Advanced Linear Algebra
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Lecture – 14
Algebra of Linear Transformations - 2

So, welcome to lecture series on advanced linear algebra. We have discussed if V and W be two vector spaces over the field say F and if V and W both are finite dimensional or dimension say n and m respectively, then you have seen L(V, W) that is collection of all linear transformation from V into W is a vector space of dimension (mn). So, this we have proved by theoretically constructing m n **(()) (01:11)** of linear transformation from V to W and showing that they are linearly independent and span the space L(V,W). Let me take an example.

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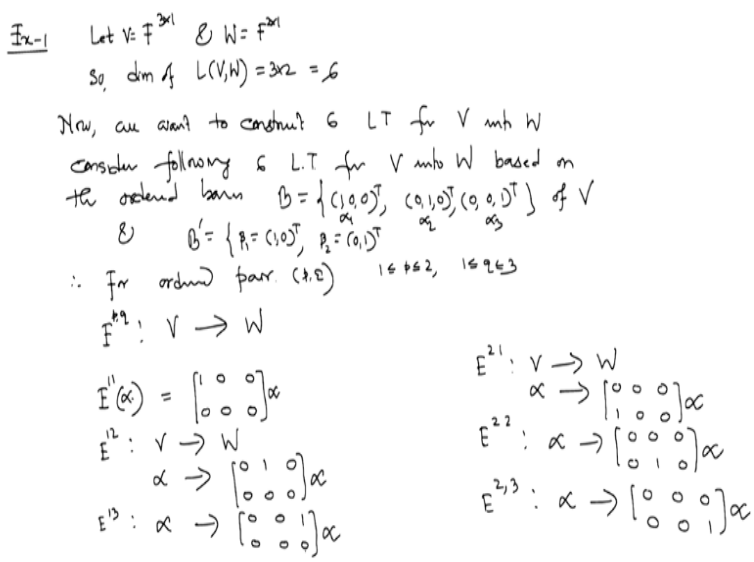

Let V = F^{3x1} and W = F^{2x1}. We know both of them are vector spaces over the field F. So the dimension of the L(V,W) = 3 x 2 = 6. Let me construct the corresponding linear transformation from V to W. So, now we want to construct 6 linear transformations from V into W which will span the space L(V,W).

So consider following 6 linear transformations from V into W based on the ordered basis B = $\{\alpha_1 = (1,0,0)^T,\ \alpha_2 = (0,1,0)^T, \alpha_3 = (0,0,1)^T\}$ this is an ordered basis of V. And B′ = $\{\beta_1 =$

$(1,0)^T$, $\beta_2 = (0,1)^T\}$. So, for ordered pair (p, q) where 1≤p≤2,1≤q≤3. I am defining $E^{p,q}: V \to W$ as follows. So, $E^{11}(\alpha) = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}(\alpha)$, $E^{12}: V \to W$, $E^{12}(\alpha) = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix}(\alpha)$
$E^{13}(\alpha) = \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\end{bmatrix}(\alpha)$, $E^{21}: V \to W$, $E^{21}(\alpha) = \begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\end{bmatrix}(\alpha)$, $E^{22}(\alpha) = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\end{bmatrix}(\alpha)$,
$E^{23}(\alpha) = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\end{bmatrix}(\alpha)$.

So, I have considered 6 linear transformations E^{11}, E^{12}, E^{13}, E^{21}, E^{22}, E^{23} as like this. So you may have the question how to guess that E^{11}, I have to define like this type of matrix.

