Advanced Engineering Mathematics

Prof. P.N. Agrawal
Department of Mathematics
Indian Institute of Technology - Roorkee

Lecture – 34
Review of Z-Transforms - II

Hello friends. Welcome to my lecture on review of Z-transforms. The second lecture on review of Z-transforms.

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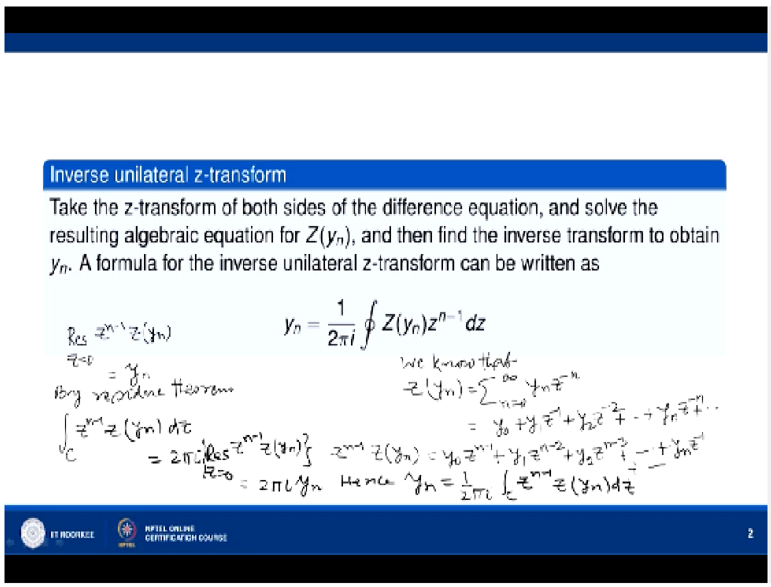

We know that the Z-transform are the discrete analog of Laplace transforms and they are used to solve difference equations. So when we are given a difference equation, we take the Z-transform of both sides of the difference equation and then solve the resulting algebraic equation for zyn. And then find the inverse Z-transform to find the sequence yn. A formula for inverse unilateral Z-transforms can be written as yn=1/2pi i integral over c Zynz to the power n-1dz.

Now let us see how this formula has been derived. We know that Zyn=sigma n=0 to infinity ynz to the power -n. So this is equal to y0+y1z to the power -1 y2z to the power -2 and so on, ynz to the power -n and so on. Now when you multiply this equation by z to the power n-1, what you get? z to the power n-1*Zyn=y0z to the power n-1+y1z to the power n-2 y2*z to the power n-3 and so on, we get yn*z to the power -1 and so on.

The right hand side here is the Laurent series. In the Laurent series, the coefficient of 1/z gives the residue of z to the power n-1Zyn. So residue of z to the power n-1*Zyn at z=0 is equal to yn, okay. Now by residue theorem, integral over c where c lies in the region of convergence of this Z-transform, okay, Z of yn, in the region of convergence of Zyn, if c is a simple closed curve, okay, then integral over c Z to the power n-1Zyndz=2pi i*residue at z=0 of z to the power n-1Zyn, okay.

So this is equal to 2pi i*yn, okay. And hence, yn=1/2pi I integral over cz to the power n-1Zyndz, where c lies in the region of convergence of Zyn. So this is inversion formula for the Z-transform using calculus of residues.

