Advanced Linear Algebra

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Lecture – 11
Linear Transformations

Welcome to lecture series on advanced linear algebra. Today, we will discuss about linear transformations. So far, we have learned what is vector space and how to compute the different subspaces, concept of basis, dimensions. Now, we want to extend this concept by introducing mapping between two vector spaces, I mean introducing a terminology linear transformation. Actually, most of the part of this course will depends on these functions. So, it is very important that we define first what is linear transformations.

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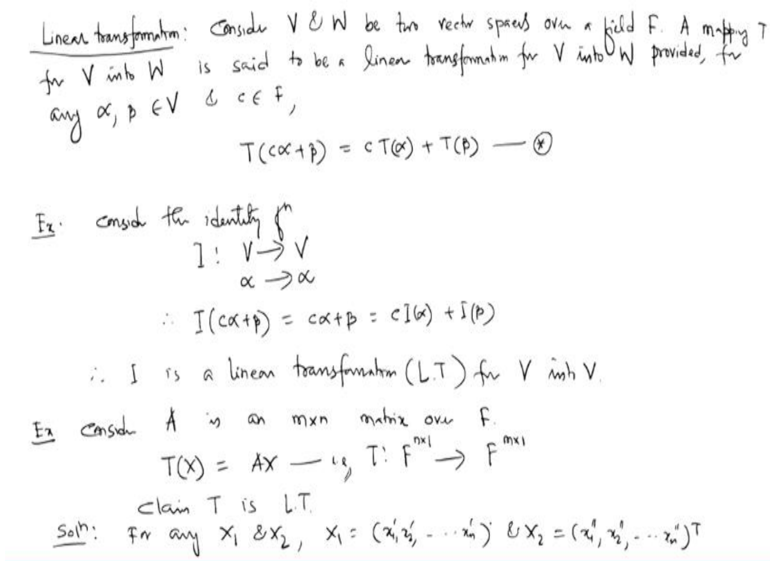

Linear transformation: Consider V and W be two vector spaces over a field say F. A mapping by function T from V into W is said to be a linear transformation from V into W provided for any $\alpha, \beta \in$ V and c $\in$ F, T(αc + β) = c T(α) + T(β). So, this is the definition of a linear transformation from vectors V and W. See here we have V and W.

Both these spaces have their individual vector additions, scalar multiplications. So, here αc + β, αc this scalar multiplication is scalar multiplication in capital and this plus a vector addition in V whereas on the right hand side where evident again time c T(α) this c times means this is a scalar multiplication in W and this addition that is T(αc + β), this plus is basically vector addition in W.

So, let me take some examples. Consider the identity mapping, identity function I : V → V. So, this mapping $\alpha \to \alpha$. So, this means that I(αc + β) = αc + β = c I(α) + I(β). So, I is a linear transformation, now onwards let I : V → V. Let me take another example. Consider A is an mxn matrix over F that is my field, a real number, may be complex number, some field F.

Then consider a function T (X) = AX. So, defining function T: F^{nx1} → F^{mx1} defined by T(X)=X. Claim T: F^{nx1} → F^{mx1}. So, like vector space V = F^{nx1}, W = F^{mx1}. Proof is very straight forward. For any X_1 and X_2 where say X_1 = $(x_1', x_2', \ldots x_n')$ and X_2 = $(x_1'', x_2'', \ldots x_n'')^T$.

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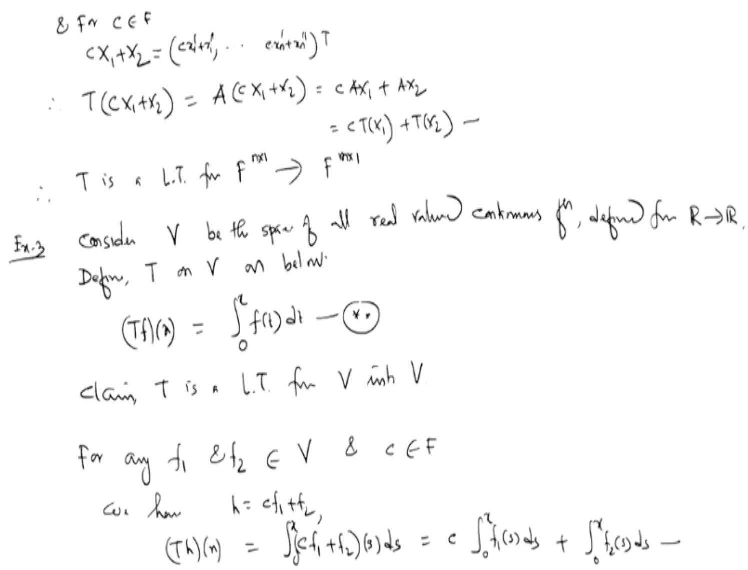

And for c ∈ F, now cX_1 + X_2 = (c $x_1' + x_1''$ ... c $x_n' + x_n''$)^T . So, T(cX_1 + X_2) = A (cX_1 + X_2). According to definitions, this is my definition star suppose, star, so I am getting this is equal to this quantity. Then if I use this even matrix multiplication also say c is the constant, T(cX_1 + X_2) = A (cX_1 + X_2) = c A X_1 + A X_2 = c T(X_1) + T (X_2).

