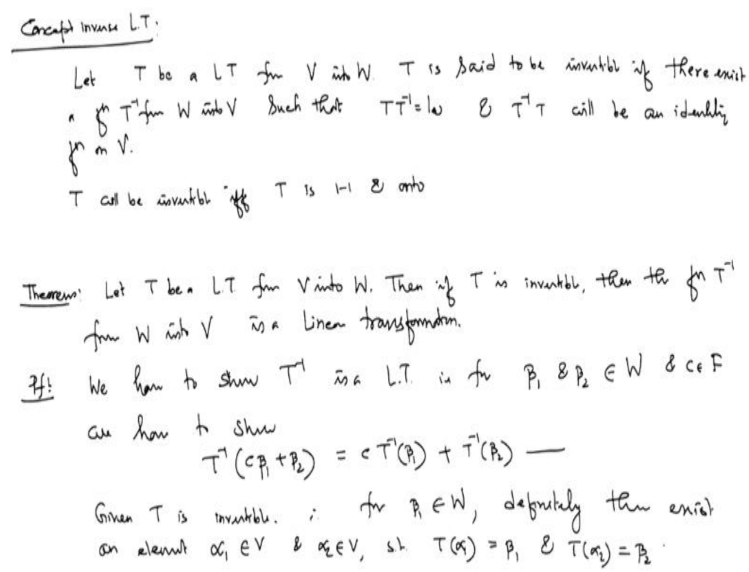
Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 15
Algebra of Linear Transformations - 3

So, welcome to lecture series. Today I will discuss about the concept of inverse linear transformations and also the concept of one-one, onto linear transformations between two spaces that is isomorphisms. We have seen what is L(V,V) the space of all linear operator on a vector V. We have seen this is also not only vector space it is also linear algebra. Now, we want to see what is meaning of a linear transformation is invertible.

(Refer Slide Time: 01:39)


Concept inverse L.T.:
Let T be a LT from V into W. T is said to be invertible if there exist a fn T^{-1} from W into V such that $TT^{-1} = I_W$ & $T^{-1}T$ will be an identity fn on V.
T will be invertible iff T is 1-1 & onto

Theorem: Let T be a LT from V into W. Then if T is invertible, then the fn T^{-1} from W into V is a linear transformation.
Pf: We have to show T^{-1} is a L.T. i.e. for β_1 & $\beta_2 \in W$ & $c \in F$ we have to show
$T^{-1}(c\beta_1 + \beta_2) = cT^{-1}(\beta_1) + T^{-1}(\beta_2)$ —
Given T is invertible. ∴ for $\beta_1 \in W$, definitely there exist an element $\alpha_1 \in V$ & $\alpha_2 \in V$, s.t. $T(\alpha_1) = \beta_1$ & $T(\alpha_2) = \beta_2$.

So, concept of inverse linear transform. Let T be a linear transformation(LT) from V into W. T is said to be invertible if there exist a function say T^{-1} from W into V such that $TT^{-1} = I_w$, TT^{-1}will be identity function, here I am denoting I_w means identity function on W, I mean from W to V and TT^{-1} will be an identity function on V.

So, TT^{-1} will be identity function on W whereas TT^{-1} will be identity function on V. In fact, if you consider under what condition a function to be invertible, we know a function will be invertible if there exists T^{-1} such that F into T^{-1} is an identity function. Here we are getting identity function over the W, identity function on the V both different. Here W is not necessarily to be equal to identity function on V.

In fact, I will say that T will be invertible if and only if T is 1-1(one-one) and onto. My claim is one thing suppose T is a linear transformation from V to W and it is invertible, then if I say that T is invertible then there exists a function, I am saying that T inverse from W to V and that function is again claim is it is a linear transformation also. So, it is a result, so you can say a small result we can write in terms of theorem.

Let T be a linear transformation from V into W, then if T^{-1} exist, if T is invertible then the function T^{-1} from W into V is a linear transformation. See here it is given me T is invertible, so T inverse exist, only thing that I have to prove that T^{-1} is a linear transformation from W into V. So, we have to prove to show T^{-1} is a linear transformation that is for β_1 & $\beta_2 \in$ W and c $\in$ F.

We have to show $T^{-1}(c\beta_1 + \beta_2) = cT^{-1}\beta_1 + T^{-1}\beta_2$. So, I have to show this one. Given T is invertible, So, for $\beta_1 \in$ W definitely there exists an element say α, I will say that is again unique element say α_1 because the function of each T is 1-1(one-one), onto, so there is this unique element $\alpha_1 \in$ V and $\alpha_2 \in$ V such that $T(\alpha_1) = \beta_1$ and $T(\alpha_2) = \beta_2$.