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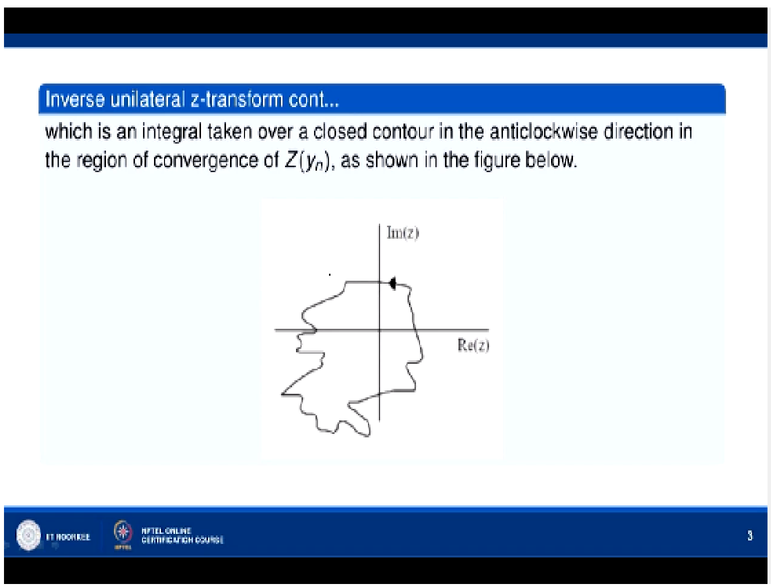

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Now here you can see, this is the region of convergence, okay. In the region of convergence, this is the simple closed curve. So which is an integral taken over a closed contour in the anticlockwise direction, in the region of convergence of Zyn as shown here in this figure.

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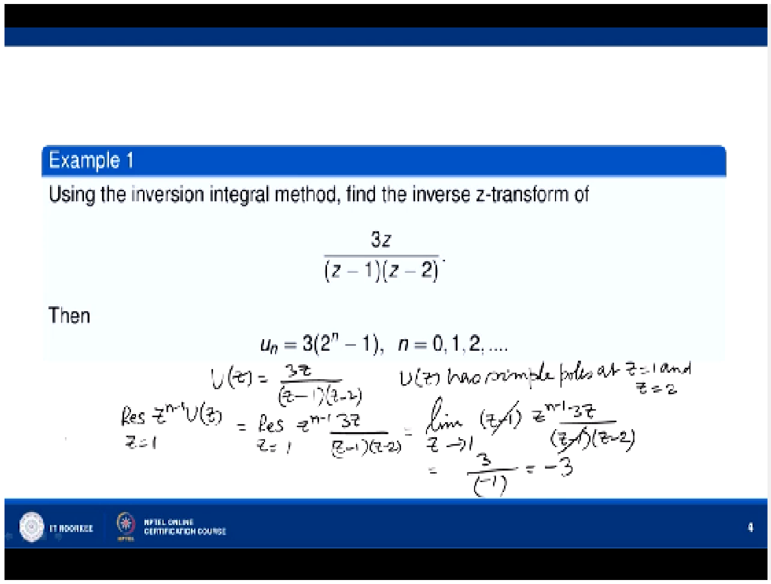

Now let us use the inversion integral method to find the inverse ztr= of 3z/z-1 z-2. We can see that here say Yz=3z or here let us take instead of Yz, let us take uz. So uz=3z/z-1*z-2, okay. Now uz has simple poles at z=1 and z=2, okay. So the region of convergence here will be mod z>2, okay. Now let us see we have the residue, let us find residue of z to the power n-1*uz at z=1. So this is a residue of z to the power n-1*3z/z-1 z-2 which is equal to limit z tends to 1 z-1*z to the power n-1 3z/z-1 z-2 because uz has simple pole at z=1.

So this z-1 will cancel and when z tends to 1, you get 1 to the power n-1*3*1, so we get 3/1-2, so -1. So it is -3, okay.

