Advanced Engineering Mathematics
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Lecture - 09
Cauchy's Integral Formula for the Derivatives of an Analytic Function

Hello friends. Welcome to my lecture on Cauchy integral formula for the derivatives of an analytic function. In this lecture, we shall discuss some important consequences of the Cauchy integral theorem. First, we discuss a corollary to the theorem which we had proved in our last lecture, we had there said that if fz is an analytic function in a simply connected domain D and z0 is any fixed point in D.

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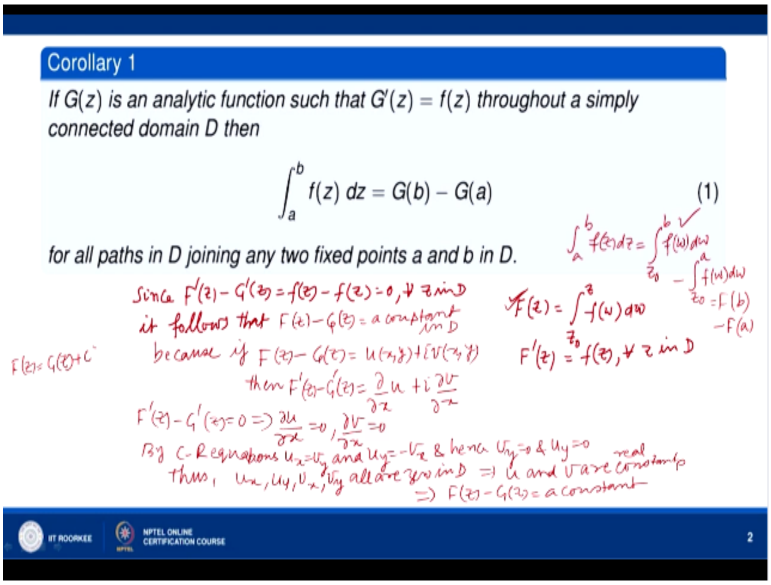

Then, if you define the function fz as integral over z0 to z fw dw. Then, we proved that fz is an analytic function in D and moreover that F prime z=fz for all z in D. So now let us prove a corollary to that theorem. If Gz is an analytic function such that G prime z is=fz throughout a simply connected domain D, then integral over a to b fz dz=Gb-Ga for all paths in D which join any two fixed points say a and b in D okay.

So now let us consider the function fz as integral over z0 to z fw dw where z0 is a fixed point in D and z is any other point in D. Then, the function fz is analytic and moreover that F prime z is=fz for all z in D. Now here we are given that G prime z is=fz, so since F prime z-G prime z is=fz-fz=0 for all z in D it follows that Fz-Gz is=to a constant in D is equal to a constant because if Fz-Gz is=say u x, y+i v x, y okay.

Then, we know that since Fz and Gz are analytic functions in D so Fz-Gz is an analytic function in D and therefore F prime z-G prime z that is the derivative of Fz-Gz is=partial derivative of u with respect to x+i times partial derivative of v with respect to x okay. Now F prime z-G prime z is=0, so F prime z-G prime z=0 then implies that ux is=0 and vx=0.

Now since u and v satisfy Cauchy-Riemann equations, so by CR equations ux is=vy and uy=-vx and there hence ux=0 gives vy=0 and uy is=0 because vx is 0. So thus ux, uy, vx, vy all are 0 in D okay and which implies that u and v are independent of x and y. So u and v are constants, real constants and therefore Fz-Gz is a complex constant. So Fz-Gz is=to a constant.

And therefore we can say that Fz is=some constant+Gz okay, where C is a complex constant. So this means that the indefinite integral Fz of the function fz is unique up to an additive constant C okay. Now hence therefore integral over a to b fz dz we may write as z0 to b fz dz which will be –integral over z0 to a fz d, fw dw should be minus okay so integral over a to b fz dz can be written as integral over z0 to b fw dw-integral over z0 to a fw dw and this quantity is=Fb okay.

This integral by our definition of Fz is Fb and this integral is Fa, so Fb-Fa. So this means that if you have a complex analytic function in a domain D and a and b are any two points in the domain D, you can take any curve joining the points a to b which lies completely inside the simply connected domain D, then integral over a to b fz dz can be evaluated by the indefinite integral of Fz.

You evaluate the indefinite integral at the upper limit that is Fb and then you evaluate fz at the lower limit that is Fa. So Fb-Fa will give you the value of the desired integral over a to b fz dz.

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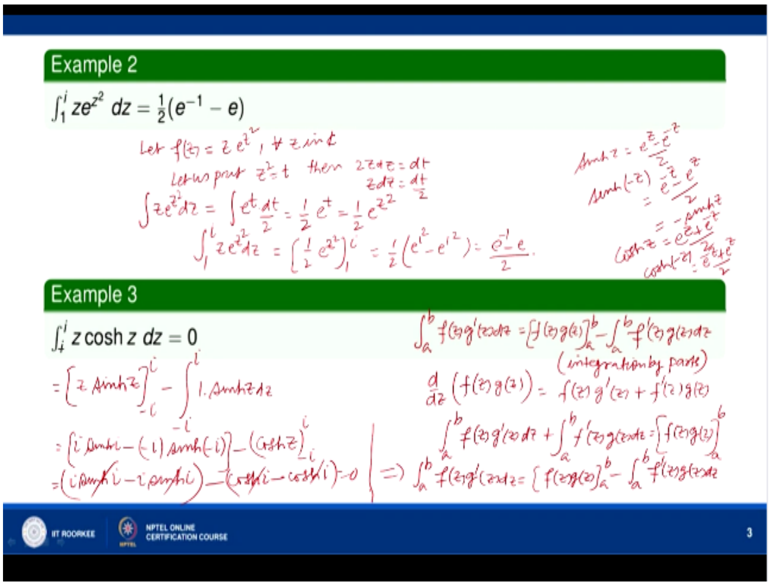

Now let us consider for example this problem integral over 1 to i ze to the power z square dz. So here we can see that let fz be=ze to the power z square. Then, obviously it is an analytic function for all z because it is infinitely differentiable for all z in the complex in C okay. So let us find indefinite integral of fz. To find indefinite integral of fz, let us put z square=t okay then 2z dz=dt or we can say z dz=dt/2.

So integral over z e to the power z square dz will be=integral over e to the power t*dt/2. So this will be 1/2 e to the power t or 1/2 e to the power z square okay. So this is your function fz and then integral over 1 to i ze to the power z square dz will be equal to 1/2 e to the power z square 1 to i. We are not writing a constant of integration here because constant of integration gets cancelled when you value at the definite integral.

