Advanced Linear Algebra
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Lecture – 13
Algebra of Linear Transformations - 1

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Let V & W be two vector spaces defined over a field say F.
Consider T & U as two LT from V into W.
Then, define addition & scalar multiplication as

$$(T+U)(\alpha) = T(\alpha) + U(\alpha) \quad \text{— (i)} \quad \text{for any } \alpha \in V$$

for any $c \in F$, $\quad (cT)(\alpha) = cT(\alpha)$ — (ii)

claim T+U is also a fn from V into W, also Linear transformation
Similarly cT " " " " " "

Consider $\alpha, \beta \in V$ & $d \in F$

$d\alpha + \beta \in V$

$$\begin{aligned}(T+U)(d\alpha+\beta) &= T(d\alpha+\beta) + U(d\alpha+\beta)\\ &= dT(\alpha) + T(\beta) + dU(\alpha) + U(\beta)\\ &= d(T(\alpha)+U(\alpha)) + T(\beta) + U(\end{aligned}$$

So, welcome to lecture series. Today we will discuss about the algebra of linear transformations that is let me V & W be two vector spaces defined over a field say F. If we consider T & U two linear transformation from V into W, then if I define some functions by an addition and I can say multiplication or scalar multiplication as (T + U) (α) = T(α) + U(α), for $\alpha \in$ to V.

Any c ∈ F, (cT) (α) = c T(α), this is some sort of scalar multiplication. Basically, if we consider V and W as two different sets and if you consider the function from V to W and we collect all the function from V to W defined by addition like one and scalar multiplication two we have seen that a vector space. But here I am not talking T and U as any functions.

I am saying that T and U are the linear transformation from V to W and we have defined sum of two linear transformations T and U as (T + U) (α) = T(α) + U(α) and for any c ∈ F, (cT) (α) = c T(α). Then claim (T + U) is also a function from V into W and also linear transformation. Similarly, (cT) is also a function of V from this and also linear transformation.

If I can somehow prove it, then if I collect all the linear transformations from V to W and if I denote that set is suppose L(V,W) then I will basically show that L(V,W) is also a vector space. I am coming to that one, before that let me see whether the way I have defined one and two whether (T + U) is a linear transformation from V to W or not and (cT) also linear transformation from V to W or not, let us quickly check that one first.

For linear transformation we have to consider, so consider $\alpha, \beta \in$ V be any two elements and say d $\in$ F be any scalar. Then (dα + β) $\in$ V and we have (T + U) ($d\alpha + \beta$)) = T($d\alpha + \beta$) + U($d\alpha + \beta$) = d T(α) + T(β) + dU(α) + U(β) because T and U are linear transformation from V to W the definition I have used. So this means that (T + U) (dα + β) = d (T(α) + U(α)) + T(β) + U(β).

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$(T+U)(d\alpha+\beta) = d(T+U)(\alpha) + (T+U)(\beta)$ —

$\Rightarrow$ T+U is a L.T. fm V into W —

Again

$(cT)(d\alpha+\beta) = c\,T(d\alpha+\beta)$

$= c\,d\,T(\alpha) + cT(\beta)$

$= d(cT)(\alpha) + (cT)(\beta)$

$\Rightarrow$ cT is a L.T fm V into W.

Consider L(V,W) as collection of all LT fm V into W.

Consider for any T & U $\in$ L(V,W)

$(T+U)(\alpha) = T(\alpha) + U(\alpha)$ — (i) for $\forall\, \alpha \in V$

& $(cT)(\alpha) = cT(\alpha)$ — (ii) for any $c \in F$

Then L(V,W) is a vector space.

This is equal to d (T + U) (α) + (T + U) (β) . I wrote (T + U)(dα + β) = d (T + U) (α) + (T + U)(β). So, this implies (T + U) is a linear transformation from V into W. Again immediately one can check (cT)(d $\alpha + \beta$) = c T(dα + β) = c d T(α) + c T(β) = d (cT)(α) + (cT)(β).