I mean I can say suppose this is $a_{11}\, E^{11}(\alpha)$, then $a_{12}\, E^{12}(\alpha)$, for $a_{13}E^{13}(\alpha)$ like that, how to know this set of matrices? So, I will discuss this issue later on, but let me check does it satisfy my definition of the linear transformation the way we introduced in our theory. So, it has to be $E^{11}(\alpha_1) = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = \beta_1$

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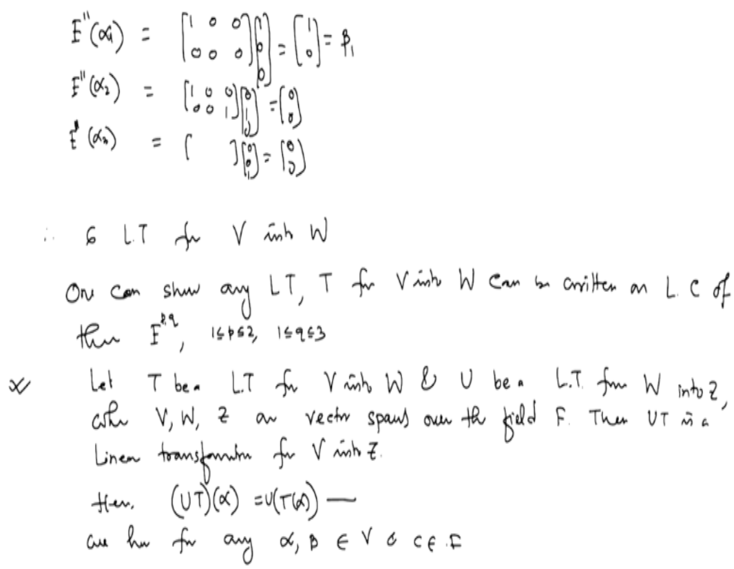

$E^{11}(\alpha_1) = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = \beta_1$, $E^{11}(\alpha_2) = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

, $E^{11}(\alpha_3) = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.

So, you can check E^{11}, E^{12}, E^{13} and E^{21}, E^{22}, E^{23}this satisfied all the criteria of the linear transformation what we have defined there. So, we have 6 linear transformations from V into W and one can show any linear transformations any (L.T.), T from V into W can be written as linear combination of this your $E^{p,q}$, where $1 \le p \le 2, 1 \le q \le 3$. So, this we can try in the home. Now, I am bit curious one thing. Suppose T is the linear transformations from V into W and U is the linear transformation from W into Z where V, W, Z are all vector spaces over the field say F, then what can to say about the multiplication of these two linear transformations? So, what is the question? Question is like this say let T be a linear transformation from V into W.

And U be a linear transformation from W into Z where V, W, Z are vector spaces over a field F. We know when we use the composition as a multiplication, the composite functions T and U is again a function from V into Z. Here instead of simple function T, I am taking T is the linear transformation from V into W and U is a linear transformation from W into Z, then what can we say about the product functions U T which is a function from V into Z.

Whether (UT) will be linear transformation or not? You can immediately check (UT) is also a linear transformation. Then (UT) is a linear transformation from V into Z, how? So here $(UT)(\alpha) = U(T(\alpha))$, so this is the definition.

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$$(UT)(c\alpha+\beta) = U(T(c\alpha+\beta))$$
$$= U(cT(\alpha)+T(\beta)) \quad \because T \text{ is a LT fr } V \text{ into } W$$
$$= c\,U(T(\alpha)) + U(T(\beta))$$
$$= c(UT)(\alpha) + (UT)(\beta)$$

$\Rightarrow$ UT is a LT fr V into Z.

$T \quad V \to W$

$U \quad W \to Z$

Linear operator: Let V be a vector space over F. A linear transformation V into V is called as a linear operator on V.

Ex Let A is 3x3 matrix over F

$$T : F^{3\times 1} \to F^{3\times 1}$$
$$X \to AX$$
$$T(cX+Y) = A(cX+Y)$$
$$= A(cX) + AY$$
$$= cAX + AY$$
$$= cTX + TY$$

Now if I define like this we have for any $\alpha, \beta \in V$ and $c \in F$, $(UT)(c\alpha+\beta) = U(T(c\alpha+\beta)) = U(cT(\alpha)+T(\beta)) = cU(T(\alpha)) + U(T(\beta)) = c(UT)(\alpha) + (UT)(\beta)$. So this implies when we use the composition as a multiplication of two functions, then this (UT) is a linear transformation from V into Z. So, we see that even if I consider product of two linear transformations it is also a linear transformation.