So this is 1/2 times e to the power i square-e to the power 1 square iota square is -1, so we get e to the power -1-e/2. So we can evaluate the value of the integral of z e to the power z square over 1 to i. Now similarly let us consider the case z cos hyperbolic z, here we are evaluating the value of z cos hyperbolic z from –i to +i okay. So what we do is here we will use the formula for integration by parts.

And integration by parts in the complex plane for complex analytic functions is also valid. We have integral over a to b fz g dash z dz=fz*gz, this is the formula for integral by parts. We can easily show the validity of this formula. Here we are assuming that fz and gz are analytic functions in a simply connected domain D and a and b are any two fixed points in the domain

D and when we evaluate the integral from a to b, we can follow any curve that lies within the simply connected domain D.

So to prove this let us consider d/dz of fz*gz. Then, this will be equal to fz*g dash z+f dash z*gz okay or we can say when we integrate we will have integral over a to b so integral over a to b fz g dash z dz+integral over a to b f dash z*gz dz will be=integral of d/dz of fz dz, so fz dz which we have to evaluate over a to b. So this gives you this formula and integral over a to b fz g dash z gz=fz*gz a to b-integral over a to b f dash z*gz dz okay.

So let us use this formula here. So integral z cos hyperbolic z. Let us take z as first function that is you are taking f as fz as z, so we can write z*cos hyperbolic z integral of the second functions, integral of cos hyperbolic z is sin hyperbolic z. Then, we have here –i to +i- -i to +i and then derivative of z is 1, derivative of cos hyperbolic z integral is sin hyperbolic z. So this is equal to we have iota sin hyperbolic term and then we have okay so - -iota sin hyperbolic-iota okay.

And then here we have cos hyperbolic z integral of sin hyperbolic z is cos hyperbolic z and we have –iota and then iota okay. Now we know that let us recall that sin hyperbolic z is=e to the power z-e to the power –z/2 okay. So sin hyperbolic iota will be equal to or you can say let me write like this. If you place z/-z here, sin hyperbolic-z will be equal to what, e to the power-z-e to the power z/2.

So this means that this is equal to –sin hyperbolic z. So if sin hyperbolic z is e to the power z-e to the power –z/2 replacing z by –z we get sin hyperbolic-z=-sin hyperbolic z and similarly cos hyperbolic z if you write it will be e to the power z+e to the power –z/2. Here when you replace z/-z, there is no change. We get e to the power –z+e to the power z/2, so cos hyperbolic-z is cos hyperbolic z okay.

So here what we get here i sin iota sin hyperbolic iota minus minus becomes plus here and sin hyperbolic-i becomes –i sin hyperbolic/i so this is –i sin hyperbolic i okay. So this is this and then here what we have cos hyperbolic i-cos hyperbolic–i which is=cos hyperbolic i okay. So this cancels with this and this cancels with this and we get the value=0 okay. So the value of this definite integral is=0.

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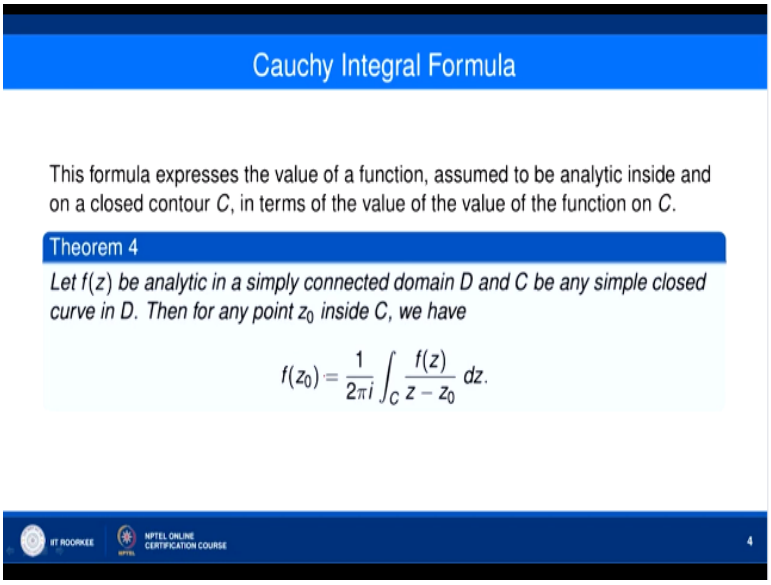

Now let us discuss Cauchy integral formula. Cauchy integral formula expresses the value of a function you can see this is Cauchy integral formula. It expresses the value of a function assume to be analytic inside and on a simple closed curve C in terms of the value of the function on C okay. So you can see z0 is a point which lies inside C, we are getting the value of f at z0, when we know the values of f on the curve C okay.

The curve C we are integrating along the curve C that means the curve C uses the values of the function, the function must be known on the curve C. So if the function is known on the curve C, we can find the values of the function at any point interior to C okay. So let fz be analytic in a simply connected domain D and C be any simple closed curve in D, then for any point z0 inside C, we have f z0=1/2 pi i integral over C fz/z-z0 dz, so let us prove this.

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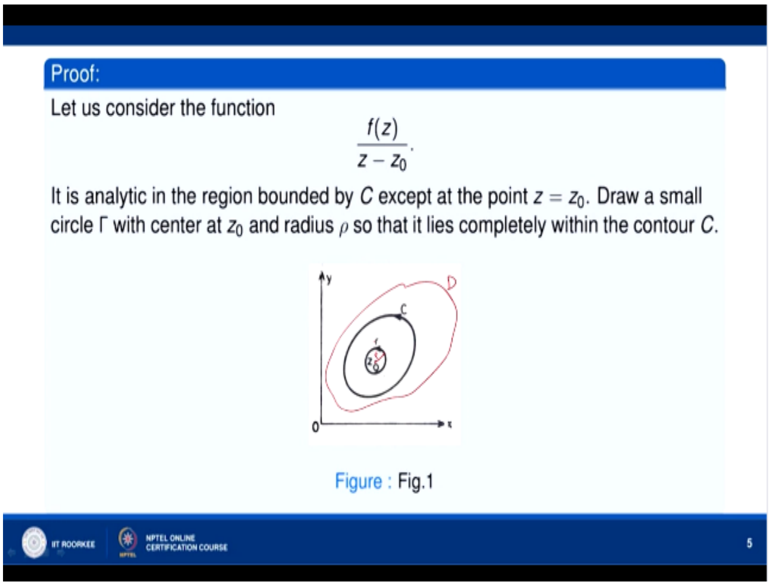

You can see let us take any simple domain okay. Let us say this is a simply connected domain okay. So inside the simply connected domain D we have a closed curve, simple closed curve C which encloses the point z0, z0 is a point lying inside C. Now let us look at the function fz/z-z0. The function fz/z-z0 is analytic everywhere inside and on the curve C except at the point z0 because fz is analytic in the whole domain D.

