Advanced Linear Algebra
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Lecture – 1
System of Linear Equations

Welcome to advanced linear algebra. In the very beginning, let me raise three questions. Why one has to study this course? What we are going to learn from this study? What will be the mode of learning the subject? The first question is basically what is the motivation behind this study and you will come to know the basic concept of linear algebra acts as a prerequisite in many courses; undergraduate, graduate and postgraduate levels also.

It may be from mathematical science, engineering science, biological science, even in sociology and economics also. Frequently one has to encounter the system of linear equations and the questions concerning to it, I mean to say existence of solutions and if it exists how to find, similarly system of differential equations, how to find a solution of it if at all solution exists, how to know the behaviour of the solutions, characteristic, etc.

To know all these things, one has to use the concept of linear algebra. So, it is a very essential subject for many courses. Regarding the second question what we are going to study that is basically course content, as you know in last two decades, the advance linear algebra is so fast, it is very difficult to consider what will be in a time limited course and what should not be.

Anyhow, we have designed the course in such a way that once if someone has done this course, he or she will be in the position to face the challenges in the linear algebra related applications in their different courses.
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Different modules:

i. Linear System of equations,
ii. Vector spaces,
iii. Linear transformations,
iv. Canonical forms and Jordan forms,
v. Inner product spaces and different operators in it,
vi. Bilinear and Quadratic forms, Orthogonal projection and Spectral theory, and
vii. Singular value decomposition.

Based on that this present course consists of 7 modules start from linear system of equations, vector spaces, linear transformations and the end singular value decomposition theorems and its application also. Regarding the last question, what will be the mode of learning the subjects, it will be through a series of lectures and solving assignments and problems which we are going to supply you, you have to solve it.

While solving you may face some difficulties, we can have email conversations or even by telephone conversations with the two teachers in this course and it call me also. Throughout this course, I mean, I will use most of the lectures will cover from basically linear algebra by Hoffman and Kunze. Other parts, I will let you know in time to time. So before starting the first modules, let us quickly recall the properties of complex and real numbers with respect to two binary operations, addition and multiplications.

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- Properties of Complex and Real numbers:

Let us represent complex number by $\mathbb{C}$ & real number by $\mathbb{R}$

Let F represent either complex number or real number.

Properties w.r.t addition: For any $x, y, z \in F$

(i) Closure property: $x \& y \in F$, $x+y \in F$

(ii) Associative ": $x + (y+z) = (x+y) + z$

(iii) Existence of zero element: $0 \in F$ & for any $x \in F$

$$0+x = x+0 = x$$

(iv) Existence of inverse: For any $x \in F$, $-x \in F$

& $x + (-x) = 0$

(v) Commutative property: For $x \& y \in F$

$$x+y = y+x$$

As you have already learnt that complex number or real numbers satisfy certain properties. So, let us represent complex number by the symbol C and real number by the symbol say R. The properties what I am going to talk for the complex and real numbers it basically same sort of properties. So, let F represent either complex number or real number. Now, let me talk about the property with respect to additions.

So properties with respect to addition for any x, y, z € F, the first one closure property, For x , y € F, x+ y € F. Second one associative property, x + (y + z) = (x + y) + z. Third one existence of zero element. You know 0 € F. It belongs to the complex number as well as it belongs to real number and for any x € F, 0 + x = x + 0 = x.

Fourth one existence of inverse. For any x € F, –x € F and x + (- x) = 0. Commutative property that is for x , y € F, x + y = y + x. The set which satisfies these first four axioms is called the abelian group. Also

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W.r.to Multiplication:

(i) closure property   for x & $y \in F$
$$x \cdot y \in F$$
or
$$xy \in F$$

(ii) Associative property: $x(yz) = (xy)z$ for $x, y, z \in F$

(iii) Existence of unit element, 1,: $1 \in F$ & for any $x \in F$
$$1 \cdot x = x \cdot 1 = x$$

(iv) Existence of inverse: For any $0 \neq x \in F$ $\exists$ $\frac{1}{x}$ s.t.
$$x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1 \qquad , \qquad \frac{1}{x} \in F$$

(v) Commutative property: For x & $y \in F$
$$xy = yx$$

(vi) Distributive prop over addition: For $x, y, z \in F$
$$x(y+z) = xy + xz$$

With respect to multiplication capital F satisfies six more axioms or properties with respect to multiplication also. The first one is closure property that is for x , y € F, x*y also € F. For sake of simplicity, I will write down simply x y or x * y € F. Here also associative property holds good. Associative property that is x (y z)= (x y) z. For x, y and z € F.

