Advanced Linear Algebra
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Lecture – 12
Concept of Rank

Welcome to lecture series on advanced linear algebra. In my last class, we have discussed about the linear transformation from a vector space to another vector space. We have also seen the existence of linear transformations from a finite dimension vector space to another vector space and we have also seen applications, some more examples of the linear transformations.

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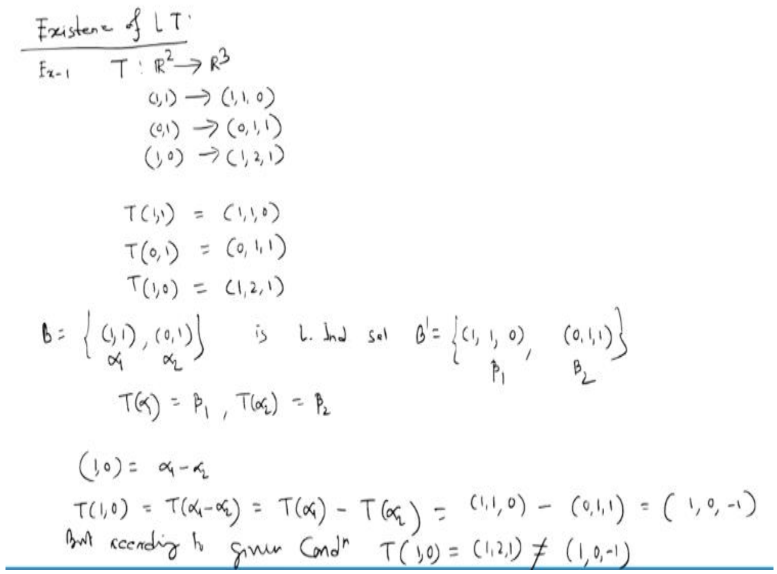

Now, let me see quickly if I consider a function from a vector space to another vector space is it necessary that always will have a linear transformation. For example, suppose I have taken here T : $R^2 \rightarrow R^3$. So my vector space V = R^2 and W = R^3. R^2 means set of all these two points, I mean you can say x-y plane, R^3 means set of all three tuples that is my Euclidean space.

It is given that T(1, 1) = (1, 1, 0) and T(0, 1) = (0, 1, 1) and T(1, 0) = (1, 2, 1). Going to check whether such type of linear transformation exist or not. First we see that the set (1, 1) and (0, 1) it is a linearly independent set and if I consider the two elements, another two elements that is B = {(1, 1), (0, 1)} and if we consider B′ = {(1, 1, 0), (0, 1, 1)}. Now, since this is linearly independent then immediately I will say that by previous theorems what we have proved there is a linear transformation.

Suppose, this is my $\alpha_1 = (1, 1)$ and $\alpha_2 = (0, 1)$ and this is my β_1 = (1, 1, 0) and $\beta_2 = (0, 1, 1)$ then we know that there exists linear transformation from R^2 to R^3 the T(α_1) = β_1 and T(α_2) = β_2. But here we have another condition that (1, 0) which is basically your linear combination that is (1, 0) = ($\alpha_1 - \alpha_2$). So, I am getting this once and but suppose this type of linear transformation is possible.

Then image of T(1, 0) = T(α_1 - α_2) = T(α_1) - T(α_2) this is okay, this is equal to how much? T(α_1) = β_1 one that is T(1, 0) = T(α_1 - α_2) = T(α_1) - T(α_2) = (1, 1, 0) - (0, 1, 1) = (1, 0, –1). So, image of T(1, 0) is like this, but to given condition image of T(1, 0) = (1, 2, 1) $\neq$ (1, 0, –1). So, such type of linear transformation does not exist.

We can see many more problems in the assignment set also. Today I will discuss about one term called rank of T, the concept of rank. Last time also I have discussed the concept of range space, null space of a linear transformation and we have seen that its range space is subspace of the corresponding codomain space of the linear transformations arise null space in the subspace of the domain space of the linear transformation.

And we have also defined what is rank of T, rank of T as basically I have defined dimension of the range space of the linear transformation, whereas nullity is basically dimension of the null space of the linear transformation.

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Does there exist any relation between rank of a L.T & nullity of that L.T?