So in particular it is analytic inside and on C okay but because of z-z0 there in the denominator here, the function fz/z-z0 is analytic everywhere inside and on C except at the point z0. Now what do we do, with z0 as center let us draw a small circle okay of radius r and radius let us say rho okay, this radius is let us say rho okay such that it lies completely inside the simple closed curve C okay.

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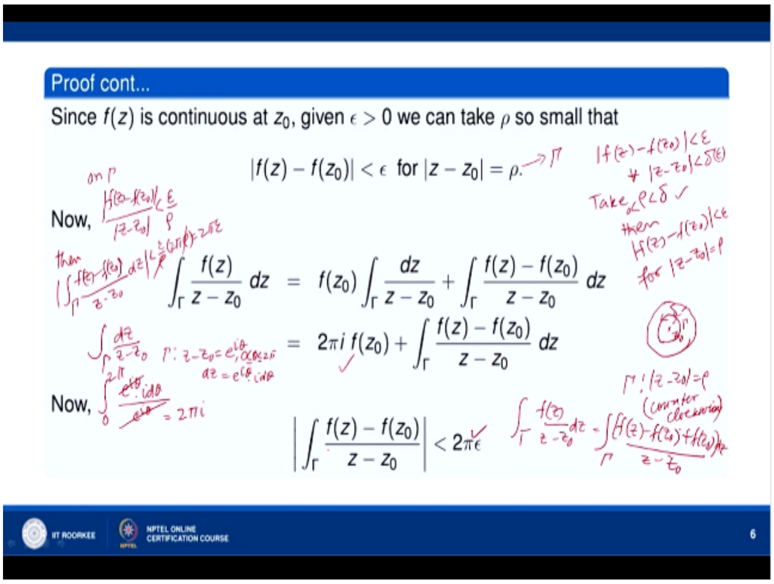

Then, what happens is that we have assumed that the function fz is analytic in D. So it is continuous everywhere in D and therefore it is continuous at the point z0. Now by continuity of the function at z0 what happens is that if you take any epsilon>0, you can find a delta such that mod of fz-f z0 is<epsilon for all z such that mod of z-z0 is<delta okay. This delta depends on epsilon.

Now what you do, the rho that you have taken earlier okay this rho, the radius of the circle gamma, this circle gamma okay with center at z0, it is radius we have taken as rho, this rho you can take to be<delta okay. So what happens is that then when you take this to be<delta, take rho to be<delta then mod of fz-f z0 will be<epsilon okay for the circle gamma okay that

is mod of z-z0 equal to because this is an open circular disk with center at z0 and radius delta okay.

This is an open circular disk and this delta okay and rho we have chosen to be<delta okay. So this circle gamma which we have drawn is this okay, this circle gamma okay. So this circle gamma because for all z which lie inside the circular disk mod of z-z0<delta, this inequality is valid, so this inequality will be valid on the points which lie on gamma okay, so mod of fz-f z0 will be<epsilon for mod of z-z0<rho.

We can choose rho to be so small okay that this happens. That means we can take rho to be<delta okay. Now what we have, let us consider the integral of fz/z-z0 on the circle gamma okay, on the circle gamma. Circle gamma is I repeat it is mod of z-z0=rho counter clockwise okay. So then I can split it into two parts fz-f z0+f z0 okay, integral over gamma fz/z-z0 dz I can write as integral over gamma fz-f z0+f z0 okay over z-z0.

So let us write it in two parts, integral over gamma fz-f z0/z-z0+integral over gamma f z0/z-z0 dz. Now f z0 is a constant because z0 is a fixed point. So f z0 being a constant will come outside the integral, so we have written it outside the integral. So f z0 integral over gamma dz/z-z0+integral over gamma fz-f z0/z-z0 dz. Now integral over gamma dz/z-z0, we have evaluated this kind of an integral earlier by using the parametric form of the curve circle.

So gamma is mod of z-z0=rho, so we can write gamma in the parametric form z-z0=e to the power i theta where 0 is<=theta<=2 pi okay. So then this will be changed to 0 to 2 pi, dz will be=e raised to the power i theta*i d theta. So e to the power i theta*i d theta/e to the power i theta. This will cancel and we will get 2 pi i okay. So value of the integral is 2 pi f z0, value of the integral is 2 pi I, so 2 pi i*f z0 we have and this integral remains the same.

Now we are going to show that since we have to prove that integral over gamma fz dz/z-z0 is=2 pi i f z0, we need to prove that the second integral on the right side is=0. So let us consider the second integral on the right. We are going to show that second integral on the right has absolute value<2 pi epsilon. Let us see how we get this. See we can prove it like this, on gamma what we have, this is circle gamma on gamma mod of fz-f z0 is<epsilon and mod of z-z0 is=rho okay.

So on gamma mod of fz-f z0/mod of z-z0 is<epsilon/rho okay, mod of z-z0 is=rho okay. So this is<epsilon/rho and the length of gamma is=2 pi rho because the center of the circle gamma is at z0 and radius is rho okay. So what we will have, then mod of integral over gamma fz-f z0/z-z0 dz is<epsilon/rho*length of gamma that is 2 pi rho okay, so rho will cancel and will get 2 pi epsilon.

So mod of integral over gamma fz-f z0/z-z0 is<2 pi rho. Now epsilon is arbitrary, we can choose it as small (()) (24:33) and therefore the value of this integral over gamma absolute value can be made arbitrarily small.