Third one existence of unit element 1 basically. See 1 belongs to capital F it may be found complex, it may be real, but 1 € F and for any x € F, 1 x = x 1 = x, so not to confuse if want to star this is fine. Fourth one existence of inverse, For any $0 \neq x$ € F, there exist $\frac{1}{x}$ such that , x $\frac{1}{x} = \frac{1}{x}$ x = 1. So, this $\frac{1}{x}$ € F.

So, even though I am not writing star it does not mean that I have not written any operation, I am writing simply x $\frac{1}{x}$ and which is equal to $\frac{1}{x}$ x. Here also commutative property also holds good that is for x,y € F, x y = y x. Finally, distributive property over addition also holds good that is for x, y, z € F, x (y+ z) = x y + x z.

So, with respect to multiplications the set capital F is the first group because satisfied all these four axioms of the group. It also satisfies the commutative property, so it is an abelian group and the last one is distributive property over addition. With this information, let me define one terminology called the field. What is field?

Field:

A nonempty set, F, equipped with two binary operations: addition, +, and multiplication, *, is said to be a field provided

(i) w.r.t. + it is a group,

(ii) w.r.t. * it is an abelian group and distributive law over addition is also satisfied.

$$\Rightarrow \quad \mathbb{C} \quad \text{is a field}$$
$$\mathbb{R} \quad \text{"} \quad \text{"}$$
$$\mathbb{Q} \quad \text{set of rational number is also a field.}$$

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Field it is a nonempty set capital F, equipped with two binary operations say addition and multiplication, is said to be field if with respect to addition it is a group and with respect to multiplication it is an abelian group and distributive law over addition is also satisfied. So, this implies the set C the complex number is a field, real number is also a field. One can also check the set of rational number is also a field.

But interestingly so Q is the subset of R and R is subset of your C and Q and R are nonempty subset also and they also satisfied the definitions, all the axioms of the field. So here Q ,I can treat that as subfield of the field R and R is a subfield of the field capital C. So, what is the definition of the subfield?

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Subfield:

A nonempty subset, F' of a field F is said to be a subfield of F provided F' satisfies all axioms of field F.

F as field for my rest of the lectures in this course.

So, subfield I am saying that it is a nonempty subset F prime of the field capital F which satisfies also all the axioms of the field capital F, then only I will say that a nonempty subset is a subfield of the field capital F. Throughout this course, I will consider capital F as field for my rest of the lectures in this course.

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System of Linear Equations :

Consider the problem of finding n scalar variables $x_1, x_2, ..., x_n, x_j \in F$, for $j = 1 ... n$ and satisfying following m equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= y_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= y_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots & \qquad -(*) \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= y_m
\end{aligned}$$

$a_{ij}, y_i \in F, 1 \le i \le m \ 1 \le j \le n.$

The above set of equations is defined as system of m-equations in n unknowns.

If $y_i = 0$ for $i = 1 ... m$, then the system is said to be **homogeneous** else it is **non-homogeneous**.

Now introducing the definition of the field, subfield let me consider a first module that is system of linear equations. What is system of linear equations? It is basically a problem of finding n variables x_1, x_2, x_n where each $x_i, x_j \in F$. The field maybe set of complex number, it may be the numbers and this n variables satisfying each equation there are m equations which I have written here in star that is I have written m equations $a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n = y_1$; $a_{21} x_1 + a_{22} x_2$

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$+\ldots+ a_{2n}\, x_n = y_2$ and similarly $a_{m1}\, x_1 + a_{m2}\, x_2 +\ldots+ a_{mn}\, x_n = y_m$.