Theorem: Let T be a Linear transformation for vector space V into W, where V & W are defined over F. Suppose dimension of V is finite, Then

Rank of T + Nullity of T = dim of V.

Pf: Let N_T represent the null space of T & R_T represent range space of T

$N_T \subset V$

Let the dimension of V is n. So dim of N_T is also finite & $\leq n$

Let $B = \{\alpha_1, \alpha_2, \dots \alpha_k\}$ be an ordered basis of N_T

So, by extension theorem, B can be extended by including n-k more LI elements of V to be a basis of V.

$B' = \{\alpha_1, \alpha_2, \dots \alpha_k, \alpha_{k+1}, \dots \alpha_n\}$ be an ordered basis of V

$\therefore \{T\alpha_1, T\alpha_2, \dots T\alpha_k, T\alpha_{k+1}, \dots T\alpha_n\}$ will span R_T

In fact, $T\alpha_i = 0$ for $i = 1$ to k, $\because \alpha_1 \dots \alpha_k$ are in N_T

Now, we want to see does there exist any relation between rank and nullity of a linear transformation. There exists any relation between rank of a linear transformation and nullity of that linear transformation that is the question. Answer is yes, we have a lesson, but for certain case I mean with some conditions that it is like this, very small more theorem type. Let T be a linear transformation from vector space V into W where V and W are defined over the field F.

Suppose dimension of V is finite, then Rank of T + Nullity of T = dimension of V. I mean to say dimension of the Range space of T + dimension of the null space of T = dimension of the V. Here I am not talking about the dimension of W, it may be finite, it may be infinite, but we are basically emphasizing on the dimension of the V. So, let me give a proof. So, this is basically condition for which there is a relation between rank and nullity of linear transformation.

Let N_T represents the null space of T and R_T represents range space of T. Certainly N_T will be subset of V. Let the dimension of V is n. Since given to us it is a finite, so I can say the dimension of V is say n. So, dimension of N_T also finite and less than equal to n. Let B $=\{\alpha_1, \alpha_2, \dots\ \alpha_k\}$ be an ordered basis of null space of T, N_T. So, by extension theorem, B can be extended by including n –k more linearly independent elements of V to be a basis of V.

So, let me consider B$' = \{\alpha_1, \alpha_2, \dots\ \alpha_k, \alpha_{k+1}, \dots \alpha_n\}$ be an ordered basis of V. So, $\mathrm{T}(\alpha_1), \mathrm{T}(\alpha_2), \dots \mathrm{T}(\alpha_k)$, $T(\alpha_{k+1})$, … $T(\alpha_n)$, so this set will span the range space R_T because any element of V can be written as a linear combination of $\alpha_1, \alpha_2, \ldots\ \alpha_n$. So, any element in the range space would be a linear combination of $\mathrm{T}(\alpha_1), \mathrm{T}(\alpha_2), \ldots \mathrm{T}(\alpha_n)$.

In fact, T(α_i) = 0 for i = 1 to k, since $\{\alpha_1, \alpha_2, \ldots\ \alpha_k\}$ are in N_T null space. So, according to the definition T(α_i) = 0 for i = 1 to k.

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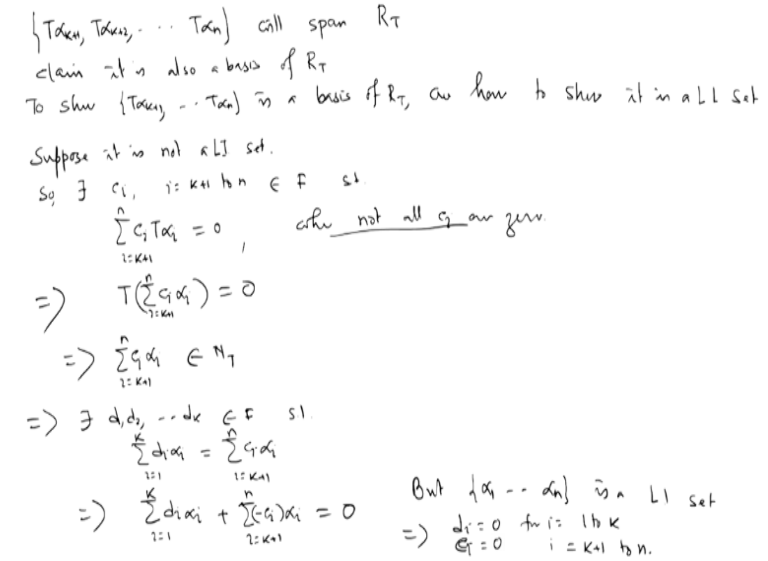