So, T is a linear transformation from F^{nx1} to F^{mx1}. Let me take another example. Consider V be the space of all real valued continuous functions defined from R real line to R, then define the mapped set T on V as below. (Tf)(x) = $\int_0^x f(t)dt$. We know this space is not a finite dimensional space although it is a vector space. So, I have taken a function T from vector space V to V defined by (Tf)(x) = $\int_0^x f(t)\,dt$.

Claim T is a linear transformation from V into V. Certainly, the right hand side of this expression, suppose this is I am giving as double star, it is again if all real valued continuous functions, so it will be again in V. So the way it has been defined, so I can say T is a mapping from V to V, now we have to prove that it is a linear transformation. For any f_1 & $f_2 \in$ V and c ∈ F, we have say h = (c f_1 + f_2).

Then (Th)(x) = $\int_0^x (c\, f_1 + f_2)(s)ds$ = c $\int_0^x f_1\,(s)ds$ + $\int_0^x f_2\,(s)ds$. So, T is also a linear transformation from V to V itself.

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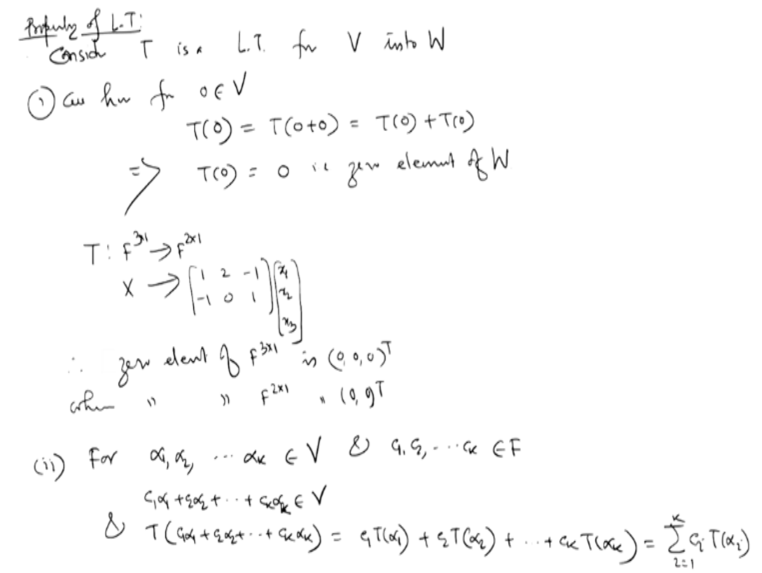
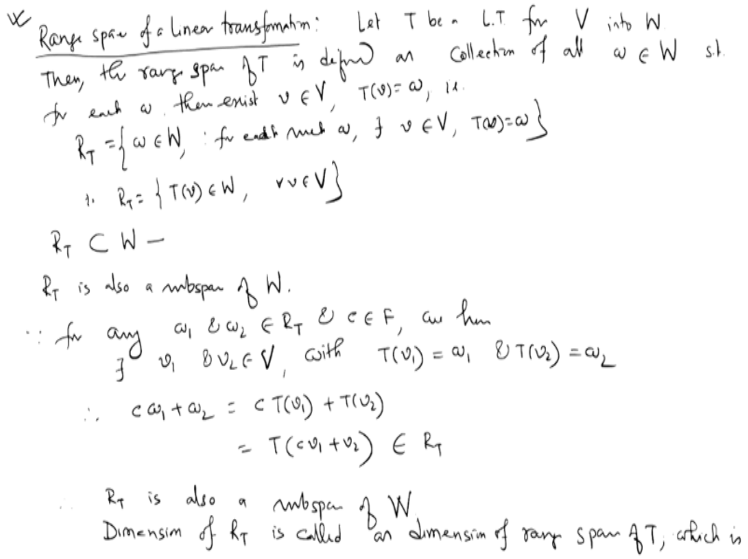


Consider T is a linear transformation from V into W. We have for 0 ∈ V, zero element of V, T(0) =T(0 + 0) =T(0) + T(0). So, this implies T(0)=0 that is zero element of W. Note that zero element of V is not necessary to be zero element of W. For example, say T: $F^{3x1} \rightarrow F^{2x1}$ defined by say X $\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

So, if I consider these ones then I see that zero element of F^{3x1} is $[0, 0, 0]^T$ whereas zero element of F^{2x1} is $[0, 0]^T$, so see that, but zero element of V goes to zero element of W is a property of this linear transformation. So, you can say the property of linear transformations. This is second property I can select this for $(\alpha_1, \alpha_2, \ldots \alpha_k) \in$ V and $(c_1, c_2, \ldots c_k) \in$ F where assume that V and W both are defined over the field F.