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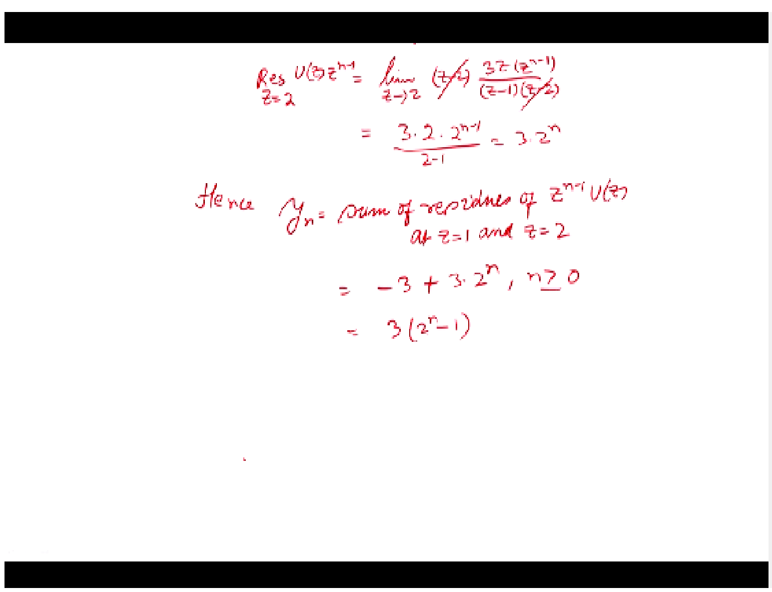

And then let us find the residue of uz*z to the power n-1 at z=2. So this is limit z tends to 2, z-2*, uz is 3z/z-1 z-2*z to the power n-1, okay. So z-2 will cancel and when z tends to 2, you get 3*2*2 raise to the power n-1/2-1. So this is 3*2 to the power n, okay. Now so we know that, we have found the residue of z to the power n-1uz. At z=1, it is -3 and we have found the residue of z to the power n-1*uz at z=2, okay.

Now let us go to the theorem. Theorem says that when yn=sum of residues, okay. So sum of residues=, hence yn=sum of residues of z to the power n-1*uz at z=1 and z=2, okay. So this is equal to, we have -3, so -3+3*2 raise to the power n, okay. So yn is, this is for n greater than or equal to 0. So we get this sequence 3*2 to the power n-3, 3*2 to the power n-3. 3*2 to the power n-1, okay. So this is how we get the inverse Z-transform here using the inversion integral method.

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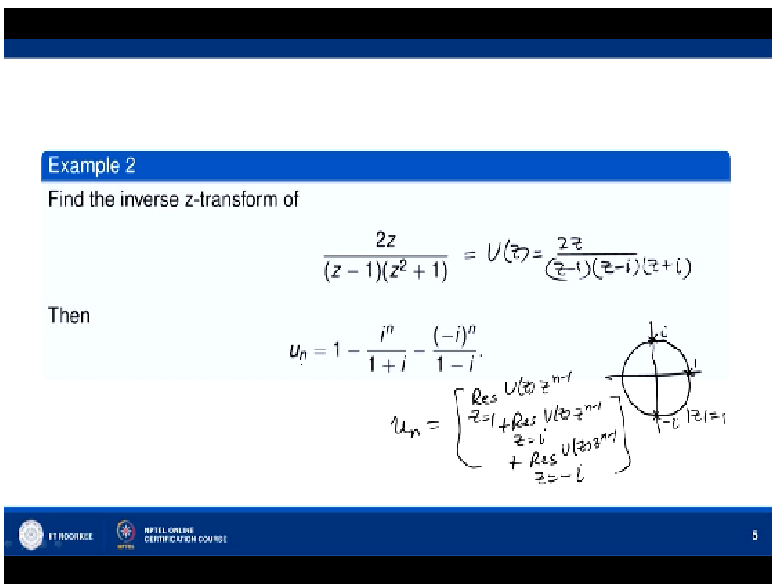

Let us take 1 more example here. Here you can see that this is uz, okay. So uz, this is uz, okay. And uz has 3 poles, okay. z=1, z=i and z=-i, okay. So we find similarly the poles lie at z=1, z=i, z=-i, okay. All these poles lie on the unit circle, on this unit circle, okay. Mod z=1. This is the z=1 here, z=i here and z=-i here, okay. So region of convergence here for this Z-transform will be mod of z>1, okay.

