have also seen that when V = W that it basically collection of linear operators from V to V or linear operator on V. Today we want to answer the following questions.

(Refer Slide Time: 01:18)

Q(1):- is it possible to represent a linear transformation(LT) by a matrix? In general, the answer is no. It depends on the vector spaces over which linear transformation(LT) it is defined. If the linear transformation is defined over the finite dimensional vector spaces, then it can be represented by a matrix. Next question will be whether this matrix representations is it unique? Q(2):- If at all it is possible to represent linear transformation by a matrix, then is it unique?

Again, answer will be it depends on the given ordered basis. So, for a given ordered basis it will be unique. Q(3):- How to find the matrix representation of a linear transformation is defined on finite dimensional vector spaces. So, consider V and W be two finite dimensional vector spaces

over a field say F with the dimension n and m respectively. Let $B = (\alpha_1, \alpha_2, \dots \alpha_n)$ and $B' = (\beta_1, \beta_2, \dots \beta_m)$ be the ordered basis for V and W respectively.

So, I have considered they are the ordered basis. Now these two ordered bases are fixed. Let T be a linear transformation from V into W that is $T : V \rightarrow W$. We want to find out a m cross n matrix which basically we will say that matrix representation of T related to ordered basis B and B prime, this one. So, for this what I will do consider $\alpha \in V$ be any element. Since $\alpha \in V$, I will be able to express α in terms of the basis element of V.

So, there exist n scalars $(x_1, x_2, \dots x_n) \in F$ such that $\alpha = \sum_{i=1}^{n} x_j \alpha_j$, that is $[\alpha]_B = X = [x_1, x_2, \dots x_n]^T$.

.

(Refer Slide Time: 07:40)

Since $T: V \rightarrow W$, now for each $\alpha_j \in V$, I will be able to express the, $T(\alpha_j) = \sum_{i=1}^{m} a_{ij} \beta_i$, the basis element W, I mean to say for each $\alpha_j \in V$, $T(\alpha_j)$ can be written as linear combinations of $(\beta_1, \beta_2, \dots \beta_m)$, this linear transformation T will be defined by the action of T on each α_j basically.

So, from this also one can define the linear transformation T. Now, what can you say about the image of alpha under T? So, $T(\alpha) = T(\sum_{i=1}^{n} x_j \alpha_j) = \sum_{i=1}^{n} x_j T(\alpha_j) = \sum_{i=1}^{n} x_j \sum_{i=1}^{m} a_{ij} \beta_i = $

$\sum_{i=1}^{m}(\sum_{i=1}^{n} a_{ij}\, x_j)\beta_i$. So, this implies $[T(\alpha)]_{B'}$ having r-th entry as r-th row of AX =

$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ ...+a_{1n}x_n \\ a_{r1}x_1 + a_{r2}x_2 + \ ...+a_{rn}x_n \\ a_{m1}x_1 + a_{m2}x_2 + \ ...+a_{mn}x_n \end{bmatrix}$. So, the r-th row of AX is basically the r-th entry $[T(\alpha)]_{B'}$. So,

this implies $[T(\alpha)]_{B'} = AX = A[\alpha]_B$, so this I can write is equal to like this.

(Refer Slide Time: 12:29)

Here A is mxn matrix over F and is known as matrix representation of T relative to ordered basis B and B'. Symbolically I can write down $[T]_{B,B'} = A$. Now, certainly you may expect that if I change the ordered basis, then what will happen? If I change the ordered basis, I will get another matrix representation. But if I fix ordered basis then I am getting unique, this will be unique matrix. Here A = $[A_1, A_2, ... A_n]$.

If I write this is the A matrix where $A_1, A_2, ... A_n$ denote the first, second and n-th column of the matrix A, then $[T(\alpha_j)]_{B'} = A_j$ because $T(\alpha_j) = \sum_{i=1}^{m} a_{ij}\, \beta_i$ this is basically saying your j-th column of the matrix A is equal to a_{ij}. So, what do we obtain? For a given linear transformation I have obtained an m cross n matrix. Now, if mxn matrix Q is given over say F, then is it possible to have a linear transformation from V into W?

Answer is yes. How? It is like this. I can define say U(α) = U($\sum_{i=1}^{n} x_j\, \alpha_j$) = $\sum_{i=1}^{m}(\sum_{i=1}^{n} q_{ij}\, x_j)\beta_i$. If I say that Q = $(q_{ij})_{mxn}$. I am getting this type of expressions. Now, this U the way I have defined this here it is a linear transformation, we can immediately check U is a linear transformation from V into W.

So, what you have seen? There is a 1-1(one-one) relation between the set of linear transformation from V to W that is from L(V,W) to set of all mxn matrices over the field F. So, let me write down in formal way what is the conclusion here.