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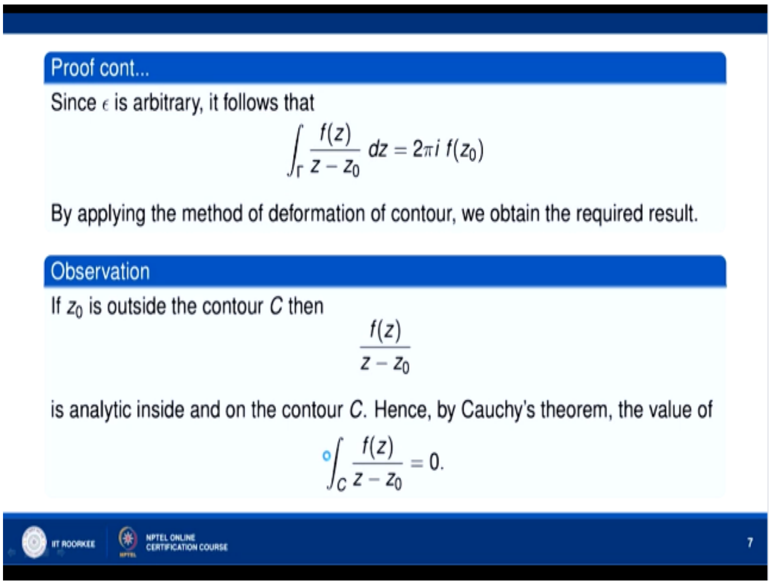

This means that integral over gamma fz-f z0 okay/z-z0 dz, this value is 0 okay and so integral over gamma fz/z-z0 dz is=2 pi i*f z0. Now let us use the method of deformation of contour okay.

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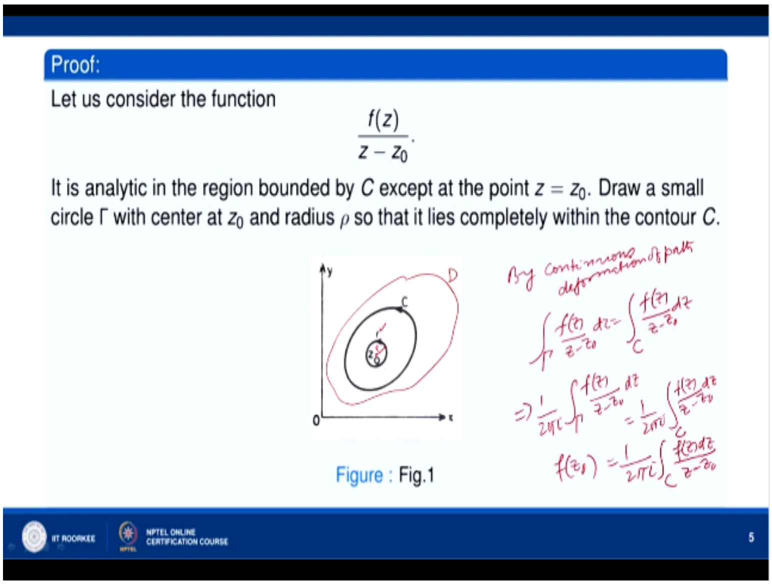

We can continuously deform this circle gamma and go to this circle over C simple closed curve. So by continuous deformation of this gamma okay we can achieve the integral over C=integral over gamma. So by continuous deformation of path because the function of z is analytic in D so we can do this integral over gamma fz/z-z0 dz is=integral over C fz/z-z0. We have seen this earlier that the integral domains unchanged okay.

So 1/2 pi i integral over gamma fz/z-z0 dz is=1/2 pi i integral over C fz/z-z0 dz. We had said that we can continuously deform the curve okay until if it does not have any similar point on the way. So there is no similar point in between gamma and C, so we can continuously deform gamma and get the integral over gamma fz/z-z0 dz=integral over C fz/z-z0 dz. Now this is equal to left side is f z0, so f z0 is equal to. So this is how we prove this Cauchy integral formula okay.

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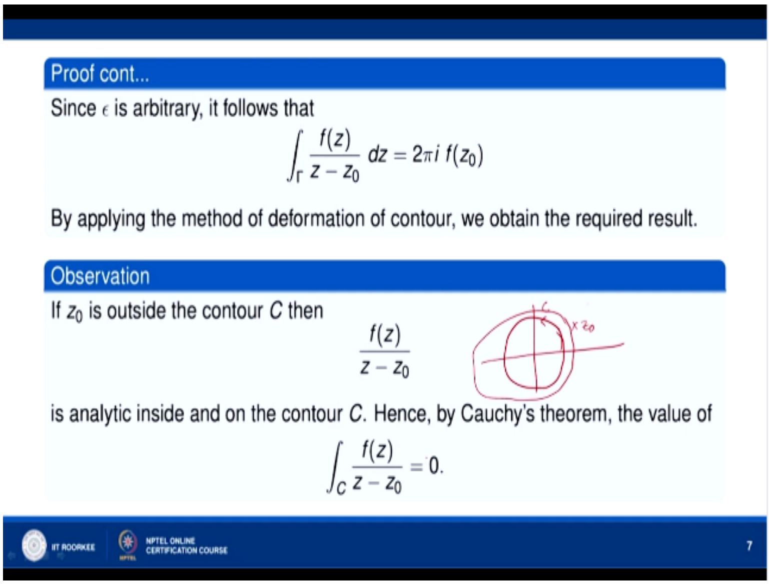

Now if it so happens that z0 lies outside the contour okay. Suppose you have a situation like this. Suppose fz okay fz is suppose we are evaluating the value of the integral of fz/z-z0 around the simple closed curve okay, around the simple closed curve and z0 point lies here okay we are integrating fz/z-z0 along this simple closed curve and the point z0 lies outside this simple closed curve C.

Then, by Cauchy integral theorem the value of the integral of fz/z-z0 along C will be=0 because you can consider a simply connected domain like this okay. So now fz is analytic in the simply connected domain D. C is any simple closed curve which lies in D okay and it is analytic everywhere inside and on C. So integral fz/z-z0 is analytic everywhere inside and on C, so its integral over C will be 0 okay by Cauchy theorem. So if z0 lies outside the simple closed curve C, then fz/z-z0 will have its integral along C=0.