(Refer Slide Time: 08:11)

$\because$ T is a L.T. from V into W

So, $T(c\alpha_1 + \alpha_2) = cT(\alpha_1) + T(\alpha_2)$ —

$= c\beta_1 + \beta_2$

$\therefore$ $c\alpha_1 + \alpha_2 = T^{-1}(c\beta_1 + \beta_2)$

$\therefore$ $T^{-1}(c\beta_1 + \beta_2) = cT^{-1}(\beta_1) + T^{-1}(\beta_2)$ —

$\therefore$ T^{-1} is a L.T. from W onto V.

Non Singular: A linear transformation T from V into W is said to be nonsingular if it is 1-1.

i.e. for α_1 & $\alpha_2 \in V$, & $\alpha_1 \neq \alpha_2$,

$\Rightarrow T(\alpha_1) \neq T(\alpha_2)$

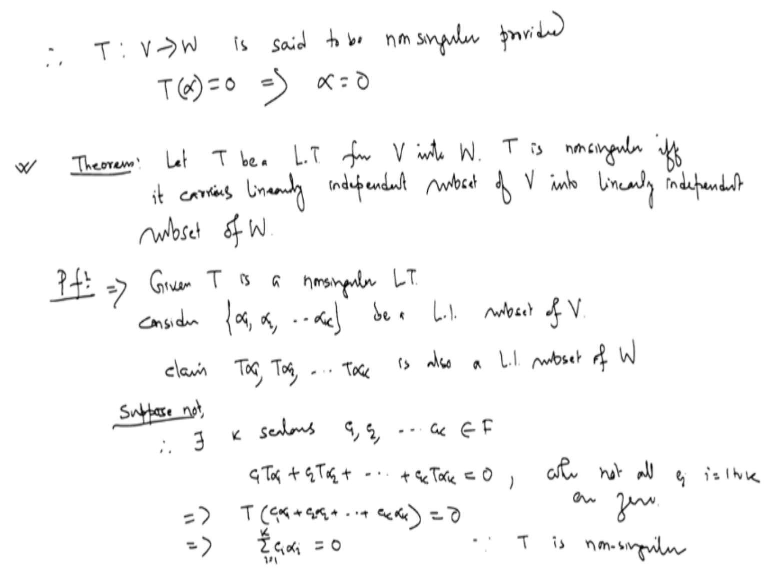
i.e. $T(\alpha_1 - \alpha_2) \neq 0$.

Since T is a linear transformation from V into W, so T($c\alpha_1$+α_2) = cT(α_1) + T(α_2)= $c\beta_1$ + β_2. So, T carries ($c\alpha_1$+α_2) to unique element. So, I have $c\beta_1$ + β_2, there is a unique element in V which image under T is $c\beta_1$ + β_2. So, this implies that ($c\alpha_1$+α_2) = T^{-1}($c\beta_1$ + β_2).

So, this is a unique element ($c\alpha_1$+α_2) which is basically image of $c\beta_1$ + β_2 under T^{-1}. So, $T^{-1}(c\beta_1 + \beta_2)$= $cT^{-1}(\beta_1)$ +$T^{-1}(\beta_2)$. So, T^{-1} is a linear transformation from W into V. So, you see that if the linear transformation T from V to W is the invertible linear transformation, then there exists a linear transformation from W to V which is the inverse of T, this is fine.

So we have seen that what is the meaning of inverse of a linear transformation. Let me define one more terminology called nonsingular. A linear transformation T from V into W is said to be nonsingular if it is (1-1)(one-one) that is for α_1 & $\alpha_2 \in$ V and $\alpha_1 \neq \alpha_2$ Implies $T(\alpha_1) \neq T(\alpha_2)$ that is $T(\alpha_1 - \alpha_2) \neq 0$.

(Refer Slide Time: 11:53)



So, T mapping from V to W is said to be nonsingular provided T(α) = 0 implies α = 0. So, this is the criteria for a nonsingular linear transformation. Using this characteristic of the nonsingular linear transformation, we can see some interesting results, what is that? Suppose T is a nonsingular linear transformation from V to W. If I consider a linearly independent subset of V, what can you say about the image of that subset under T, will it be linearly independent or not?

This question one can answer again by using the definition of nonsingular linear transformation. So, let me just write in a formal way, this is small theorem. Let T be a linear transformation from V into W. T is nonsingular if and only if it carries linearly independent subset of V into linearly independent subset of W. So, let me quickly prove this one. It will be basically coming from the definition itself only. So, I have to prove both the ways if, otherwise only.

If because it is given to us that T is a linear transformation from V to W and T is nonsingular if and only if it carries linearly independent subset of V into linearly independent subset of W. So, let me do first if part. It is given T is a nonsingular linear transformation. Consider $\{(\alpha_1, \alpha_2, \dots\ \alpha_k)\}$ be a linearly independent subset of V. Claim $\{\mathrm{T}(\alpha_1), \mathrm{T}(\alpha_2), \dots \mathrm{T}(\alpha_k)\}$ is also a linearly independent subset of W.