So, $\{T(\alpha_{k+1}), T(\alpha_{n+2}), \ldots\ T(\alpha_n)\}$ will span the range space R_T. So, this is n – k elements basically which span in R_T . Claim; it is also a basis of; so to be basis we know it has to span the space R_T and it has to be also linearly independent. Already we have seen $\{T(\alpha_{k+1}), T(\alpha_{n+2}), \ldots\ T(\alpha_n)\}$ span the R_T, now to show it a basis I have to basically show this $\{T(\alpha_{k+1}), T(\alpha_{n+2}), \ldots\ T(\alpha_n)\}$ linearly independent sets.

So, now to show $\{T(\alpha_{k+1}), T(\alpha_{n+2}), \ldots\ T(\alpha_n)\}$ is a basis of R_T, we have to show it is a linearly independent set. Suppose not, suppose, it is not a linearly independent LI set. So it means that it will be linearly dependent, there exist c_i, i = k+1 to n ∈ F such that $\sum_{i=k+1}^{n} c_i\, T(\alpha_i)$ = 0 where not all c i are 0. So this implies T ($\sum_{i=k+1}^{n} c_i\, \alpha_i$) = 0. This is because from the definition of T only we have like this.

So, this implies ($\sum_{i=k+1}^{n} c_i\, \alpha_i$) ∈ N_T because according to the definition of null space so I have ($\sum_{i=k+1}^{n} c_i\, \alpha_i$) ∈ N_T. So, since an element of N_T, so I will be able to express as a linear combination of the basis element of N_T. So it means this implies $(d_1, d_2, \ldots\ d_k)$ ∈ F such that $\sum_{i=1}^{k} d_i\, \alpha_i =$

$\sum_{i=k+1}^{n} c_i \, \alpha_i$.

So, this implies that one can rewrite sigma $\sum_{i=1}^{k} d_i \, \alpha_i + \sum_{i=k+1}^{n} (-c_i) \, \alpha_i$. But $\{\alpha_1, \alpha_2, \dots \, \alpha_k\}$ is the linearly independent set. So, this implies that each d_i = 0 for i = 1 to k and c_i = 0 for i = k+1 to n. This contradicts that in the very beginning we assumed that not all c_i are 0.

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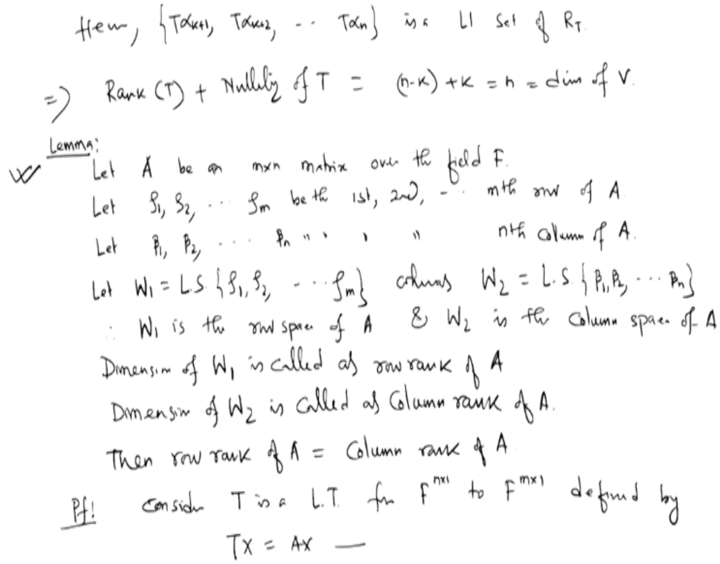

Hence, $\{T(\alpha_{k+1}), T(\alpha_{n+2}), \dots \, T(\alpha_n)\}$ is a linearly independent set of R_T. This implies Rank(T) + Nullity(T) = (n – k) + k = dimension of V. So, you have seen that rank of linear transformation T + nullity of the T = dimension of the V. See if the dimension of V is not finite can I conclude like that? So, in that case we cannot do it.