Here a_{ij}, y_i € F where $1 \le i \le m$ and $1 \le j \le n$. The above set of equation can be defined as the system of m equations in n unknowns. One can define it as system of m equations in n unknowns. When the right hand side of the set of equations are 0, we wish to say that the system of m equation is homogeneous, otherwise it is nonhomogeneous.

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Solution of system of equations:

Any n-tuple $(x_1, \ldots, x_n)$, $x_i \in F$, $1 \le i \le n$, which satisfies each equation of $(*)$ is said to be a solution of the system.

And any n-tuple x_1, x_2, x_n or x_i € F and i from 1 to n, if satisfied each equation of m equation in n unknow system, then I will say that this n-tuple will be a solution of the system. Now let us try to understand the solution as well as existence of solutions from our elementary 11th-12th concept also, then I will go for the generalization case.
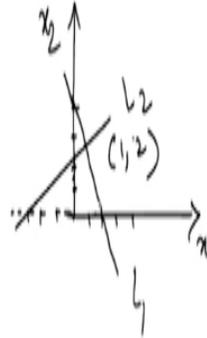
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Geometrical interpretation:

Ex. 1:

$$2x_1 + x_2 = 4 \qquad \text{(i)} - L_1$$
$$x_1 - 2x_2 = -3 \qquad \text{(ii)} - L_2$$

Let me take very simple example that is $2\ x_1 + x_2 = 4$ and $x_1 - 2\ x_2 = -3$. So, geometrically what I can say it here so this first equation is nothing in the x_1, x_2 plane if I consider this is x_1 and this is my x_2, so in x_1, x_2 plane the first equation represents basically a line, again simply this one okay. So, when x_1, $=0$ that is equal to this point and $x_2=0$, so I can say a line is like the first line L_1, this is L_1.

And second one say that $x_1=-3$ here and when $x_2=0$, $x_1=-3$, so this point and when $x_1=0$ it is $\frac{3}{2}$, so it will be $\frac{3}{2}$ here. So it is something like this. So, this is my L_2,, this is my L_1,. Now, what are the meaning of the solutions? Solution is basically intersection of these two lines. Now these two lines meet at the point here as (1, 2), you can check it this is (1, 2), yeah I see that satisfy the surface.

So, these two lines if they are parallel then certainly we will not get solutions. So, the existence of solution depends on whether lines are parallel or they intersect each other or not. It may be possible that two lines also overlap and lie on each other also. So, then also systems have solution, but if they are parallel certainly no.
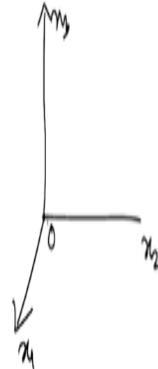
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Ex. 2:

$$x_1 + x_3 = 2 \qquad \text{(i)}$$

$$x_1 + 2x_2 = 3 \qquad \text{(ii)}$$

$$3x_2 + 2x_3 = 5 \qquad \text{(iii)}$$

Similarly, if I go for the next example in my three-dimensional space say x_1, x_2, x_3 we see that I have considered three equations here. Now, the first each equations represent a plane $x_1 + x_3 = 2$ means it is a plane you know parallel to x_1, x_3 plane where x_2 is perpendicular plane. you can say, Now, similarly, $x_1 + 2 x_2 = 3$ is another plane in three-dimensional space, similarly third one.

Now, this system will have solution provided not a single plane is parallel. If one plane is parallel to any of these two, definitely this system will not have solutions. Existence of solution depends on whether planes are parallel or not.

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Solution in terms of column structure:

Ex. 1:

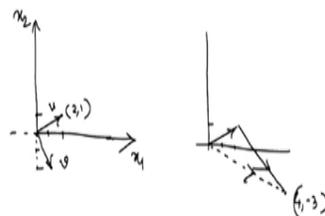$$2x_1 + x_2 = 4 \qquad \text{(i)}$$

$$x_1 - 2x_2 = -3 \qquad \text{(ii)}$$

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$ —

$$u = (2,1)^T$$
$$v = (1,-2)^T$$

$$x_1 = 1 \quad \& \quad x_2 = 2$$

$\Rightarrow$) We are looking for possible linear combination of u & v which will give w.