And we will find these residues, residue at z=1 of uz*z to the power n-1+residue of uz*z to the power n-1 at z=i and then residue of uz*z to the power n-1 at z=-i. So then un will be the sum of the residues of uz*z to the power n-1 at z=1, z=i and z=-i. So we know we have seen just now how to find the residue at a simple pole. Suppose we want to find the residue at z=1, then you multiply uz*z to the power n-1/z-1, z-1 will cancel with the z-1 in the denominator of uz and then you take the limit as z tends to 1.

So we get the residue at z=1. And similarly, we can find the residue at z=i and -i. Then the sum of the residues at 1, i and -i will give you the desired inverse Z-transform which is the sequence un.

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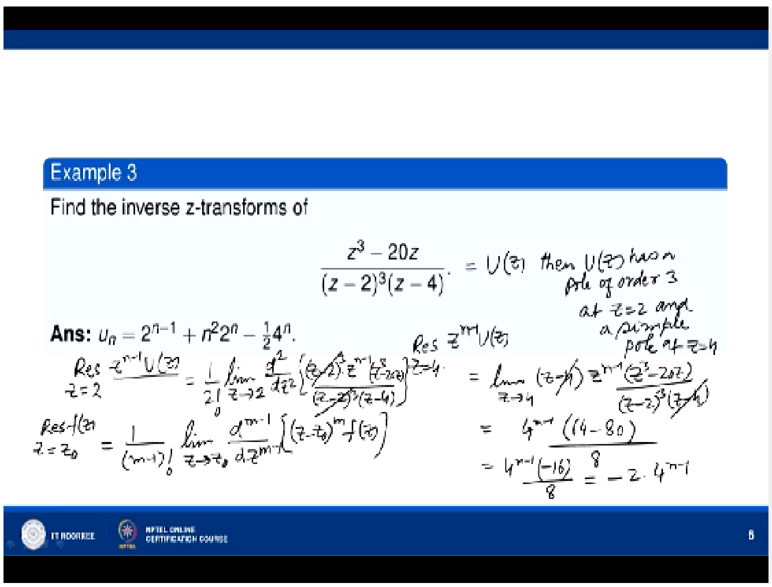

Now let us go to find the inverse Z-transform of z cube-20z/z-2 whole cube*z-4. Now here you can see, we have a pole of order 3 at z=2 in the denominator and a pole of order 1 at z=4. So if this is uz, so this is equal to let us say this is uz, okay. Then uz has a pole of order 3 at z=2 and a simple pole at z=4, okay. So we will find the residue of z to the power n-1*uz, okay, at z=4 which is a simple pole.

This will be limit z tends to 4, z-4*z to the power n-1*uz. uz is z cube-20z/z-2 whole cube*z-4. So you can see z-4 will cancel. Now we can find the limit as z tends to 4. So 4 to the power n-1 and then we have 4 cube which is 64, -80/4-2 cube, so that is 2 to the power 3 which is 8, okay. So 4 to the power n-1, here we have -16/8. So this is -2*4 to the power n-1. So we have found the residue at the simple pole z=4.

Now we will have to find the residue at z=2 of z to the power n-1uz. So here we have pole of order 3. So we know that the residue in case of a pole of order more than 1 is found by using this formula. Residue of fz, let we have this formula at z=suppose z0. The formula is 1/m-1 factorial limit z tends to z0 dm-1/dzm-1 z-z0 to the power m*fz. This is the formula for the residue in case fz has a pole of order m at z=z0.

So here we have z to the power n-1*uz which is the function having pole of order 3 at z=2. So this will be equal to 1/, 3-1 factorial that is 2 factorial limit z tends to 2, okay, z- is 2 here. Then d square/dz square and then we have z-2 raise to the power 3*z to the power n-1*uz which is z cube-20z/z-2 whole cube*z-4, okay. So this z-2 whole cube will cancel. Now we have to differentiate z to the power n-1*z cube-20z/z-4 twice with respect to z, take the limit as z tends to 2 and then multiply by 1/2 factorial that is 1/2, will get the residue at z=2.

