Advanced Linear Algebra
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Lecture – 10
Computations Concerning Subspaces

Welcome to our lecture series on advanced linear algebra. So, far we have discussed about the system of equations and their existence of solution and how to find a solution, what is the criteria for that. Then we have also defined the vector space and details of it also. Now, let us conclude what we have learned by solving one problem and consider the matrix and related to that matrix how to compute different subspaces and answering different questions concerned to that matrix.

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$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Q1: Find an invertible matrix P such that PA is a row-reduced echelon matrix R.
Is it unique?
Q2: Find an ordered basis B for the row space W of A.
Q3: Say which vector $\beta = (b_1, b_2, b_3, b_4, b_5)$ in W
Q4: Find the coordinate matrix of β w.r.t. B
Q5. What is the general element of W?
Q6: Give an explicit description of the vector space V of all 5x1 column matrix X such that
Ax = 0
Q7: Find a basis for V
Q8: For what 5x1 column matrix Y the system AX = Y has solution X?

So, let me take a 5x5 matrix A equal to like this. So it is a 5x5 matrix over the, so field is my real number here. Then I have raised a couple of questions here. The first question is, find an invertible matrix P such that PA is a row-reduced echelon matrix R. Then there is also question whether that matrix P is unique or not? Second question find an ordered basis B for the row space of A.

Third which vector say $\beta = (b_1, b_2, \; b_3, b_4, b_5)$ in this subspace W and find the coordinate matrix of β with respect to ordered basis B. Then four more questions what is general element of W and what will be the structures of this solution X of this system AX= 0? What will be basis for the solution space of the system AX = 0 and for what set of 5x1 column matrix the system AX=Y will have solution X?

Now, to answer these questions, I will consider an augmented systems and do the elementary row operations for the matrix A. So, let me write down.

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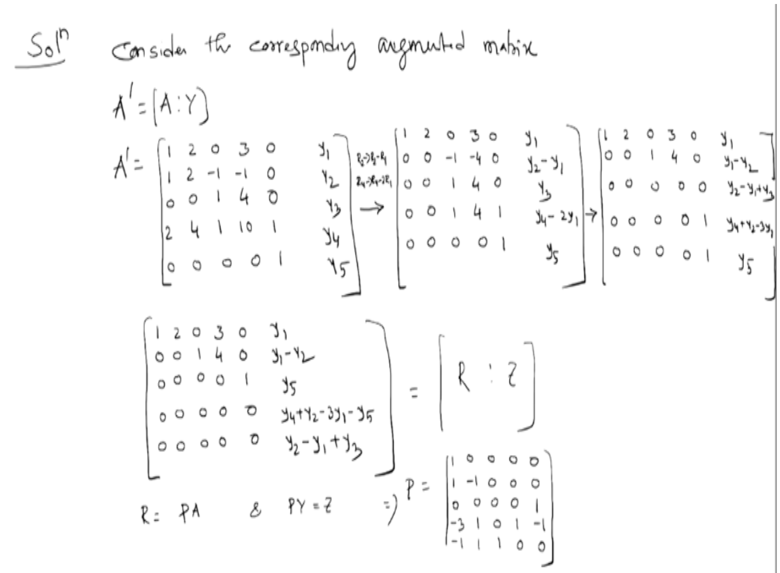

So, it is solutions. So consider the corresponding augmented matrix say $A' = [A : Y]$. So I can write down this $A' = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 1 & 2 & -1 & -1 & 0 & y_2 \\ 0 & 0 & 1 & 4 & 0 & y_3 \\ 2 & 4 & 1 & 10 & 1 & y_4 \\ 0 & 0 & 0 & 0 & 1 & y_5 \end{bmatrix}$. So this is the augmented matrix corresponding the system AX=Y. Since we have to find an invertible matrix P such that PA equal to row-reduced echelon matrix, so I have to do the elementary row operations and get that matrix also.

So, let me do it. So, if I do it I will have this thing. The first row I will keep as it is (1 2 0 3 0 :y_1) because the first nonzero entry appears in first column which is equal to 1 that is leading entry, so it is fine. Now, to satisfy the second criteria for the row-reduced echelon forms, all the entry of the first column must be, I mean apart from the first one all has to be 0. So that is why I am just writing that $R_2 \rightarrow R_2 - R_1$ and $R_4 \rightarrow R_4 - 2R_1$.

If I do it what I am getting, I am getting here so $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 0 & 0 & -1 & -4 & 0 & (y_2 - y_1) \\ 0 & 0 & 1 & 4 & 0 & y_3 \\ 0 & 0 & 1 & 4 & 1 & (y_4 - 2y_1) \\ 0 & 0 & 0 & 0 & 1 & y_5 \end{bmatrix}$. So this is the updated one. But I see that this part inside the part is 5x5 part, this is not a row-reduced echelon matrix, so we need further elementary row operations we have to apply. So if I do it, then I am getting 1 2 0 3 0 y_1, this is as it is.