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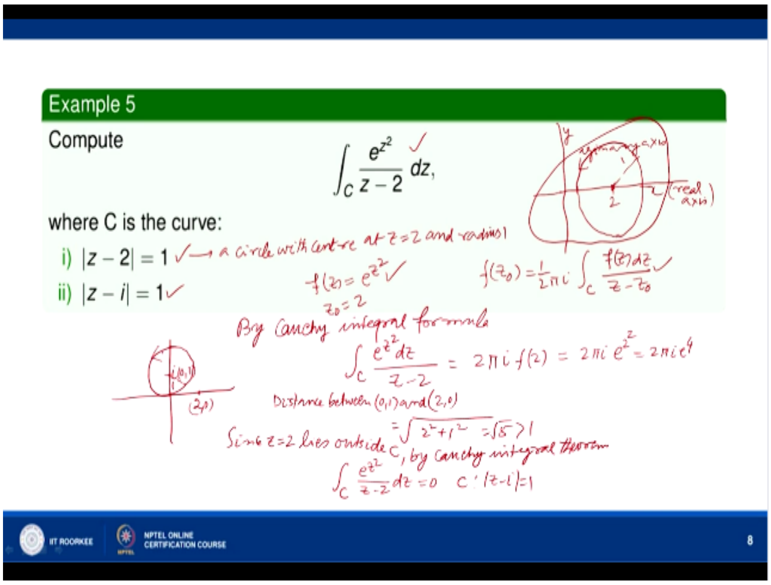

Now let us calculate this integral okay, integral over C e to the power z square/z-2 dz where C is the curve mod of that so let us take this curve first. You can see here, suppose this is complex z plane okay. So this is x axis I mean real axis and this is y imaginary axis okay. So here we are having the simple closed curve mod of z-2=1 which represents a circle this center at z=2 and radius 1 okay.

So let us draw this circle okay like this okay. So this is a circle with center at 2, z=2 and radius 1 okay. Now if you look at this function compare this function with the Cauchy integral formula, here you notice that Fz is=e to the power z square and z0 is=2. The Cauchy integral formula is f z0=1/2 pi i integral over C fz dz/z-z0. So let us compare this integral with this integral okay. You can see Fz=e to power z square and z0=2.

Now we are evaluating its integral, integral of this curve around the simple closed curve C. This is a simple closed curve because it is a circle and when we do not write the orientation of the curve we always mean it is anti-clockwise okay unless otherwise stated we will always mean that the orientation of the simple closed curve is in the positive direction okay. So this is the direction in which we have to integrate.

And now you can see z0=2, this z0=2 lies inside the simple closed curve that is the circle, it is at the center of the circle okay. Therefore, by Cauchy integral formula, therefore Cauchy integral formula we have to check the analyticity of the function fz, fz is e to the power z square e to power z square is analytic for all finite z, it is analytic in the whole complex plane so in particular it is analytic in any simply connected domain D which contains this simple closed curve C that is the circle mod of z-2=1.

You can take any simple closed curve okay. Let us take any simple closed curve okay, any simple closed curve which contains your circle C mod of z-2=1. So fz is=e power z is analytic there and therefore integral over C e to the power z square dz upon z-2=2 pi i*f z0 that is f at 2 z0 is 2 here. So fz is e power z square, so 2 pi i*e to the power 2 square okay that means 2 pi i e to the power 4 that is the value of the integral.

Now let us take the other problem mod of z-i=1. Here also we have circle. This time the center of the circle is that z=iota and radius is 1. So let us draw this circle, iota means the point z=i means 0, 1 point, 0, 1 point means iota is here okay. That is in complex z plane z=iota means in the Cartesian plane it is 0, 1 point okay. Now we take 0, 1 or z=i as center radius 1 and draw the circle.

So we get this circle okay and you can see z=2 is here okay z=2 this is z=2 this means the point 2, 0 okay. The point 2, 0 has its distance from the center 0, 1. How much is the distance? Distance between z=i that is 0, 1 and 2, 0 is=under root 2 square+1 square. So this is equal to square root 5 and square root 5 is >1 okay. So the radius of the circle which is 1 okay means that the circle does not enclose the point 2, 0 or z=2.

And therefore since z=2 lies outside C okay by Cauchy integral theorem, the integral over C e to the power z square over z-2 dz=0 okay. The C is mod of z-i=1 and we are integrating again in the anti-clockwise direction okay.

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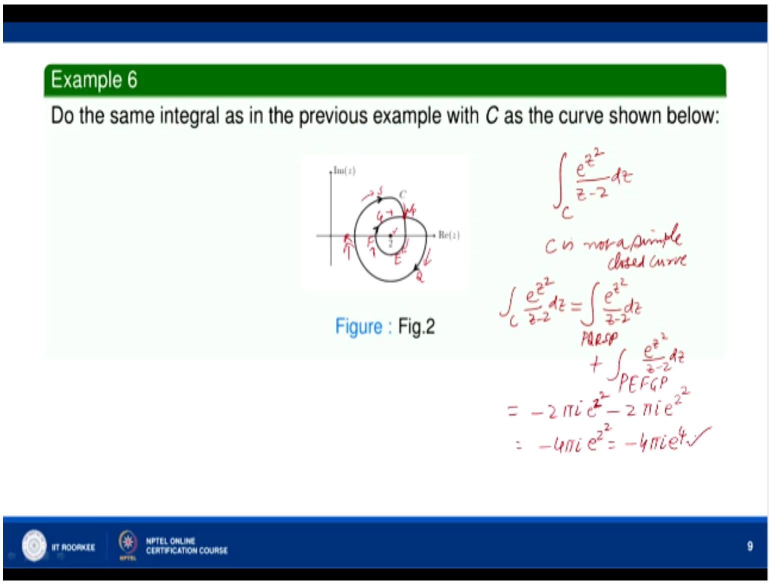

Now let us do one more example here. Let us do the same integral as in the previous example with C as the curve shown below. So we are again integrating e to power z square over z-2 but this time the curve C is different. Let us consider integral over C e to the power z square over z-2 dz okay. So we have to integrate e to power z square over z-2 along C but C is not a simple closed curve you can see.

Let us say suppose we start from here, then we move like this okay. Let us start form here we go this way, this way and then this way and then reach here and then we move this way. We start let me let us start like this. So let us start from here okay and then we go this way, then this way, then this, when we reach here we come this way okay. We go this way, this way and then reach here. So we are moving around the point z=2 twice okay.

Around the z=2 point we are moving twice but we are moving in the clockwise direction and then we see that it is not a simple closed curve. This is a closed curve but it is not a simple closed curve because it intersects itself at this point. This point it intersects itself, so C is not a simple closed curve. So how to apply the Cauchy integral formula here, what we will do, we will break it into two parts okay.

We will break it into two parts. First, we will integrate from along the suppose this point is say P okay PQR and then SP. So integral over C can be written as integral over C e to the power z square/z-2 dz can be written as integral over PQRSP e to the power z square/z-2 dz. Now you can see, from here if we go PQRSP it is a simple closed curve, it does not intersect itself and we have reached the initial point.

Now from here, we then move this way PEF okay, so PEFG let me write. So then we have the second part as integral over PEFGP okay. Now the curve PQRSP okay the function e to the power z square is analytic in the whole complex plane, e to power z square/z-2 is analytic everywhere in the complex plane except at the point 2 okay and PQRSP is the simple closed curve okay.