If I write (UT) the domain this will be mapping from V into Z, but if I write (TU) instead of (UT), let me check. See B is V to W it is my T and if I consider U that is W to Z, is it possible to define (TU)? I mean first operating on U, when turns U as the first function, so (TU) is not defined because (TU)($\beta \in$ W, U(β) $\in$ Z, but T is mapping from V to W, so therefore it cannot be defined.

Linear operator; what is linear operator? When a linear transformation is defined from a vector space V into the same vector space then it is a linear operator. So, let V be a vector space over the field F. A linear transformation V into V is called as a near operator on V. So, examples we can see like linear transformation here also I can give. Let A is a say 3x3 matrix over the field F. I mean to say the entries of A is from field F.

So, now let me consider T: $F^{3x1} \rightarrow F^{3x1}$ defined by any element say X $\in F^{3x1}$ column vector, X $\rightarrow$ AX, of 3x1 order so, X and Y $\in F^{3x1}$ and c is any constant over F, then we see T(c X + Y) = A (c X + Y) = A (c X) + AY = c AX + AY = c TX + TY.

So, this implies T is a linear operator on F^{3x1}. If, I consider A is a zero matrix then also I will have another example of T: $F^{3x1} \rightarrow F^{3x1}$, which is mapping any element of zero element of F^{3x1}.

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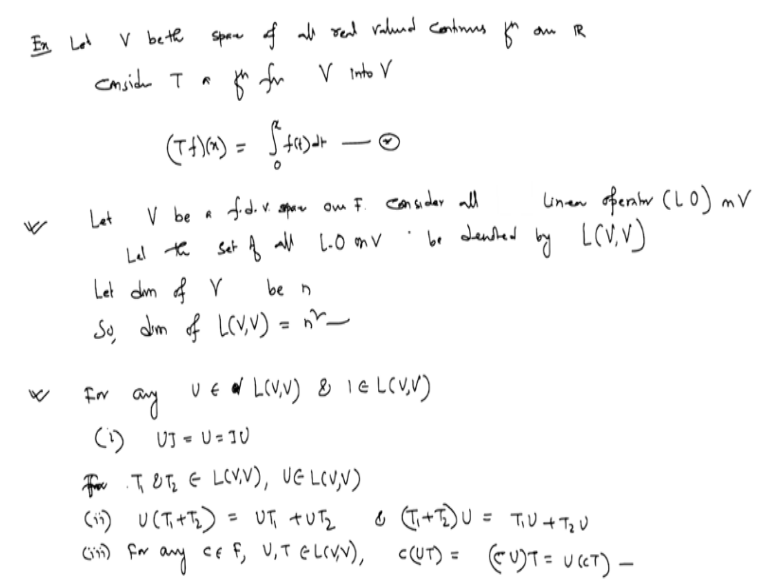

You can take another example let V be the space of all real valued continuous function over say real line R. So, you know that V is again a vector space. So, consider T a function from V into V defined by (Tf)(x) $=\int_0^x f(t)dt$. So, if I define like this, then I see that this function T is a function from V into V and it is also linear transformation, in fact it is a linear operator because V = W.

Now, let me go to this specific linear operator, I mean linear operator of finite dimension. Let V be finite dimensional vector space over field F. Consider all linear operator (LO) on V. Let the set of all linear operator(LO) on V be denoted by L(V,V). So, it is basically set of all linear transformation from V into V. Let dimension of V = n. So dimension of L(V,V) = n^2, according to our last results here also one can construct n^2 linear operator on V which will act as a basis for the space L(V,V) basically.