So I will do R_2 divided by –1, so 0 0 1 4 0. we will have $(y_1 - y_2)$ and R_3 I simply add with R_2, so I will have 0 0 0 0 0 and $y_2 - y_1 + y_3$. Similarly if I go for the R_4, R_4 replacing by $R_4 + R_2$ then I am getting 0 0 this will be 0 0 1 y_1, $y_4 + y_2 - 3y_1$ and last row will remain as it is, so this is y_5 like this $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 0 & 0 & 1 & 4 & 0 & (y_1 - y_2) \\ 0 & 0 & 0 & 0 & 0 & (y_2 - y_1 + y_3) \\ 0 & 0 & 0 & 0 & 1 & (y_4 + y_2 - 3y_1) \\ 0 & 0 & 0 & 0 & 1 & y_5 \end{bmatrix}$

We see still it is not row-reduced echelon matrix. So what I will do, again I have to do the next step. So for that I am getting again 1 2 0 3 0 y_1 and 0 0 1 4 0 $(y_1 - y_2)$. And I will replace fifth row by third row and third row by fifth row. So if I do it then I am getting 0 0 0 and 0 1 and y_5 and fourth row I am subtracting by fifth row. So if I do that one and as it is 0 0 0, so I will have $y_4 + y_2 - 3y_1 - y_5$ and last one since I have shifted third row to the fifth row so this will be 0 0 0 0 0 $y_2 - y_1 + y_3$. So, I have got a new matrix like this thing $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 0 & 0 & 1 & 4 & 0 & (y_1 - y_2) \\ 0 & 0 & 0 & 0 & 1 & y_5 \\ 0 & 0 & 0 & 0 & 0 & (y_4 + y_2 - 3y_1 - y_5) \\ 0 & 0 & 0 & 0 & 1 & (y_2 - y_1 + y_3) \end{bmatrix}$ = [R : Z]. Now, you see that the first five columns, they satisfy the condition of the row-reduced echelon form of the matrix and it is also row-reduced matrix.

So, if I can write this is equal to something like you know R and Z like this. So here R is equal to have got a finite sequence of elementary matrix has been multiplied from the left side to the matrix A and if I denote that is equal to P, then PA=R and PY=Z. So this implies P matrix is basically coefficient matrix of the Z columns. This means that P will be here 1 0 0 0 0.

Second row will be 1 –1 because coefficient, so coefficient of y_1, y_2, y_3, y_4, y_5 we have to check it. Coefficient of y_3, 0 should be 0, similarly like this and then 0 0 0 0 1 and fourth row will be –3 1 0 1 and –1, fifth one will be –1 +1 +1 0 0. So, this is the P = $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -3 & 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & 0 & 0 \end{bmatrix}$. See, I can write down fifth row as the fourth row or fourth row as fifth row. So, this means that I can have different P matrix, so therefore this P matrix what we have obtained here is not unique. So, P is not unique, but P it is clearly invertible matrix.

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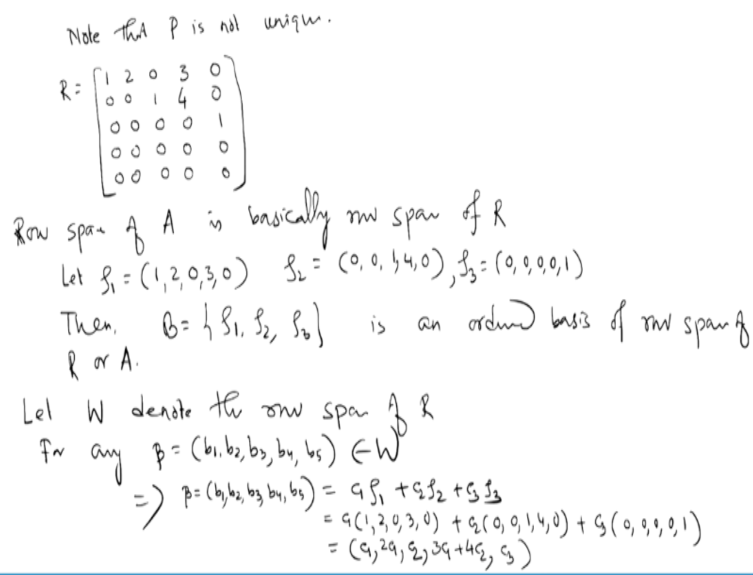

So, note that P is not unique, I hope this is clear why it is not unique. So, this is the answer to the first question what we raised in the very beginning find an inevitable matrix P such that PA is a row-reduced echelon matrix R. So let me rewrite this one. So we have R = $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 1 2 0 3 0 and then 0 0 1 4 0, 0 0 0 0 1, 0 0, so this is my R = $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. But this R is unique. For a given matrix A we will have only one row-reduced echelon matrix.