So, this means that (cT) is also linear transformation from V into W, it is fine. So, you have seen that if T is a linear transformation from V to W and U is also linear transformation from V to W, then (T + U) is also linear transformation from V to W and (cT) also linear transformation from V to W. Now consider L(V,W) as a collection of all linear transformation from V into W. Consider for any T & U $\in$ L(V,W); (T + U)(α) = T(α) + U(α).

And (cT)(α) = cT(α) or all $\alpha \in$ V for any c $\in$ F that is basically the V and W are defined over the field F that we have considered. Then L(V,W) is a vector space. Now to prove it is a vector space, I have to prove all the axioms of the vector space, I meant to say I have to show that L(V,W) is an abelian group with respect to addition which define like one and it satisfied five more axioms with respect to scalar multiplication defined as two.

Here that zero transformations V to W is basically zero element of this set L(V,W). The way it has been defined, we see that the L(V,W) is closed with respect to additions and another important thing is that see we have already seen if I consider the collection of all functions from a set V to another set W defined by say (f_1 + f_2)(s) = $f_1(s)$ + $f_2(s)$ and (c $f_1)(s)$ = c $f_1(s)$. So, with respect to that type of vector addition and scalar multiplication your set of all the functions from V to W will be a vector space.

Now, my collection L(V,W) here is a collection of all linear transformations, this is also function, but is a special characteristic, it having special nature, what is that? That for each function belongs to L(V,W) it is basically linear transformation. So, this will be a subset of the vector space consisting of all the function from V to W. We have to show that in that set of all this linear transformation all the function which is basically functional linear transformation will be a subspace of that set.

One can think that way, but without going there side also one can again prove independently that L(V,W) is a vector space. Closure property holds good with respect to addition that is from the definition itself already seen. Associative property will also hold good because we have already seen the set of function from a set V to W satisfied associative property. So, that associative nature will also hold good for this special kind of function that is linear transformation also.

Next one will be the question of existence of zero element. Zero element means basically zero linear transformations, so that is also there. Next will be the additive inverse. So for any linear transformation T from V to W, one can again define –T from V to W so that T + –T = 0 transformation, so that is also there. And commutative property holds good because the T (α) or U(α), see this is our element of the vector space which is also satisfied commutative property.

So, (T + U) (α) = T(α) + U(α) = U(α)+ T(α) also because these are the elements in the vector space W, so which satisfied the commutative property. So, therefore it satisfies all these axioms of the abelian characteristic of the space L(V,W) with respect to addition. Similarly, one can check five more axioms with respect to scalar multiplications.

So, I am leaving it as a homework for you please, you can check it that L(V,W) is a vector space with respect to the binary operation that is vector addition defined as one and scalar multiplication defined as two. Let me see can I give some dimension for the L(V,W).

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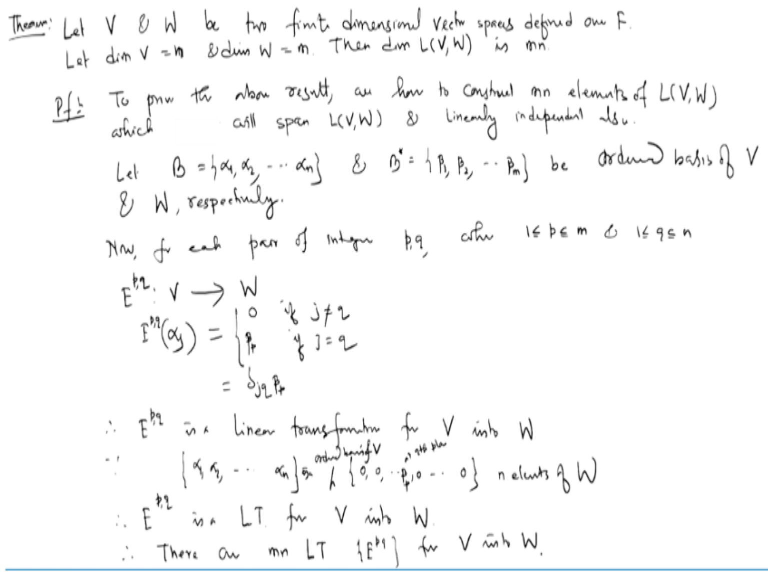

Let V and W be two finite dimensional vector spaces defined over the field F. Let dimension of V = n and dimension of W = m. Then dimension of L(V,W) = (mn). So, we have picked up a question that say like this V and W be to finite dimensional vector spaces over the field F and dimension of V = n and dimension of W = m, then dimension f L(V,W) = (mn). So, you can see this as the theorem.