Integral over e power z square/z-2 dz, then we can find by the Cauchy integral formula but we notice here that we are moving along the simple closed curve PQRSP in the clockwise direction. So this will be=-2 pi i f z0, f z0 means e to the power 2 square, z0 is=2 and fz=e to power z square, so this is e to the power 2 square okay and then here again when we move from PEFGP, this is again a simple closed curve and we are taking a round about the point z=2.

So by Cauchy integral formula, this is also=-2 pi i e to the power 2 square okay. So what we get -4 pi i okay e to the power 2 square, e to the power 2 square this also same value we have, so -4 pi i e to the power 4 okay. Along both the parts of the curve C okay we are moving clockwise. So we will put a negative sign. So that is why we get the value this -4 pi i e to the power 4. So this is how we do this problem where we took C to be any closed curve which is not simple closed.

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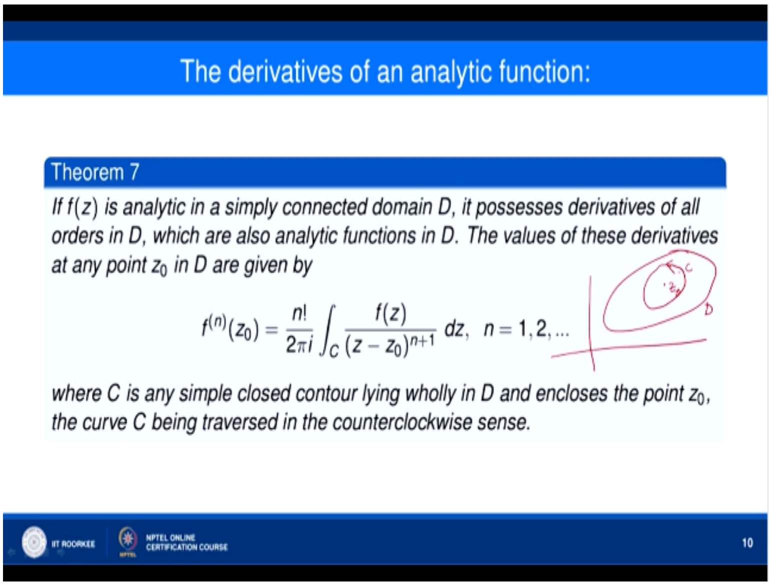

Now let us consider the theorem which tells us that if fz is an analytic function then all other derivatives of fz are also analytic. Now this result is very important if you see it, compare it with the corresponding result in real analysis, there if fz is a function by equal to fx is differentiable we cannot say anything about the distance of the second order derivative but here if the function fz is analytic okay if fz is differentiable in some domain, all order derivatives of fz are also differentiable, all order derivatives of fz are analytic.

That means all order derivatives of fz exists and are therefore analytic functions. So this way the complex analytic functions we have in a very simple manner. Then, as far as the derivatives are concerned compared to their counterparts in real analysis. So if fz is analytic in a simply connected domain D, it possesses derivatives of all orders in D. This result we had used when we proved the fact that the real and imaginary parts of an analytic function are harmonic functions.

There we had said that later on we will prove that if fz is analytic in D then all order derivatives of fz are also analytic and so fz is infinitely differentiable and therefore the real and imaginary parts u x, y and v x, y of fz are continuous okay second order partial derivatives of fz, second order partial derivatives of u x, y and v x, y are continuous and that is why the mixed derivatives u x, y and u y, x are equal okay.

And so we were able to prove that u is the solution of the Laplace equation. So this result we had used that time, now we are going to prove it. So if fz is analytic in a simply connected domain D, it possesses derivatives of all orders in D, which are also analytic functions in D. The values of these derivatives at any point z0 in D are given by fn z0 the nth derivative, fn z0=n factorial/2 pi i integral over C fz/z-z0 to the power n+1 dz where n takes values 1, 2 and so on.

Now C is any simple closed curve lying only in D and encloses the point z0, the curve C being traversed in the counterclockwise sense. So let me say suppose your domain is like this okay, this is your domain D and this is your point z0 okay. C is any simple closed curve okay which encloses z0 and lies completely inside D okay and we are moving along C in the counter clockwise direction okay.

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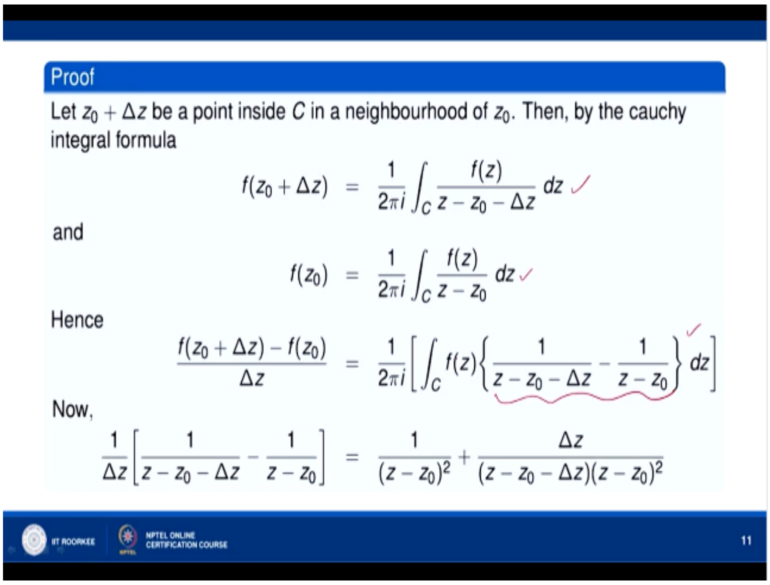

Now let us take the point z0+delta z inside C okay in a neighborhood of z0. So let us take a point z0+delta z here. Let us take a point z0+delta z in a neighborhood of z0 inside C okay. Then, we can apply Cauchy integral formula. Cauchy integral formula when we apply, we get the value of f at z0+delta z=1/2 pi i integral over C fz/z-z0-delta z dz okay and when we apply Cauchy integral formula for f at the point z0, we get f z0=1/2 pi i integral over C fz/z-z0 dz.

Now let us subtract the second equation from first equation, so f z0+delta z-f z0/delta z is=1/2 pi i integral over C fz and then 1/z-z0-delta z-1/z-z0 delta z. Now this expression we can write in this manner. You can verify that they are equal, 1/delta z we are multiplying this expression by 1/delta z, so 1/delta z 1/z-z0-delta z-1/z-z0 this expression inside the curly bracket multiplied by 1/z.