Now, my second question was find an ordered basis B for the row space of A. Already we have proved the row space of A is basically row space of R. Now, in R there are three nonzero rows. We have already proved that all the nonzero rows of row-recued echelon matrix will be linearly independent and span the row space of R. So therefore, if I denote let $\rho_1 = (1,2,0,3,0)$ and $\rho_2 = (0,0,1,4,0)$ and $\rho_3 = (0,0,0,0,1)$.

Then B = $\{\rho_1, \ \rho_2, \ \rho_3\}$ is an ordered basis of row space of R or A which is the answer to the second questions. Now, what is the third question? Say which vector $\beta = (b_1, b_2, \ b_3, b_4, b_5)$ within W? Let W denote the row space of R. For any $\beta = (b_1, b_2, \ b_3, b_4, b_5) \in$ W implies that I will be able to express beta as LC of $\{\rho_1, \ \rho_2, \ \rho_3\}$, I mean to say there are constant c_1, c_2, c_3 over the field real number R.

Such that your $\beta = (b_1, \ b_2, \ \ b_3, \ b_4, \ b_5) = c_1\rho_1 + \ c_2\rho_2 + c_3\rho_3 = \ c_1 \ (1,2,0,3,0) + \ c_2 \ (0,0,1,4,0) + c_3(0,0,0,0,1) = (c_1, 2c_1, c_2 \, , 3c_1 + 4c_2, \, c_3)$. So, general element is of this form. So, this implies that I can immediately compare the coefficients from the both sides.

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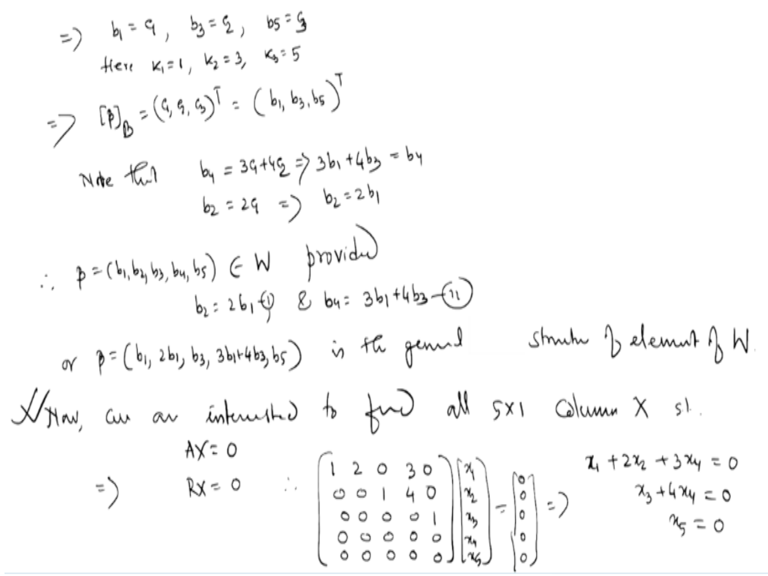

So, this implies that $b_1 = c_1$, $b_3 = c_2$ and $b_5 = c_3$. Already we have proved in our last class for a general mxn matrix what will be the relation between coefficient $c_1, \, c_2, \, c_3, \, c_4$ and the entry of the beta matrix. So, generally this $c_1 = b_1$, $c_2 = b_2$ and $c_3 = b_5$ in that way also one can directly write down the solutions without comparing the coefficient also. But anyhow this is a very 5x5 problem, so it is not difficult for us to calculate the coefficient of c_1, c_2, c_3.

So, immediately I see that $c_1 = b_1$, $c_2 = b_3$ and $c_3 = b_5$. Here k_1 = 1, k_2 =3, k_3 = 5. So, the column number in which leading entries appear in the row-reduce echelon form is 5. So, this implies the $[\beta]_B = (c_1, c_2, c_3)^T = (b_1, b_3, b_5)^T$. So, this is the coordinate matrix of any element beta belongs to W.

And what will be the general element in W? Note that $b_4 = 3c_1 + 4c_2 = 3b_1 + 4b_3 = b_4$ So, this is one and another one is your $b_2 = 2c_1$ this implies that what your $b_2 = 2b_1$. So, this implies $\beta = (b_1, b_2, b_3, b_4, b_5) \in$ W provided $b_2 = 2b_1$ and $b_4 = 3b_1 + 4b_3$. So, this is the condition for that is b_1, b_2, b_3, b_4, b_5 so that β will be in W.