And then we aid to this residue, the residue which we have already found at z=4, we will get the required answer.

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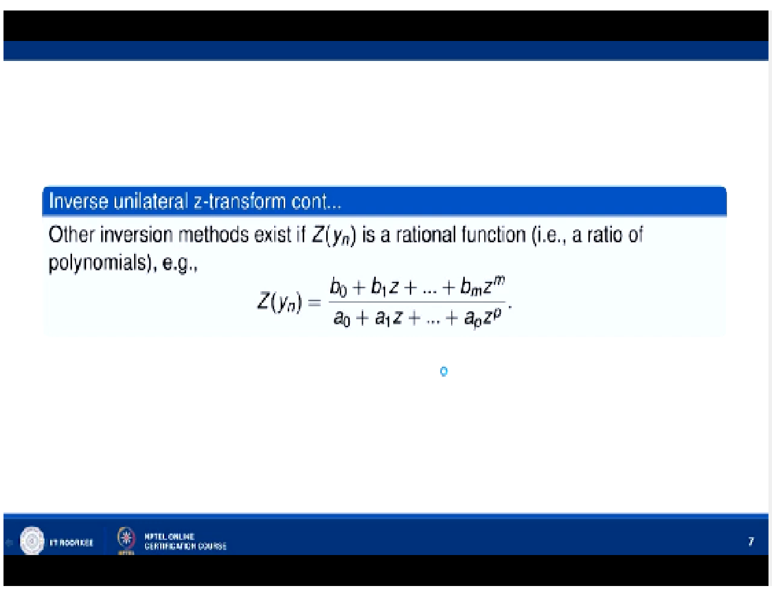

So inverse unilateral Z-transform. Now there are other inversion methods. Let us say if Zyn is a rational function, okay. Then how we will find the inverse Z-transform? So if Zyn is a rational function, that is it is a ratio of 2 polynomials, say b0+b1z and so on bmz to the power m/a0+a1z and so on apz to the power p.

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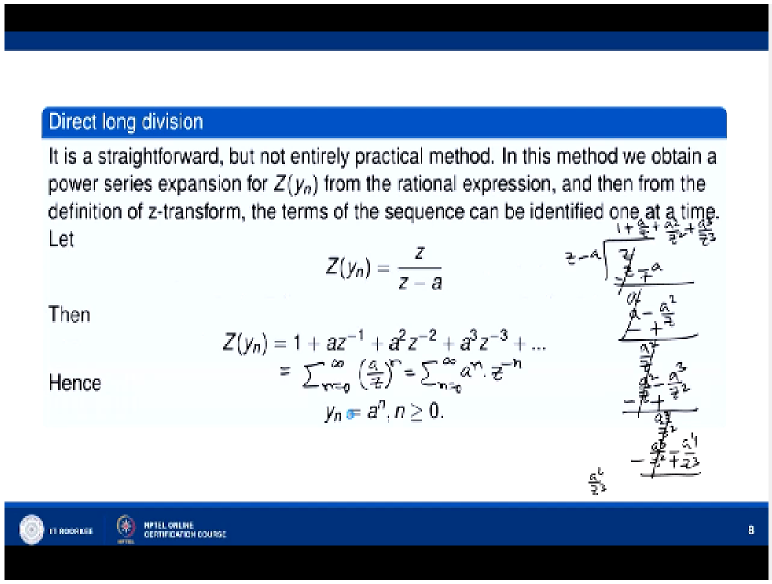

Then one method is that direct long division method, okay. It is a straightforward but not entirely practical method. In this method, we obtain a power series expansion for Zyn from the rational expression. And then from the definition of Z-transform, the terms of the sequence can be identified one at a time. let us say Zyn is z/z-a. So what do you do? This is your z, you are dividing z/z-a.