If delta z is=1/z-z0 whole square+delta z/z-z0-delta z*z-z0 whole square. We are writing this expression inside the curly bracket in a particular form because we want to prove the Cauchy integral formula this one for n=1 and when you put n=1 here, fn z0 you get fn z0 is 1 factorial/2 pi i that is 1/2 pi i integral over C fz/z-z0 whole square dz.

So the right side here we are trying to bring in the form integral over C fz/z-z0 whole square that is why we have written this as 1/z-z0 whole square. Now let us replace this value here okay.

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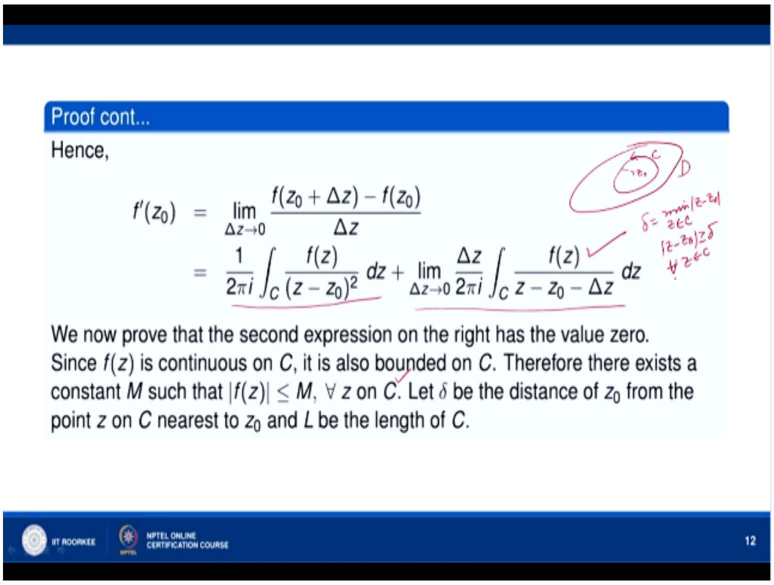

So when we put this value there what we get and take the limit as delta z tends to 0 okay. So when we take the limit as delta z tends to 0, this gives you f prime z0, so f prime z0 is=f z0+delta z-f z0/delta z which is equal to 1/2 pi i integral over C fz/z-z0 whole square dz+limit delta z tends to 0 delta z over 2 pi i. So this is limit delta z tends to 0 delta z/2 pi i integral over C fz/z-z0-delta z dz okay.

Now f prime z0 is=this, we want to prove that f prime z0 is=this value, this means we have to show that this value is 0 okay. So let us now show that the second expression on the right side. This is second expression on the right side, it has value 0. So since fz is analytic inside in D, it is analytic inside and on C okay, in particular fz is continuous on C. So since fz is continuous on C, it is a bounded quantity on C, it is absolute value, mod of fz is<=some constant M on C okay.

So therefore we can find a constant M such that mod of fz is<=M for all z on C. Now let us say that delta be the distance of z0 from the nearest point z on C okay. So what we have, suppose this is your domain D and this is your simple closed curve okay. See z0 is some point here okay, so then we are taking delta to be the minimum value of mod of z-z0 okay where z belongs to C okay.

So delta is with distance of z0 from the point z on C which is nearest to z0 okay, so minimum of mod of z-z0 z belonging to C and L be the length of C. This means that mod of z-z0 will be >=delta for all z belonging to C okay and L is the length of C okay.

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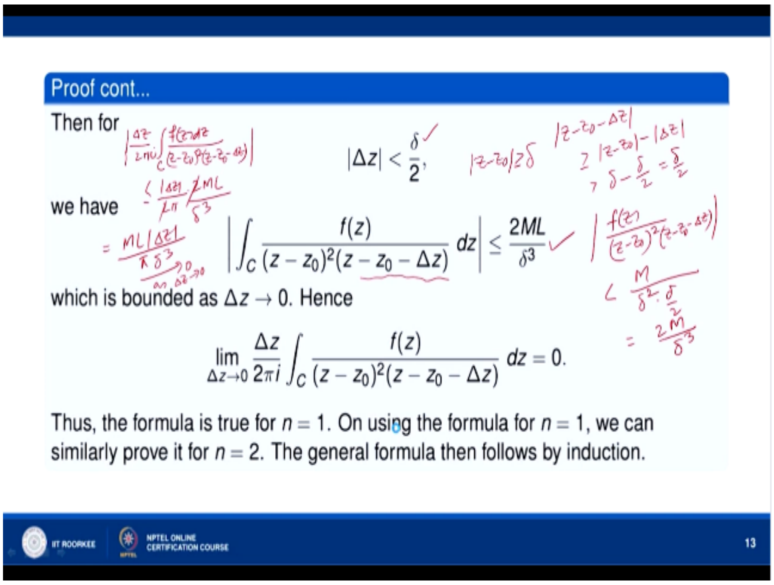

Then, what we have, let us choose about delta z to be smaller than, mod of z to be smaller than this delta/2, this delta is the minimum of the minimum the shortest length between the distance of the point z0 from the nearest point on the boundary of C okay. So mod of delta z let us take to be<delta/2 then what we will get, mod of z-z0-delta z okay. Let us calculate this. Mod of z-z0-delta z is>=mod of z-z0-mod of delta z.

Now z belongs to z is moving along the curve C so mod of z-z0 is>=delta and mod of delta z is<delta/2, so this is delta/2 here. You can put a negative sign here or you can put a greater than sign here. So this is>delta-delta/2, so this is delta/2. So what we can say, mod of fz/z-z0 whole square*z-z0-delta z mod of this is<=mod of fz is<=M on C. Mod of z-z0 is>=delta okay.

So this is delta square and then we have mod of z-z0-delta z which is>delta/2 so this is into delta/2. So let us write it as less than here okay, so then this is equal to 2M/delta cube okay and L is the length of C. So we can write here, maximum value of or estimate of mod of fz/z-z0 whole square*z-z0-delta z modulus of this quantity integrant is<=2M/delta cube and length of C is L, so we have 2ML/delta cube okay.

Now from this what do we notice, we have to calculate, we have to show this goes to 0 okay. So let us consider mod of delta z/2 pi i okay integral over C fz dz/z-z0 whole square*z-z0 delta z okay mod of this, this is<=mod of delta z/2 pi i oh sorry mod we have used, so this will be mod of delta z/2 pi*2ML/delta cube okay. So this cancels with this and we get ML times mod of delta z/pi delta cube.