If the dimension of the V is not finite then we cannot conclude like this type of conclusion. So, we have seen what do you mean by rank of a linear transformation. Soon we shall see the linear transformation is nothing it basically give me some matrix. So, it will have a matrix structure. So, we have seen the meaning of the rank of T. Now, let me introduce the concept of rank of a matrix. We have already defined a row rank of the matrix.

So, let me recall and give similar definition of the rank of a matrix and let us discuss that. Let A be an mxn matrix over the field F. Let $(\rho_1, \rho_2, \dots \, \rho_m)$ be the first, second and m-th row of matrix A. Let $\beta_1, \beta_2, \dots \, \beta_n$ be the first, second and n-th column of matrix A. Let $W_1 = LS \; \{(\rho_1, \rho_2, \dots \, \rho_m)\}$ whereas $W_2 = LS \; \{\beta_1, \beta_2, \dots \, \beta_n\}$. So, W_1 is the row space of A and W_2 is the column space of the matrix A.

Dimension of W_1 is called as row rank of matrix A and dimension of W_2 is called as column rank of matrix A. Then row rank of A = column rank of A. So, this is a problem I have considered. It is basically I have considered mxn matrix and I am basically showing that for a m cross and n matrix defined over the field day F row rank of A equal to column rank of A. So, this is a proof type, this is some sort of Lemma type we can say.

I will apply the above results this thing. So, consider T: $F^{nx1} \rightarrow F^{mx1}$ defined by TX = AX. I have already mentioned F^{nx1} is basically space of column matrix of n cross 1 order and F^{mx1} is basically space of column matrix of mx1 order. They are all vector spaces over the F. So, I have defined T: $F^{nx1} \rightarrow F^{mx1}$ by T(X) = AX.

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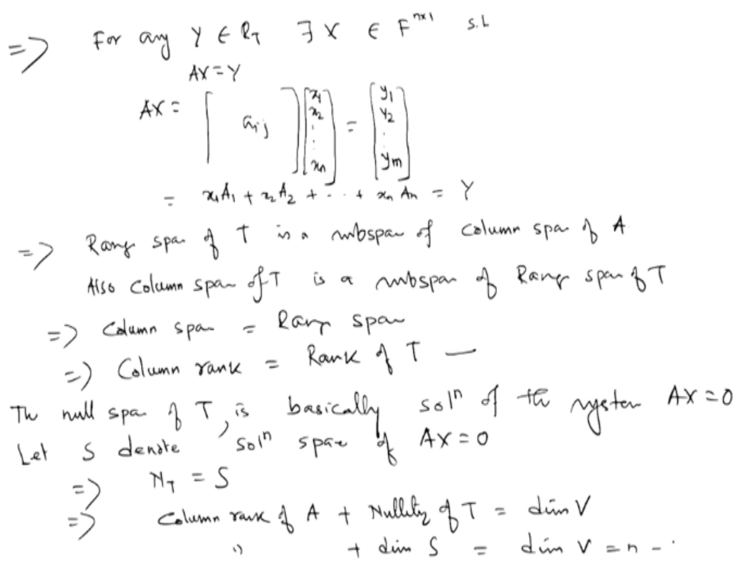

So, this implies for any Y ∈ R_T, ∃ X ∈ F^{nx1} such that your AX=Y, but

AX= $[a_{ij}] \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_m \end{bmatrix} = x_1A_1 + x_2A_2 + ... + x_n = Y.$ So, this implies range space of T is the subspace of column space of A. Also column space of T is a subspace of range space of A because if I take any linear combination of the column of A that is basically it is writing AX, so it will be in the Range space. So, this implies that column space = Range space. So, this implies column Rank = Rank of T. Now, if I see the null space of this, the null space of T is basically what?

Solution of the system AX=0 So, let S denote solution space of AX=0 So, this implies that my N_T = S. This implies that null space of T is basically S. So, this implies by the previous theorem which I am saying also rank nullity theorems, previous theorem says that rank of T + nullity of T = dimension of the space V. So, this implies column rank of A + Nullity of T = dimension of V or column rank of A + dimension of S = dimension of V = n.