Then we see $(c_1\,\alpha_1 + c_2\,\alpha_2 + \ldots c_k\alpha_k) \in$ V and T$(c_1\,\alpha_1 + c_2\,\alpha_2 + \ldots c_k\alpha_k) = (c_1\,T(\alpha_1) + c_2\,T(\alpha_2) + \ldots c_k T(\alpha_k)) = \sum_{i=1}^{k} c_i\,T(\alpha_i)$ So, this is because the definition that T(αc + β) = c T(α) + T(β). **(Refer Slide Time: 15:50)**



So, now I will introduce some more terminology range space of a linear transformation. What is that? Let T be a linear transformation from V into W. Then the range space of T is defined as collection of all smaller w ∈ W such that for each small w, there exist v ∈ V, T(v) = w. That is mathematically if I write range space depending on R_T= {T(v) =w ∈ W, v ∈ V}, both are same. I mean collection of all the image elements of V, R_T is subset of W, R_T is also a subspace of vector space W, how? Because for any w_1 & $w_2 \in R_T$ and c ∈ F, ∃ $v_1, v_2 \in$ V with T(v_1) = w_1 and T(v_2) = w_2. So, now c w_1 + w_2 = c T(v_1) + T(v_2) = T($c\,v_1 + v_2$) $\in R_T$. So, R_T is also a subspace vector space W. Let me introduce one more terminology the dimension of R_T is called as dimension of range space of T.

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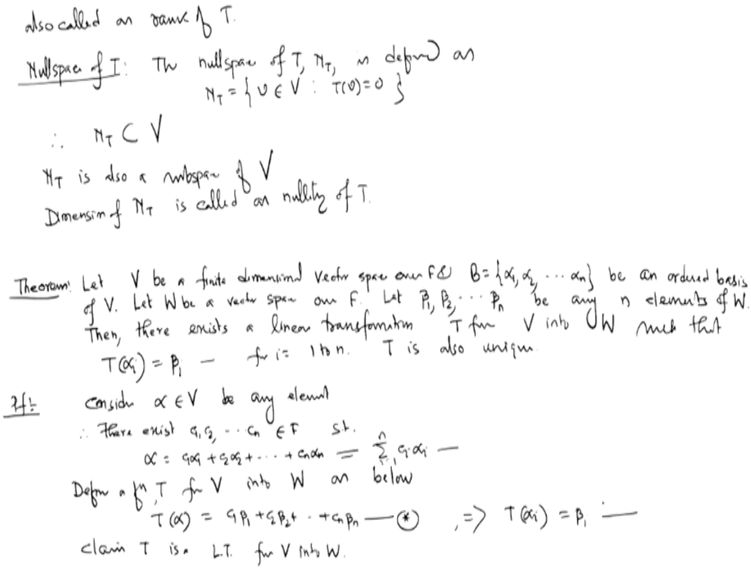

Which is called as rank of T. And the terminology let me introduce this is called null space of T. The null space of T which I am denoting as $N_T = \{v \in V, T(v) = 0\}$. This is called the null space of T. So, certainly N_T is subset of V. N_T is also a subspace of V. So R_T is subspace of W, whereas R_T is a subspace of V. Dimension of N_T is called as nullity of T.

So before proceeding more about these two subspaces and whether any possible relation is there or not, let me prove one nice result that is this theorem that is existence of linear transformation. Let V be a finite dimensional vector space of our F and B = $(\alpha_1, \alpha_2, \ldots \alpha_n)$ be an ordered basis of V. Let W be a vector space over F. Let $\beta_1, \beta_2, \ldots \beta_n$ be any n elements of W.

Then there exist a linear transformation T from V into W such that $T(\alpha_i) = \beta_i$ for i = 1 to n and T is also unique. So, this is a very important result. The proof is not difficult, but most of the time students used to forget this issue and face lot of difficulties to solve the problem. So, this existence of linear transformation is very important result in the linear algebra also. Here you see we have mentioned about the dimension of V as a finite.