And this quantity goes to 0 as delta goes to 0, so as delta goes to 0 okay this quantity goes to 0 and so limit delta tends to 0 delta z/2 pi i fz/z-z0 whole square*z-z0-delta z dz is=0. So this quantity becomes 0 and what we have is then so this formula okay so then we have f prime z0=this quantity okay. So this means that the formula holds for n=1 okay. Following this proof okay, we can similarly prove for n=2 okay.

So on using the formula for n=1 we can similarly prove for n=2. We have earlier used this value that is f z0 equal to this, f z0+delta z=this, now you use this formula for f prime z okay. So write f prime z0+delta z and f prime z0 and then divide f prime z0/delta z-f prime z0/delta z, consider over delta z and take delta z tends to 0 we get f double prime z0. So we can get the similar same proof we have for n=2 okay.

So on using the formula for n=1 we can similarly prove for n=2 and then general formula follows by induction okay.

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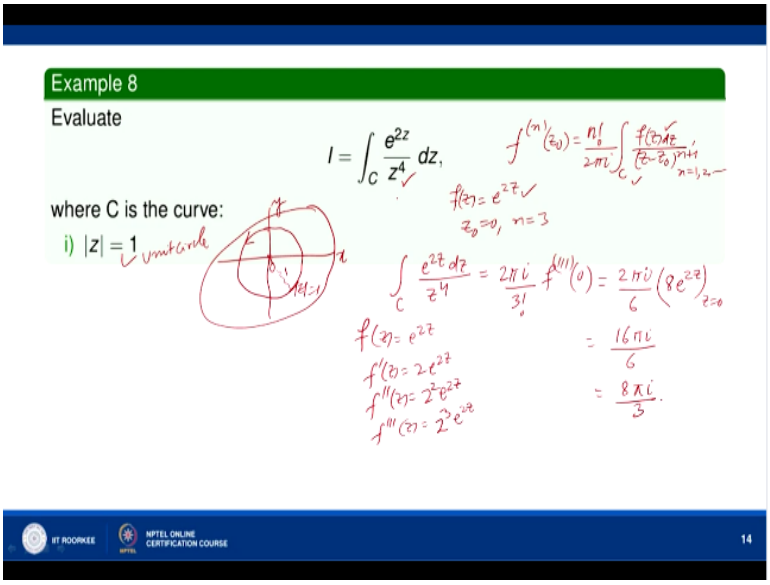

Now let us consider this I=integral e to power 2z/z to the power 4. So let us compare it with this formula fn z0=n factorial/2 pi i integral over C fz dz/z-z0 to the power n+1 where n takes values 1, 2, 3, and so on okay. So here if you compare this integral with this integral, we have fz=e to the power 2z okay and z0 is=0 and n is=3 alright. Now let us see the curve along which we have to integrate.

The curve C is mod of z=1 this means that the center of the circle is at 0, z=0 and radius is 1, so we are considering unit circle mod z=1 this center is z=0 and radius is 1 okay and this is your z plane okay real axis, imaginary axis. So this is unit circle okay unit circle. Now fz=e to power z is analytic for all z okay and so it is analytic in any simply connected domain you can consider which contains mod z=1 okay.

Mod z=1 is a simple closed curve okay. It is a circle, so it is a simple closed curve. So we are moving along the simple closed curve in the counter clockwise direction and therefore if you follow this formula we have integral over C e to the power 2z/z to the power 4 dz will be equal to 2 pi i, take n=3 so 3 factorial and then f triple prime okay z0, z0 is 0 okay. Now fz is e to the power 2z, so you can find its derivative f prime z it is 2 times e to the power 2z.

Then, f double prime z, so 2 square e to the power 2z and then f triple prime, so it is 2 cube e to the power 2z okay. So this will be equal to 2 pi i/6, 2 to the power 3 means 8e to the power 2z at z=0. At z=0, e to the power 2z is 1, so we have 16 pi i/6 which is=8 pi i/3 okay. So this is the value of the given integral okay.

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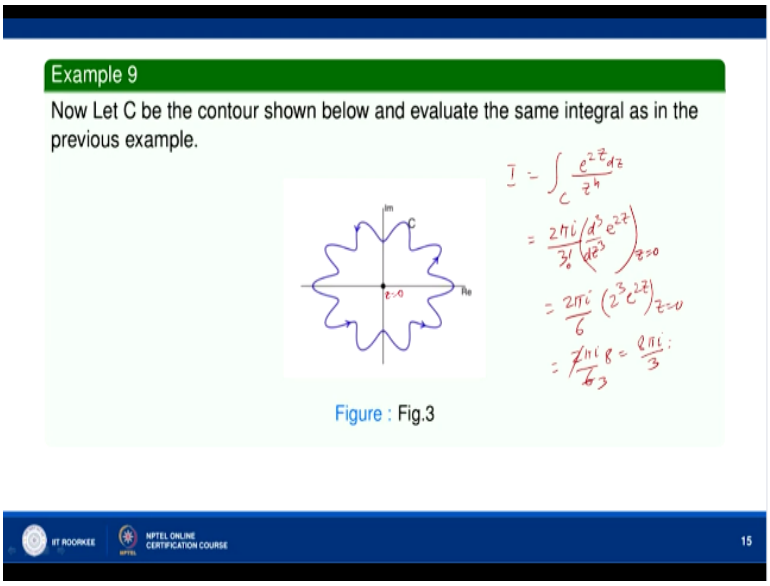

Now let C be the contour shown below and evaluate the same integral as in the previous example. Now you can see here, this is z=0 okay and we are getting I=integral over C e to the power 2z/z to the power 4, so I=integral over C e to the power 2z/z to the power 4 dz okay. C is a simple closed curve here okay. We are going along the simple closed curve C okay. The function e to power 2z is analytic in the whole complex plane.

So we can apply the formula and this is 2 pi i/3 factorial okay and then d cube /dz cube of e to the power 2z at z=0. So it will be same, we get 2 pi i/6 and we get 2 cube e to the power 2z at z=0. So this is 2 pi i/6 and we had 8 okay, e power 0 is 1, so we get 8 pi i/3 okay. So we get the same value here because the curve C is a simple closed curve which encloses z=0 and fz=e to power 2z as we know it is analytic in the whole complex planes.

So this is how we evaluate this integral. With this I would like to end my lecture. Thank you very much for your attention.