To prove the above results, we have to construct (mn) elements of L(V,W) which will span L(V,W) and linearly independent also. So, let B ={(α_1, α_2, ... α_n)} and B*={(β_1, β_2, … β_n)} be ordered basis of V and W respectively. So, I have considered ordered basis for W first. Now, I have to construct (mn) functions from V to W. Now for each pair of integer p, q where , $1 \le p \le m, 1 \le q \le n$.

Consider a function, I have considered function $E^{p,q} : V \rightarrow W$ defined by $E^{p,q}(\alpha_j) = \begin{Bmatrix} 0 \ ; j \neq q \\ \beta_p \ ; j = q \end{Bmatrix}$ $= (\delta_{jq}\, \beta_p)$, where $\delta_{jq} = \begin{Bmatrix} 0 \ ; j \neq q \\ 1 \ ; j = q \end{Bmatrix}$. So $E^{p,q}$ is a linear transformation from V into W, how?

How can I say that $E^{p,q}$ is a linear transformation from V into W the reason is we have already seen in the existence of linear transformation from vector V to W for a given ordered basis if you consider same number of elements in the corresponding W space, then there exists a unique linear transformation which will map to α_j to β_j type. So, if I consider here the set is basically your $(\alpha_1, \alpha_2, \ldots \alpha_n)$ this is one set basically ordered basis in V.

And other set is basically I am saying that which is your (0, 0, β_p, 0,… 0). So this is a subset of W, this is subset of V, this subset is linearly independent. We know there is a unique linear transformation with mapping α_i to this set. And if I consider this say $(\gamma_1, \gamma_2, \ldots \gamma_n)$ then there is a linear transformation mapping α_1 to γ_1, α_2 to γ_2 like that.

So, based on that philosophy, I can see that here $E^{p,q}$ the way it has been defined from V to W is a linear transformation. Since, this is an ordered basis of V and n elements of W and this function is basically mapping between these two sets and so $E^{p,q}$ is a linear transformation from V into W. So, this is true for any ordered pair p, q when $1 \le p \le m, 1 \le q \le n$.

So, there are (mn) functions of the form of $E^{p,q}$ from V into W. I will say this is a linear transformation. There are (mn) linear transformations $E^{p,q}$ from V to W.

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claim $\{E^{p,q},\ 1 \le p \le m,\ 1 \le q \le n\}$ spans $L(V,W)$

Let T be a L.T. fm V into W. i.e $T \in L(V,W)$

For $\alpha_j \in V$

$T(\alpha_j) \in W$

$\therefore$ Then exist m scalars A_{ij} $i = 1$ to m $\in F$ such that

$$T(\alpha_j) = \sum_{i=1}^{m} A_{ij}\beta_i \quad \text{---} \ (*)$$

Claim $T = \sum_{p=1}^{n} \sum_{q=1}^{n} A_{pq} E^{pq}$ —

Let $U = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{pq}$

$\therefore U(\alpha_j) = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{pq}(\alpha_j)$

$= \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} \delta_{jq} \beta_p$

$= \sum_{p=1}^{m} A_{pj} \beta_p = T(\alpha_j)$

$E^{pq}(\alpha_j) = \delta_{jq} \beta_p$

Claim; this $E^{p,q}$ 1≤p≤m,1≤q≤n. spans L(V,W). Let T be a linear transformation from V into W. I have considered any arbitrary linear transformation T from V into W. I mean to say T an arbitrary element of L(V,W) that is T ∈ L(V,W). My claim is that T can be written as linear combinations of this m n linear transformations. For α_j ∈ V, this α_j is basically basis element of the V.