So, these are the two conditions, one and two I can say like this or I can say that β will be of the form of $(b_1, 2b_1, b_3, 3b_1 + 4b_3, b_5)$ is the general structure of elements of W that is row space of your R. Interested to find all 5x1 column X such that AX=0 having solution space of the system AX=0. Now, the solution space of the system AX=0 is equivalent to solution space of the system RX=0.

Therefore, let us quickly see what is RX=0. So RX=0, means we are having

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, this means that I am getting the system $(x_1 + 2x_2 + 3x_4) = 0$ and $(x_3 + 4x_4) = 0$ and $x_5 = 0$. So, I am getting three equations. Now, I can immediately solve it and I can tell the solution space of this system RX=0.

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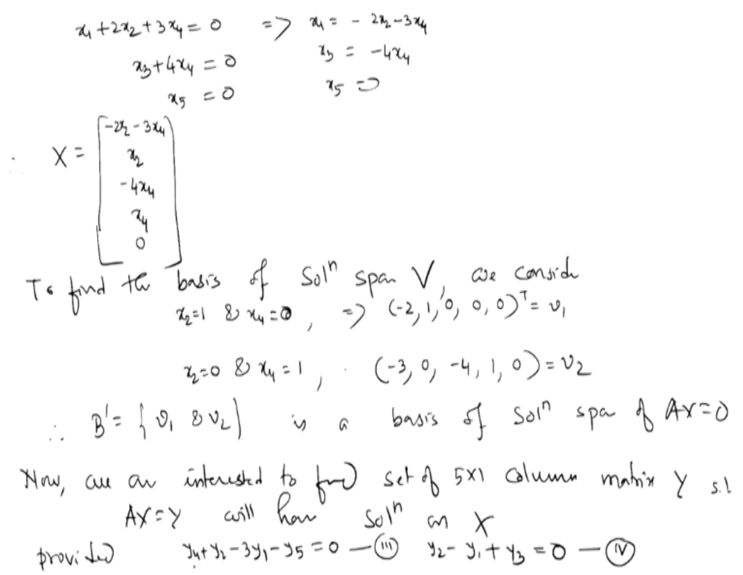

We have written x_1 + 2 x_2. So I have this one. So, this implies that I can write down x_1 as a explicitly function of x_2 and x_4, x_3 as a function of x_4 and x_5 = 0. So, this means that I am getting x_1 = –2x_2 –3$x_4$and x_3 = –4 x_4 and x_5 = 0. So, this implies general solution X=$\begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ -4x_4 \\ x_4 \\ 0 \end{bmatrix}$. So, this is the basically the general structure of the solution of the system AX=0.

Now, to find the basis for the solution space, what I will do? Since there are two variables, which are x_2 and x_4 which are considered as the three variables, so to find the basis of solution space V, we consider x_2 = 1 and x_4 = 0. So, this means that I am getting x_1= –2, then x_2 = 1, I have considered, x_3 = –4 x_4, so x_4 = 0 and next is also 0 , this transpose, so this is one solution $(-2, 1, 0, 0, 0)^T =$ v_1.

And now again putting x_2= 0 and x_4 = 1, this will give me putting x_2 = 0 so –3, 0, -4, 1 and 0. So, this $(-3, 0, -4, 1, 0)^T =$ v_2. So, then B'= { v_1 & v_2 } is the basis of solution space of AX=0. Since, it has two free variables, so the dimension of the solution space will be 2. So, we need to linearly independent elements which will span this solution space that is b_1 and b_2 we have calculated.

So, the next question was for what 5x1 column matrix Y the system AX=Y has solutions? Now, we are interested to find a set of 5x1 column matrix Y such that the system AX=Y will have solution as X. Already we have seen the criteria for the existence of solution that since in the row-reduced echelon form of the given matrix A having three nonzero rows, so therefore in the augmented systems that ($y_4 + y_2 - 3y_1 - y_5$) = 0.

Similarly, in the fifth row ($y_2 - y_1 + y_3$) = 0. Other one is ($y_2 - y_1 + y_3$) = 0. So, these are the two conditions 3 and 4. So, all set of Y which entries are y_1, y_2, y_3, y_4, y_5 satisfies condition 3 and 4 for that set of Y the system AX=Y will have solution as X provided the entry satisfies these two relations. So, this is all about the different questions I raised related to the given matrix. I hope this will give a clear-cut picture about what we have learned so far related to this course. Thank you.