And this L(V,V) is a vector space, this already we have done, it is exactly the same the way we did for the L(V,W) also. So, I am not repeating that one, so you can check it, so it is a vector space and dimension of this space will be n square. See in last result what we have proved if T is a linear transformation from V into W and U is a linear transformation from W into Z, then you (UT) is a linear transformation from V into Z.

Now, if I consider V = W = Z then I can say that here U and T both of them are basically linear operators on the vector space V. So, (UT) is defined, it will be again a linear operator on V. So, apart from vector addition and scalar multiplication to L(V,V) if I consider multiplication of two vectors, here vectors mean basically I am talking about the linear operator, then also we see that product of two vectors is again a vector type.

This is basically my composition is multiplication as used to do with the case of product of two functions. Apart from this vector addition and scalar multiplication on the L(V,V) if I add another binary operation that is multiplication of two vectors then I can say for any U ∈ L(V,V) and identity function from V to V, so I ∈ L(V,V), universal linear operator I also belongs to L(V,V). We have (i) UI = (UI) = U.

(ii)T_1 & T_2 ∈ L(V,V) and U ∈ L(V,V), we have U(T_1 + T_2) = UT_1 +UT_2 & (T_1 + T_2)U = T_2U +

T_2U. (iii) c ∈ F and (UT) ∈ L(V,V) then c (UT) = (cU) T = U (cT), we can quickly check that as used to do from the case of the function here also the function U and T here, not only function it is a linear operator with respect to composition multiplication.

We see that distributive law over the addition also holds good. And we also see that there is the identity operator on L(V,V) such that UI = U = IU. So, we can quickly check that this type of results also holds good. I mean to say U(T_1 + T_2)(α) = U(T_1)(α) + U(T_2)(α), you can check it. Similarly, you can also check that other two axioms of properties with respect to multiplication.

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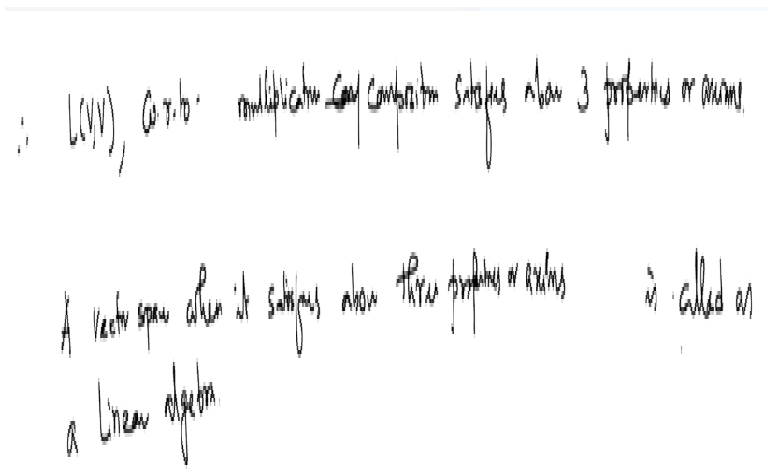

So L(V,V) with respect to multiplication comparisons satisfied about three properties or axioms. A vector space when it satisfies above three properties or axioms these are called as linear algebra. So, L(V,V) is a linear algebra. So, let me conclude. So, what we have learned here? So, we have seen that what is linear operator and similar to L(V,W), L(V,V) is also a vector space and if the dimension of vector space V is finite say n the dimension of L(V,V) = n^2.

And in the vector space L(V,V) one can also introduce multiplication compositions and with respect to that composition it satisfies three more axioms that is the existence of identity operators such that I into U = U, distributive property over addition also holds goods and one more that if a constant is constant c and U and T and any two linear operators then c(UT) = (cU)T = U(cT).

So, this type of characteristics or properties also holds good with respect to multiplication. And a vector space satisfying these three characteristics or properties with respect to multiplication is called a linear algebra. So, L(V,V) is a linear algebra.