Now, let me represent again, let us discuss this in a bit different notation. Different notation means

one can see this and let me consider the same equation the first example and I also got the second examples, let me see solution of the systems in a bit column structure. What I meant to say you the same equation one can write down $x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$. So if you represent this $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ column as my vector say u vector and the $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ as v vector.

So, these equations represent basically some sort of linear combination of the vector u and v which will give me another vector on the right hand side. So, if I quickly sketch it, so what I have done so let be considered here this is my x_1 and this is my x_2. Now I am writing the vector u, let me consider u is equal to as you $[2,1]^T$ I can say that column vector hypothesis or if you recall our 11th-12th as vectors having in this say force velocity there is a vector.

So we can write in 2i+1j type that is also possible. But anyhow, I hope you will not confuse with these notations. So, let me write the vector u equal [2,1]. So, this is basically the vector u, the vector having coordinate is basically [2,1]. Then another vector v, v is equal to I am saying that $[1, -2]^T$, so this one unit this side and two units bottom, [1,2] here. So, it is here, this is my v, so these two vectors.

Now, suppose somehow I know that for $x_1=1$ and $x_2=2$, it is a solutions, $x_1=1$ and $x_2=2$, suppose we have the solutions, let me see it what does it means. So, this is u, so let me write down here also, sketch it, so first 1, 2 one unit, yeah this is my u, then it has gone to the next is u v, so it will be something like this. Then two times of this, so I can extend it, you will see that this point is going to basically [4,-3]. so 3 units from the bottom and this side is again 4 units.

So, this means that this one, so one time of u, two times of v equal to this one. So, this is basically your right hand side vector. So, this means that we are looking for the possible linear combinations of u and v which will give right hand side vector. So, this let me consider w which will give w. So, it may have linear combination possible in a combination followed you will have for right hand side vector, you may not also.

Now the question still remains unsolved. So, it is not, we have not answered, we have just

explained the meaning of existence of solution for this system.
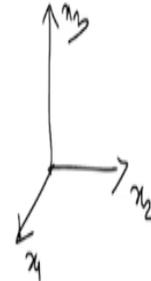
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Ex. 2:

$$x_1 + x_3 = 2 \qquad \text{(i)}$$
$$x_1 + 2x_2 = 3 \qquad \text{(ii)}$$
$$3x_2 + 2x_3 = 5 \qquad \text{(iii)}$$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

One can also represent for the second examples over the three-dimensional space that what I have considered here as x_1, x_2 and x_3 over this also. You will see that here also we are looking for the possible linear combination of the u, v, w which will give me say something z, you may have solutions, you may not have solutions. Now still under what condition the system will have solution and how to find a solution this is still , you know left out.

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- How to know whether a given system of linear equations has solution or not?

- If yes, then how to find it?

Certainly, we have to follow elimination procedure.

Let us see the question how to know whether a given system of linear equation has solution or not and if yes it has solution, then how to find it, this question we are going to answer. Certainly, one

has to follow the elimination procedure what we have learned in our 11th-12th course. In the elimination procedure when we are eliminating one by one exactly what we are doing? We are basically doing some linear combinations of the given systems and introducing another system. So, what is linear combination of equation?

Linear Combination of equations

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = y_1$$
$$a_{21} x_1 + \quad \cdots + a_{2n} x_n = y_2$$
$$\vdots$$
$$a_{m1} x_1 + \quad \cdots + a_{mn} x_n = y_m$$

(*)

Multiplying of c_1, of c_2, $\cdots$ of c_m to 1st, 2nd, $\cdots$ mth equn, respectively & add it then we will have a new equation

$$\left(\sum_{i=1}^{m} a_{i1}\right) x_1 + \left(\sum_{i=1}^{m} a_{i2}\right) x_2 + \cdots + \left(\sum_{i=1}^{m} a_{in}\right) x_n = \sum_{i=1}^{m} c_i y_i \; —$$

which is called as linear combination of equs of (*)

So, if I recall my system of equation star that is $a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = y_1$; $a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = y_2$ and $a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = y_m$ if you recall the systems of equations for which I have considered as you know star, multiplying $0 \neq c_1$, $0 \neq c_2$ and $0 \neq c_m$ to first, second and mth equations respectively and add it, we will have a new equation of the form of like you know I can write down.