(Refer Slide Time: 17:18)

Theorem: Let V & W be two f.d.v. spaces over f, with dimension n & m respectively. Let B & B' be ordered bases of V & W respectively. Then for each L.T. T: V → W, there exists a unique mxn matrix A over F such that for each $\alpha \in V$

$$[T(\alpha)]_{B'} = A[\alpha]_B$$

Furthermore, for T → A is one one.

i.e. if S denote set of all mxn matrices over f, then there is 1-1 relation between L(V,W) & S.

So, I can write like this. Let V & W be two finite dimensional vector spaces over a field F with dimension n & m respectively. Let B and B′ be ordered bases of V and W respectively. Then for each linear transformations T: V → W, there exist a unique mxn matrix A over F such that for each $\alpha \in$ V,$[T(\alpha)]_{B'} = A[\alpha]_B$.

Furthermore, for T→ A for each linear transformation there exist a mxn matrix from T → A, this mapping is one-one. I mean to say if I consider mxn matrix over the field F, then also one can find out a linear transformation from V to W. That is if S denote set of all mxn matrices over F then there is a 1-1 relation between L(V,W) and S. So, that is what we have proved.

(Refer Slide Time: 20:46)

Ex-1. Consider a L.T. $T: F^3 \to F^2$, defined by

$$T(x_1, x_2, x_3) = (x_1 - 2x_2, x_2 + x_3) \quad —(*)$$

Consider $B = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ & $B' = \{e'_1 = (1,0), e'_2 = (0,1)\}$

Then $[T]_{B,B'} = ?$

$Te_1 = (1, 0) = a e'_1 + b e'_2 = a(1,0) + b(0,1)$

$\Rightarrow a = 1, b = 0$

$\therefore [T]_{B,B'} = A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$Te_2 = (-2, 1) = a(1,0) + b(0,1) = (a, b)$

$\therefore a = -2, b = 1$

$Te_3 = (0, 1) = 0 e'_1 + 1 e'_2$

Now, let me consider some examples. Consider a linear transformation of $T: F^3 \to F^2$, defined by $T(x_1, x_2, x_3) = (x_1 - 2x_2, x_2 + x_3)$, suppose I have taken like this randomly. So, we can check this is a linear transformation $T: F^3 \to F^2$. Consider $B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$. This is a standard ordered basis on F^3, space of 3 tuples.

And $B' = \{e'_1 = (1, 0), e'_2 = (0, 1)\}$ some name, we can also give $(\alpha_1, \alpha_2, \alpha_3)$ instead of e_1, e_2, e_3 and then $\beta_1, \beta_2, e'_1, e'_2$ also. Then what is the matrix representation of T relative to ordered basis B and B' ($[T]_{B,B'}$) first we have to learn this one. To find that one what I have to do? According to the definition we have to see the image of e_1, e_2, e_3 under T also.

First, $T(e_1) = (1, 0)$, because $(x_1 - 2x_2, x_2 + x_3)$, , so $T(e_1) = (1, 0) = c_1 e'_1 + c_2 e'_2 = c_1(1, 0) + c_2(0, 1)$. So this implies that, $c_1 = 1$ and $c_2 = 0$, $T(e_2) = (-2, 1) = c_1(1, 0) + c_2(0, 1) = (c_1, c_2)$, $c_1 = -2$ and $c_2 = 1$, $T(e_3) = (0, 1) = 0e'_1 + 1e'_2$, $c_1 = 0$ and $c_2 = 1$, this means $[T]_{B,B'} = A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

So, the matrix representation of this $T: F^3 \to F^2$ is given like this, A equal to this one. Here one more thing that you let me tell you, now instead of the standard ordered basis if we take any other ordered basis, then you can also find out the matrix representation corresponding to that ordered basis. This B, I will replace by say my $\alpha_1 = (1, 1, 1)$; $\alpha_2 = (1, 1, 0)$, $\alpha_3 = (1, 0, 0)$ and B' I will keep it as it is.

Then also you will have a new matrix. The matrix representation suppose in that ordered basis

instead B*and then other one is B′, then matrix representation of T related to B* and B prime will be different. You may again cross check is it possible to correlate the two matrices again, please take it as a homework and see it. We will come to this question maybe in the next class.

(Refer Slide Time: 26:27)

Let me take another example. Let V be the space of all polynomial function of ≤ 3 over a field is F. So, this vector space is also finite dimensional of dimension 4. So, consider basis as B = {1, x, x², x³} Let me consider W also equal to W = V & B′ = B. So let T be a linear transformation from V into V that is here T is a linear operator.

For any p ∈ V, p = $p_o + p_1 x + p_2 x^2 + p_3 x^3$, something like this because polynomial of ≤ 3, so like that. So, let me define $(Tp)(x) = p_1 + 2p_2 x + 3p_3 x^2$. So, basically the T, I have considered is the differential operator. Now, what can we say about the matrix representation of T ($[T]_{B,B}$ =?), Now again we can proceed by earlier method.