So it will go 1 times, so z-a, so we will - + here z, so this will be a. Now then we will take, we have to make this a 0. So we multiply a by a/z. So a/z when you multiply to z-a you get a-a square/z, okay. So this will cancel. Now you have a square/z, okay. So in order to cancel this a square/z, we have to multiply by a square/z square. So we get a square/z-a square/z square. This cancels and we get now a square/z square and to cancel a square/z square, now we multiply a square/z, here this will be a cube/z square.

So now we have to multiply by, a cube/z cube. Then this will be a cube/z square-a4/z cube. We get a4/z cube and so on, okay. So we get the series 1+a/z+a square/z square+a cube/z cube and so on, continuing this division method, okay. Now you can see, this is 1+a/z+a square/z square+any, so this is nothing but sigma n=0 to infinity a/z raise to the power n, okay. Or sigma n=0 to infinity a to the power n*z to the power -n.

So this is the Z-transform of a to the power n. So yn=a to the power n. So it is not practical method because we can do this only for a finite number of steps and it is sometimes very complicated.

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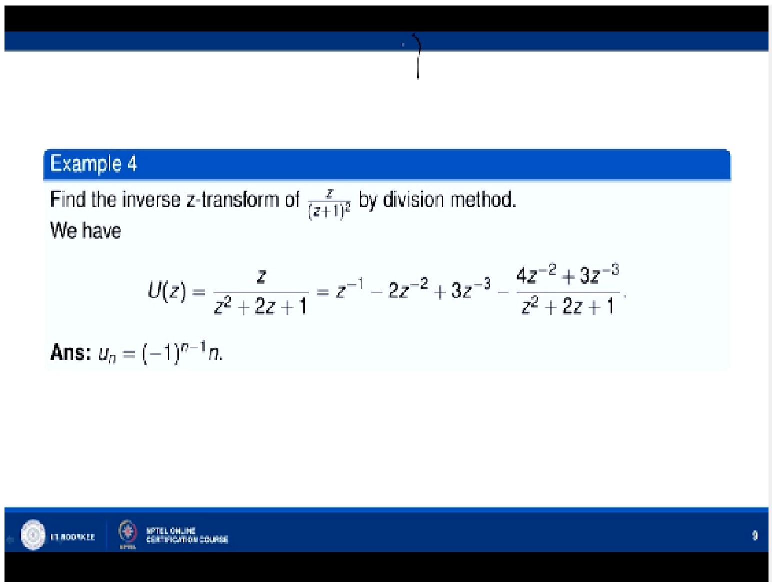

Now let us find the inverse Z-transform of z/z+1 whole square say for example. So z/z+1 whole square is z/z square+2z+1. Let us do the same thing as we did in the previous example.

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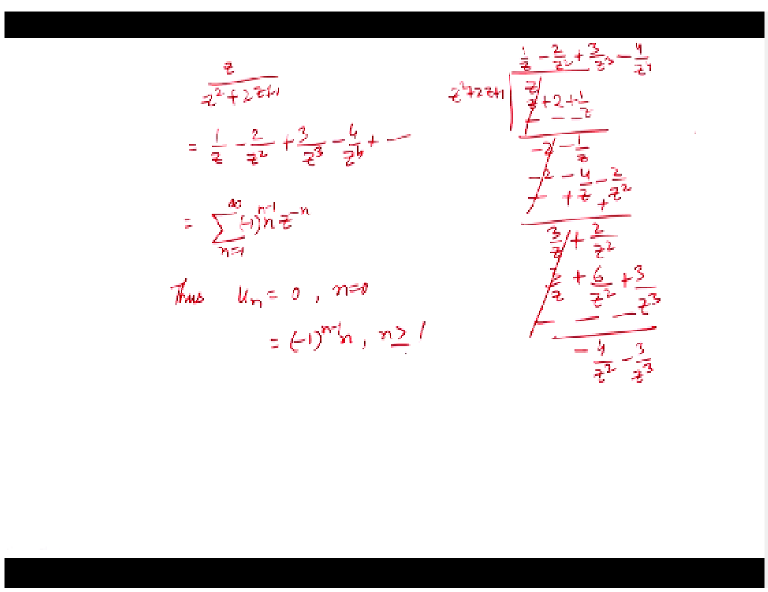