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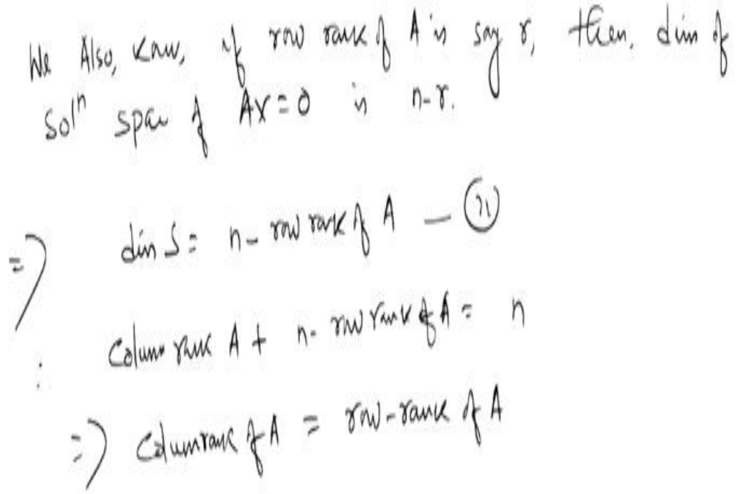

Again, we know if row rank of A is say R then dimension of solution space of AX = 0 is what? So, in this case the dimension of the solution space of AX = 0 will be number of free variables. Since A is a mxn matrix and if row rank of A is r, then number of free variables will be n – r. So, this implies dimension of S = n – r, r is what, r is basically row rank of A. So, we have if I substitute this one in previous expressions, this is my 2 and previous equation this is 1 one.

Then we have that column rank of A + n – row rank of A = n. So, this lies that column rank of A = row - rank of A. So, this is very interesting result. For a mxn matrix defined over a field F, the row rank of the matrix is equal to column rank of the matrix also. So, we have seen that for a given mxn matrix, the column rank of the A = row rank of A.

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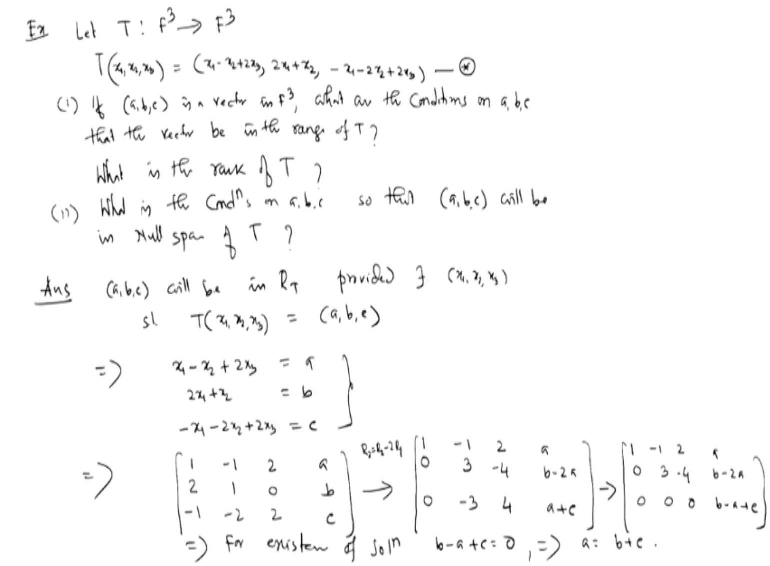

Let me take an example and see here I have taken a function T : $F^3 \rightarrow F^3$ defined by T$(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$. So, this is the definition suppose. And then I raised two questions. The first question is if (a, b, c) is the vector in F^3 what are the conditions on (a, b, c) that the vectors be in the range of T? And part of this question what is the rank of the transformation T.

And second is what is the condition on (a, b, c) so that (a, b, c) will be in the null space of T? So, let me answer one by one. So, we want to know under what conditions on a, b, c this will be in the range space of (a, b, c). So, (a, b, c) will be in range space R_T provided $\exists$ (x_1, x_2, x_3) such that T(x_1, x_2, x_3) = (a, b, c). So, this implies that what? So, this implies that $x_1 - x_2 + 2x_3$ = a, $2x_1 + x_2$ = b and $-x_1 - 2x_2 + 2x_3$= c.