We have not mentioned about the dimension of W, nothing. We have mentioned $(\alpha_1, \alpha_2, \ldots \alpha_n)$ as linearly independent elements of V whereas $\beta_1, \beta_2, \ldots \beta_n$, be any arbitrary elements of W. We have not mentioned anything whether they are linearly independent or dependent, nothing. So, the beauty of this result that for any ordered basis B on V if one give n elements of W, one can have a unique linear transformation from V to W such that T(α_i)= β_i.

Let me quickly sketch the proof. Consider $\alpha \in$ V be any element. Since $\alpha \in$ V can be written as a basis element. So $\exists$ $(c_1, c_2, \ldots\ c_n) \in$ F such that your α = $(c_1\ \alpha_1 +\ c_2\ \alpha_2 +\ \ldots\ c_n\alpha_n)$ = $\sum_{i=1}^{n} c_i\, \alpha_i$ Now define a function say T from V into W as below. Of course, this is equal to I can write down as T(α) = $c_1\ \beta_1 +\ c_2\ \beta_2 +\ \ldots\ c_n\beta_n$. So, defining this is by definition of the function. Claim T is a linear transformation from V into W. See the function from V into W that is clear because β_1, β_2, … β_n there are elements of W and in taking the linear combination of β_1, β_2, … β_n that will be certainly will be in the W. So, it is a function from V to W is clear and is also well-defined rule. So, this implies also I can say T(α_i)= β_i. I have also considered this thing.

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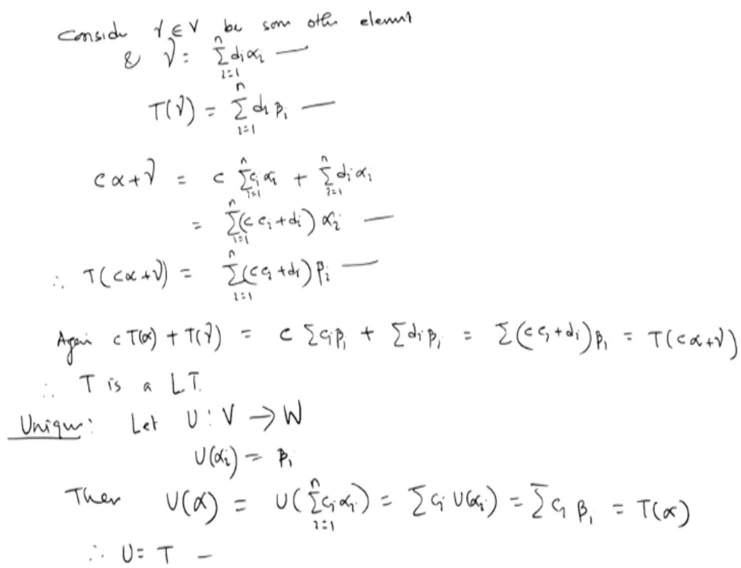

Consider $\gamma\ \in$ V be some other element and γ = $\sum_{i=1}^{n} d_i\, \alpha_i$ because B is the ordered basis for the space V and $\gamma\ \in$ V, so γ can be written as like this thing. So according to the definitions T(γ) $=$ $\sum_{i=1}^{n} d_i\, \beta_i$. Now you see what is c α + γ = $c\ \sum_{i=1}^{n} c_i\, \alpha_i$ + $\sum_{i=1}^{n} d_i\, \beta_i$ = $\sum_{i=1}^{n}(c\ c_i +\ d_i\ )\, \alpha_i$, then T(c α + γ) = $\sum_{i=1}^{n}(c\ c_i +\ d_i\ )\, \beta_i$. Again c T($\alpha$)+ T(γ) = c $\sum_{i=1}^{n} c_i\, \beta_i$+ $\sum_{i=1}^{n} d_i\, \beta_i$ = $\sum_{i=1}^{n}(c\ c_i +\ d_i\ )\, \beta_i$= T(c α + γ). So T is a linear transformation. Now claim is it is unique. Suppose not, let U: $V\ \rightarrow W$ U(α_i)= β_i. Then U(α) = U($\sum_{i=1}^{n} c_i\, \alpha_i$) = $\sum_{i=1}^{n} c_i\, U(\alpha_i)$ = $\sum_{i=1}^{n} c_i\, \beta_i$ = T(α). So, this implies U = T. So, we see that there exists a unique linear transformation from V into W.

So, we see that for a given vector space V of finite dimensional space and when the ordered basis is given to it suppose n elements to prove and is given from the other space W, then there exists a linear transformation which is also unique from V to W such that have a relation that T(α_i)= β_i, this relation holds. Thank you.