Say $\sum_{i=1}^{m} a_{i1} x_1 + \sum_{i=1}^{m} a_{i2} x_2 + \ldots \sum_{i=1}^{m} a_{in} x_n = \sum_{i=1}^{m} c_i y_i$. So I will have a new equation as a linear combination of the equations of star which is called as linear combination of equations of star. Even if ,I want to multiply the first equations by some constant c and get a new system, there I am getting basically a linear combination of the existing system.

Equivalent system:

Two system of equations are said to be equivalent if each equation of one system is linear combination of equations of other system.

$$
\begin{aligned}
b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n &= z_1 \\
b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n &= z_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad & \qquad -(**) \\
b_{m1}x_1 + b_{m2}x_2 + \ldots + b_{mn}x_n &= z_m
\end{aligned}
$$

$b_{ij}, z_i \in F, 1 \le i \le m\ 1 \le j \le n.$

❖ **Equivalent systems of linear equations have exactly the same solutions.**

And through these procedures, I can write in a very generalized form and from the system star we have obtained again m equations as linear combinations of the equations of star and we have a double star, here also I am getting that m equation. Here also all b_{ij}, z_i € F and $1 \le i \le m$, $1 \le j \le n$. I will say that these two systems star and double star are equivalent systems equation will say equivalence.

Two equations are said to be equivalent provided each equation of the one system has been obtained as a linear combination of the equation of another system. So, I can say double star and star are equivalent system. Now, certainly there will be questions whether both the system will have same solutions? Answer is yes. Let me take very simple example.

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$$c(a_{11} x_1 + a_{12} x_2) = c y_1 \rightarrow L_1$$
$$a_{21} x_1 + a_{22} x_2 = y_2 \rightarrow L_2$$

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = y_1 : p_1 \left. \begin{array}{l} \text{— plane in 3dim span} \end{array} \right\}$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = y_2 : p_2 \qquad \text{1)}$$

$$b_{11} x_1 + b_{12} x_2 + b_{13} x_3 = z_1 \left.\right\} \text{equivalent}$$
$$b_{21} x_1 + b_{22} x_2 + b_{23} x_3 = z_2 \qquad \text{System}$$

Let me consider say $a_{11} x_1 + a_{12} x_2 = y_1$; $a_{21} x_1 + a_{22} x_2 = y_2$. So if we multiply the first equation by some constant c_1 to both the sides, question is whether the system will be changed. I mean system will change, but do you think solution will be changed? Now, if we multiply with some constant c, suppose c_1 both sides left hand side of this one, then we see there is no change. This is a line say line 1, this is line 2.

So, the line 1 and line 2 remain as it is even if we multiply some constant c_1 or c_2. So similarly in three-dimensional space suppose if I consider two planes, say say $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = y_1$; $a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = y_2$. Here each of this equation represents a plane, the plane in three-dimensional space, this is p_1 and this is say p_2, plane two this one. Now multiplying by some nonzero constant to any of these two equations on both sides, it does not change the plane at all.

So, all the points on the plane will remain same. So, by this process if I multiply suppose from this I am multiplying c_1 to say c_2 to this is system one and suppose I have got the new system as I say $b_{11} x_1 + b_{12} x_2 + b x_3 = z_1$ and $b x_1 + b_{22} x_2 + b_{23} x_3 = z_2$. where I have multiplied c_1 to say first equation, but second equation I have kept as it is my previous one p_2, so now this system is a new system.

But both these systems again giving me the same plane what I have introduced in my first system, existence of solution depends whether this plane are parallel or not. If they are parallel, certainly

there will be no solution. If there is no parallel, then certainly they will meet, there will be a common point, so we will have a solution. Now, it is true for first case, even equivalent system case also, so solution will not change.

So, what we have learnt so far? We have learned what is system of m question in unknowns, what do you mean by solutions and what is the geometrical meaning of the solutions in our existence concept, what you have learned in 11th-12th's through the vector structures, through the geometrical structures and what is equivalent systems and we have seen that equivalent systems have the same solutions. Thank you.