So, this condition has to be satisfied because T(x_1, x_2, x_3) is given in star. So, I have used that definition and I have these three equations because (a, b, c) will be in the range real space of T provided this system has solution. So, this implies that if I write down, $\begin{bmatrix} 1 & -1 & 2 & a \\ 2 & 1 & 0 & b \\ -1 & -2 & 2 & c \end{bmatrix}$. Now I have to check whether the existence of solution criteria is satisfied or not. So, let me go for the elementary row operations for this. I am getting see if I do it first row I will keep as it is 1 –1 2 a. Second I am replacing basically $R_2 = R_2 - 2R_1$, if I do it so this is going to be 0. Second row the first entry will be 0 and 1 – 2 = 3 and then 0, this is –4, then b – 2 a . And third row is going to be just add $R_3 = R_3 + R_1$. So if I replace R_3 by $R_3 + R_1$. then I am getting 0 -3 +4 and here I am getting a + c. $\begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 3 & -4 & b-2a \\ 0 & -3 & 4 & a+c \end{bmatrix}$, Well, so one more iteration is required, I mean one more step. So I will have 1 –1 2 a, 0 3 –4 b – 2 a. Now if I simply add second and third so I will have 0 0 0, I will have b – a + c. So, this implies for existence of solution we need b – a + c = 0. $\begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 3 & 4 & b-2a \\ 0 & 0 & 0 & b-a+c \end{bmatrix}$. Although I have not made this matrix as a row-reducing echelon form, but I can immediately see the first and second are linearly independent and so no need to proceed for the next step or to find basically the rank of the given matrix.

The rank of the given matrix is equal to it is obvious it is 2 because first and second row is basically linearly independent. So, this implies that a has to be equal to (b + c). So, this is the required condition for any 3-tuple (a, b, c) to be in the range space of the linear transformation T.

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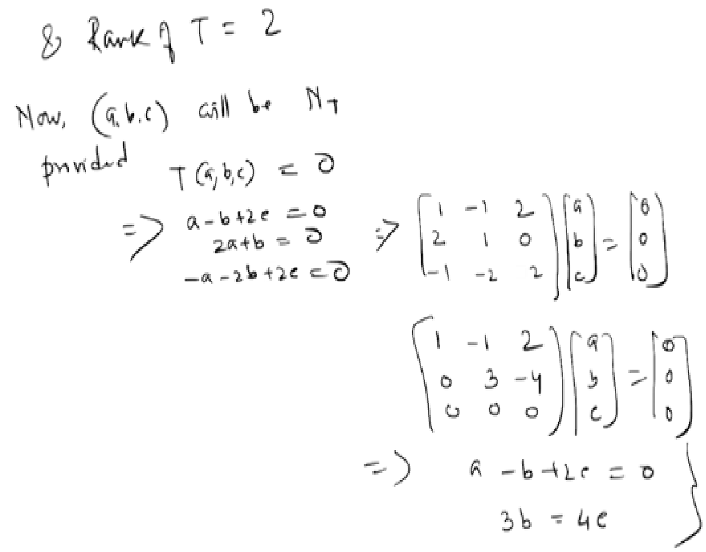

So, Rank of T = 2 because we have two linearly independent rows so that will basically give the row space of the given matrix or I can say that so this will basically act as the basis for the linear transformation space T. This is for the range space of T. Rank of this matrix here and since here I am getting two linearly independent rows, so this means that row rank equal to 2. So row rank equal to 2 means column rank of the same matrix equal to 2.

So I can say that rank of the T = 2. Now (a, b, c) will be in N_T, null space provided T(a, b, c) = 0. So, T(a, b, c) = 0 means this implies that a – b + 2c = 0, 2a + b = 0, –a – 2b + 2c = 0. So, this is the criteria for the point (a, b, c) will be null space of the linear transformation T. Indirectly you can say that solution space of the system, this is basically $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ if you solve the system like (a, b, c) if we consider this system. So, we are looking for the solution of this system. So, this means that the solution of this system is basically equivalent to we can say that if you consider the row-reduced echelon matrix, corresponding row-reduced echelon matrix is your $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ looking for the solution of this system. So, this means that (a – b + 2c) = 0, other one is (3b = 4c). So this is the required condition for (a, b, c) to be in the null space of linear transformation T.