Here we have basis B = {1, x, x², x³}. So, now I have to see what is the T(1)? T(1) will be equal to what? So, this is the definition, if I use this definition T(1) = 0 = $c_1.1 + c_2 x + c_3 x^2 + c_4 x^3$ this implies,$c_1 = c_2 = c_3 = c_4$=0,this is equal to like this thing.

So, suppose this is equal to matrix say A, then the first column will be 0, 0, 0, 0. What about T x? T(x) = 1 = (1.1+ 0. x + 0. x² + 0. x³). So, this will be (1, 0, 0, 0). Similarly T(x²) = 2x, so this will

be again (0, 2, 0, 0) and third $T(x^3) = 3x^2$. So, the corresponding matrix representation of this

operator will be this matrix $[T]_{B,B} = A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(Refer Slide Time: 31:35)

Now, we are curious to know if I consider S as a set of all the mxn matrix over the field F and this is one set, another one is L(V,W) where V having dimension n and W is having dimension m, is it possible to have an isomorphism relation between these two sets? Let S denote set of all mxn matrices over F and claim there exist a 1-1 linear transformation from L(V,W) into S. 1-1 we have already checked.

Only we have to check that if I consider φ: L(V, W) → S. T→ $[T]_{B,B'}$, which is basically saying. Fixing ordered basis B and B' of V and W respectively, this is fixed. Now, is it possible to have a function from L(V,W) to S which will be 1-1 and linear transformation.

So, this we can answer if I somehow show that if T_1 and T_2 are two linear transformations belonging L(V,W) and if the matrix representation of T_1 is a A and matrix representation of T_1, of course let this is fixed ordered basis is B, then matrix representation of any constant $(cT_1 + T_2) = (cA + B)$. If somehow, I can show this one then this question is answered because anyhow we have shown that the function mapping T to the corresponding matrix representations, this is a 1-1 mapping, we have seen it.

Now, just we have to say that it is a linear transformation. So, if T_1 and T_1 are any two linear transformations from V into W. Let B and B$'$ be the ordered bases of V and W respectively. Let dimension of V and W be n and m respectively. Let $[T_1]_{\mathrm{B,B'}} = A = (a_{ij})_{mxn}$ and $[T_2]_{\mathrm{B,B'}} = B = (b_{ij})_{mxn}$.

Then any constant $[(cT_1 + T_2)]_{\mathrm{B,B'}} = cA + B$, this is the claim. Consider $B = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ and $B' = \{\beta_1, \beta_2, \ldots \beta_m\}$. So, this is my ordered basis V and W respectively.

(Refer Slide Time: 37:14)

$$\text{We know } [T_1(\alpha_j)]_{B'} = A_j \quad \& \quad [T_2(\alpha_j)]_{B'} = B_j$$

$$\therefore \quad (cT_1 + T_2)(\alpha_j) = cT_1(\alpha_j) + T_2(\alpha_j)$$

$$= c\sum_{i=1}^{m} a_{ij}\beta_i + \sum_{i=1}^{m} b_{ij}\beta_i$$

$$= \sum_{i=1}^{n}(c\,a_{ij} + b_{ij})\beta_i$$

$$\Rightarrow \left[(cT_1 + T_2)(\alpha_j)\right]_{B'} = cA_j + B_j$$

$$\therefore \quad \left[cT_1 + T_2\right]_{B,B'} = cA + B \quad —$$

Now, we know $[T_1(\alpha_j)]_{B'} = A_j$ & $[T_2(\alpha_j)]_{B'} = B_j$, it is given to us. Now, $(cT_1 + T_2)(\alpha_j) = c(\alpha_j) + T_2(\alpha_j) = c\sum_{i=1}^{m} a_{ij}\beta_i + \sum_{i=1}^{m} b_{ij}\beta_i = \sum_{i=1}^{m}(ca_{ij} + b_{ij})\beta_i$. So, this impiles $[(cT_1 + T_2)(\alpha_j)]_{B'} = c\,A_j + B_j$. So this implies $[(cT_1 + T_2)]_{B,B'} = c\,A + B$.

So, what we have learned? We have seen that there is a one-to-one correspondence between the set L(V,W) to this set of all mxn matrices S. And this one-to-one function psi this will be also a linear transformation. I mean $\varphi(cT_1 + T_2) = c\varphi T_1 + \varphi T_2 = c$ A + B. So, what we have learned? So, today we have learned first is that there is a 1-1, 1-2 mapping.

Not only 1-1, 1-2 is linear mapping, between the set of all mxn matrices and set of all linear transformations from V into W when dimension of V and dimension of W is n and m respectively,

so that is the very interesting result. So, this will give an opportunity in our next lectures that instead of talking linear transformation, one can also deal with the corresponding matrices also. Thank you.