Suppose not, we will prove it by contradiction not. So, ∃ k scalar $(c_1, c_2, \dots\ c_k)$ belongs to the field or with vector space V and W are defined. Assume that V and W here are defined over the field F. So, I have taken k scalar $(c_1, c_2, \dots\ c_k) \in$ F such that $c_1\mathrm{T}(\alpha_1) + c_2\mathrm{T}(\alpha_2) + \dots + c_k\mathrm{T}(\alpha_k) = 0$ where not all c_i, i= 1 to k are 0. This is the hypothesis we have considered. Now this implies that one can write down $\mathrm{T}(c_1\ \alpha_1 +\ c_2\ \alpha_2 +\ \dots\ c_k\alpha_k)= 0$. This implies $\sum_{i=1}^{k} c_i\ \alpha_i = 0$, why? The reason is T is it is given to us it is nonsingular. So according to definition it is nonsingular T(α) = 0 implies α = 0, so based on that I am getting this one.

(Refer Slide Time: 18:02)

But $\sum_{i=1}^{k} c_i \alpha_i = 0 \Rightarrow c_i = 0$ for $i = 1$ to k $\because \{\alpha_1, \dots \alpha_k\}$ is a L.I. subset of V.

which contradicts our hypothesis.

$\therefore \{T\alpha_1, \dots T\alpha_k\}$ is also a L.I. subset of W.

$\Leftarrow$: Given T maps every L.I subset of V to L.I subset of W

$\therefore$ for $0 \neq \alpha \in V$, certainly $\{\alpha\}$ is a L.I subset of V

$\therefore$ $T\alpha$ will be also L.I.

$\Rightarrow T(\alpha) \neq 0$

i.e. if $T(\alpha) = 0 \Rightarrow \alpha = 0$

$\Rightarrow$ T is non-singular.

But $\sum_{i=1}^{k} c_i\, \alpha_i = 0$ implies $c_i = 0$ for i = 1 to k. Since $\{\alpha_1, \alpha_2, \dots \alpha_k\}$ is a linearly independent subset of V that is why each coefficient equal to 0 which contradicts our hypothesis. So, $\{T(\alpha_1), T(\alpha_2), \dots T(\alpha_k)\}$ the set is also a linearly independent subset of W. Now, let me go for the only if. Given T maps every linearly independent subset to linearly independent subset of W.

So, for $0 \neq \alpha \in V$, certainly the singleton element $\{\alpha\}$ is a linearly independent subset of V. So, T α will be also linearly independent. So, this implies that $T(\alpha) \neq 0$ or I can say that if $T(\alpha) = 0$ implies that $\alpha = 0$. This implies if $T(\alpha) = 0$ implies $\alpha = 0$. So, this implies T is nonsingular. So, we see that if T is a linear transformation from V into W, T will be nonsingular if and only if it carries linearly independent subset of V into linearly independent subset of W.

(Refer Slide Time: 21:17)

Isomorphism: Let T be a L.T. from V into W. T is said be an isomorphism from V into W provided T is one-one & onto. Then V is said to be isomorphic to W.

$\therefore$ If V is isomorphic to W, then W is also isomorphic to V.

Ex Let V be a f.d.v.s over F & dim V = n. Then V is isomorphic to F^n

Soln: Given V an f.d.v. space of dim n. Let $B = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ be an ordered basis of V.

For any element $\alpha \in V$, $\exists$ n scalars $x_1, x_2, \dots x_n \in F$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$$

$\therefore$ For α, there is a coordinate $(x_1, x_2, \dots x_n) \in F^n$, $x_i \in F$

$$\therefore \quad \varphi : V \to F^n$$
$$\alpha \to (x_1, x_2, \dots x_n)$$

Let me define one more terminology, which is also very important to understand the linear transformation or linear operator over a vector space. It is called concept of isomorphism. Let T be a linear transformation from V into W. T is said to be an isomorphism from V into W provided T is one-one and onto. If T is one-one and onto, then I will say that T is an isomorphism from V into W. And then V is said to be isomorphic to W.

If a linear transformation is one-one, onto then it is invertible. So, T^{-1} exist. So, T^{-1} this means T^{-1} is also linear transformation from W into V. So, T^{-1} will be also one-one and onto, so T^{-1} will be also isomorphism from W into V. So, if V is isomorphic to W, then W is also isomorphic to V. Let me consider some examples. First example is let V be a finite dimensional vector space(f.d.v.s.) over F and dimension of V = n.