Now, T(α_j) ∈ W. So, this means that I will be able to express the α_j as a linear combination of the basis element W, means there exist m constants. So, exist m scalars say A_{ij}, i = 1 to m ∈ F such that your T (α_j) = $\sum_{i=1}^{m} A_{ij}\,\beta_i$. So, I will have like this thing. So for any p, q T(α_j) I have picked up from the set of ordered basis which belongs to V and we have retained T(α_j) as like this thing.

Claim T = $\sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\, E^{pq}$. Claim T is basically linear combinations of this (pq) linear transformation. with the coefficient is here A_{pq} type. Let U is equal to say let it is not p, it is equal to some linear transformation from V into W, let me give the name as U. So, U = $\sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\, E^{pq}$. Now if I see the U($\alpha_j$) = $\sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\, E^{pq}(\alpha_j)$.

Already we know the definition of E^{pq}, $E^{pq}(\alpha_j) = (\delta_{jq}\,\beta_p)$. So, U($\alpha_j$) $= \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\ \delta_{jq}\,\beta_p = \sum_{p=1}^{m} A_{pj}\ \beta_p = T(\alpha_j)$.

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$\Rightarrow T = U = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\, E^{pq}$ —

L.I of $\{E^{pq}, 1 \le p \le m, 1 \le q \le n\}$:

for $\sum_{p=1}^{m}\sum_{q=1}^{n} a_{pq}\, E^{pq} = 0$

$\Rightarrow \left(\sum_{p=1}^{m}\sum_{q=1}^{n} a_{pq}\, E^{pq}\right)(\alpha_j) = 0$ for $\forall\, \alpha_j \in B$

$\therefore \sum_{p=1}^{m} \sum a_{pq}\, \beta_p\, \delta_{qj} = \sum_{p=1}^{m} a_{pj}\, \beta_p = 0$

$\Rightarrow a_{pj} = 0$ for $p = 1$ to m $\because \{\beta_1, \beta_2, \dots \beta_p\}$ is a L.I. set of W

$\therefore \{E^{pq}, 1 \le p \le m, 1 \le q \le n\}$ is L.I

$\therefore$ it is a basis of L(V,W)

So, this implies T = U = $\sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\, \mathrm{E}^{pq}$. I have shown that any linear transformation from V into W can be written as a linear combination of these (mn) linear transformations. Now, I have to show that this m n linear transformation $\mathrm{E}^{p,q}$ is linearly independent. Now, I will show that linearly independent LI of $\{\mathrm{E}^{p,q}\ 1 \le \mathrm{p} \le \mathrm{m}, 1 \le \mathrm{q} \le \mathrm{n}\}$, so I have to prove this one. See the way we have written here see T = $\sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\, \mathrm{E}^{pq}$.

So, now for $\sum_{p=1}^{m}\sum_{q=1}^{n} a_{pq}\, \mathrm{E}^{pq} = 0$. If I do it, this means it is zero linear transformation. So, this means $(\sum_{p=1}^{m}\sum_{q=1}^{n} a_{pq}\, \mathrm{E}^{pq})(\alpha_j) = 0$. for all $\alpha_j \in$ B. $\sum_{p=1}^{m}\sum a_{pq}\, \beta_p\, \delta_{qj} = \sum_{p=1}^{m} a_{pj}\ \beta_p = 0$ So, This implies $a_{pj} = 0$ for p = 1 to m since $\beta_1, \beta_2, \dots\, \beta_p$ is a linearly independent set of W such that $a_{pj} ==$ 0. This we have seen for any arbitrary j. So, this implies that if I take any linear combination of the $\mathrm{E}^{pq} = 0$.

So $\mathrm{E}^{p,q}\ 1 \le \mathrm{p} \le \mathrm{m}, 1 \le \mathrm{q} \le \mathrm{n}$ is linearly independent. So it is a basis of L(V,W). So, we have seen that for the vector space L(V,W) when V and W is a finite dimensional of dimension n and m respectively, then the dimension of the vector space L(V,W) it will be finite and it is equal to (mn) and for a given ordered basis versus one can easily construct the basis element also this way. So, we will continue in the next class, today that is all. Thank you.