Then V is isomorphic to the space of n-tuple over F, V is isomorphic to F^n. Let me solve it that how V can be isomorphic to F n. Given V as a finite dimensional vector space of the dimension n. So, let B = $(\alpha_1, \alpha_2, \dots \alpha_n)$ be an ordered basis of V. Therefore, any element α ∈ V I can associate n-tuples, how? For any element α ∈ V, I will be able to express α as a linear combination of basis element that is $(\alpha_1, \alpha_2, \dots \alpha_n)$.

So, I can say for any element α ∈ V, ∃ n scalars $(x_1, x_2, \dots x_n) \in$ F such that α = $(x_1\,\alpha_1 + x_2\,\alpha_2 + \dots + x_n\alpha_n)$. So, for α there is a coordinate $(x_1, x_2, \dots x_n)$ where $x_i \in$ F. So, for each α, I am getting a coordinate. This coordinate I can say belongs to n-tuple space, belongs to F^{n}. So, for each α there is a unique coordinate $(x_1, x_2, \dots x_n) \in \mathrm{F}^{n}$.

So, this implies one can have a function say φ : V → F^{n} such that for each α, I will have a coordinate, the corresponding coordinate i.e. α → $(x_1, x_2, \dots x_n) \in \mathrm{F}^{n}$. Let me say here on thing. We have defined coordinate matrix of any vector with respect to given ordered basis. Coordinate matrix there we have taken in as a transpose of this coordinate, means basically column vector, but here I am talking coordinate of a vector α, means basically n-tuple space here, so please do not confuse with this coordinate and coordinate matrix.

(Refer Slide Time: 28:19)

We can show
φ is a L.T. & also 1-1 & onto.

For any $\beta \in V$, Let $(y_1, y_2, \dots y_n)$ be the coordinate of β w.r.to given ordered basis B. Let c be any element from F

$$\therefore \quad c\alpha + \beta = c\sum_{i=1}^{n} x_i\alpha_i + \sum_{i=1}^{n} y_i\alpha_i = \sum_{i=1}^{n}(cx_i + y_i)\alpha_i$$

$$\Rightarrow \text{coordinate of } c\alpha+\beta = (cx_1+y_1, cx_2+y_2, \dots (x_n+y_n))$$
$$= c(x_1, \dots x_n) + (y_1, \dots y_n)$$
$$\Rightarrow \quad " \quad c \text{ coordinate of } \alpha + \text{coordinate of } \beta$$
$$\therefore \quad \varphi(c\alpha+\beta) = c\varphi(\alpha) + \varphi(\beta)$$
$\therefore$ φ is a L.T. from V with F^n

So, I see that if I define a function φ : V → F^{n} like α → $(x_1, x_2, \dots x_n)$ we can show φ is a linear transformation and also one-one and onto so that φ will act as an isomorphism between V into W. So, linear transformation characteristic of φ you can check it like this. For any other $\beta \in$ V there will be again coordinate of β. Let $(y_1, y_2, \dots y_n)$ be the coordinate of β with respect to given ordered basis B.

Let c be any element from the field F. So (cα+ β) = c$\sum_{i=1}^{n} x_i\,\alpha_i$ + $\sum_{i=1}^{n} y_i\,\alpha_i$ = $\sum_{i=1}^{n}(cx_i + y_i)\alpha_i$. So, this implies that coordinate of (cα+ β) = (cx_1+ y_1, cx_2 + y_2,…cx_n + y_n) = $c(x_1, x_2, \dots x_n)$ + $(y_1, y_2, \dots y_n)$ So, this implies that coordinate of this is equal to c coordinate of α+coordinate of β.

So this implies φ(cα+ β) = cφ(α) + φ(β). So, φ is a linear transformation from V into F^n.

(Refer Slide Time: 31:31)

φ is 1-1 & onto (H.W

φ is one-one and also onto. So, you can do it as a homework. So, we have seen that any finite dimensional vector space of diamonds an n over a particular field say F, the vector space is isomorphic to n-tuple space over the F. So, what are the benefit we are going to have through introducing this concept? Sometimes it is easy to talk the characteristics of this space, instead of the corresponding space you go to the characteristic of corresponding isomorphic space it will be easy.

So, even though both the vector spaces they are vector additions, maybe difference, their characteristics maybe different, but they will preserve certain things, what are the things? If a subset is linearly independent in V, its image under the map a, φ which is basically isomorphisms φ of that subset will be again linearly independent in F^n also, already we have checked that one.

Because φ is one-one and onto, it will carry the linearly independent subset to linearly independent subset. So, it may be easy to check linearly independence of the subset S, not in V, you can check the linearly independence of φ of S which is in the W. So, that is why this concept is very important. We will talk more about this and utilize this concept in future lectures. Thank you.