So we have z/z square+2z+1, okay. So we have z/z square+2z+1, okay. So we will multiply this z square/1/z so that we have here z square*1/z as z, then 2z*1/z is 2, then +1/z. When we subtract, we get -2-1/z, okay. Now we want to cancel -2. So we multiply by -2/z square. So we get -x, then -2/z square*2z, so we get -4z/z square, so -4/z, okay, -4/z -2/z square. So this cancels, 4/z-1/z is 3/z and then we get +2/z square.

So now in order to cancel 3/z, we multiply by 3/z cube. So we get 3/z here, then we get 2z*3, that is 6z/z cube, so 6/z square. Then we get 3/z cube. So this is -4/z square-3/z cube. Now in order to cancel -4/z square, we will have to multiply by, z square we have to multiply by 4/z to the power 4, -4/z to the power 4, okay. So we can see this is nothing but 1/z-2/z square, okay, 3/z cube-4/z to the power 4 and so on.

This we can write that sigma n=1 to infinity z to the power -n -1 to the power n-1*n, okay. So if you take n=1, you get z to the power -1. When you take n=2, you get -2*z to the power -2 and then you take n=3, you get 3/z cube and so on. So thus we have, thus un here is equal to 0 when n=0 and it is -1 to the power n-1*n when n is greater than or equal to 1. So you can see here, this un is 0 when n=0 and when n is greater than or equal to 1, un is -1 to the power n-1.

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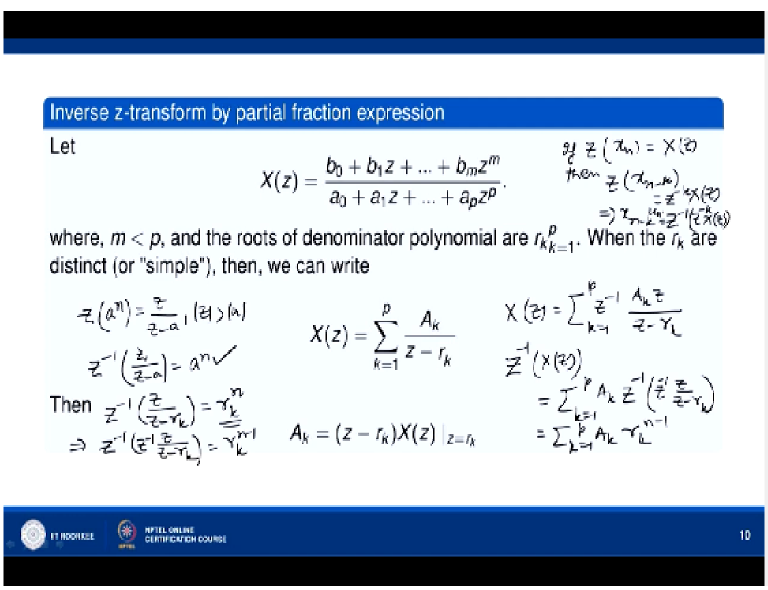

Now inverse Z-transform by partial fraction, okay. So Xz is b0+b1z+ and so on bmz to the power m/a0+a1z+apz to the power p where m<p. And the roots of denominator polynomial are rkk=1 to p. When the rk's are distinct and simple, then we can write Xz as sigma k=1 to pAk/z-rk, okay.

So since rk's are distinct, okay, the Ak can be found out directly. Ak will be equal to z-rk*Xz at z=rk, okay.

So we know the value of Ak's now. Now what you do in order to find the inverse Z-transform, Xz, inverse Z-transform of Xz, what we do is? We have to bring it to the standard form, the right hand side to be in the standard form. We know that Z-transform of a to the power n=z/z-a where mod of z>mod of a, okay. So z inverse of z/z-a will be equal to a to the power n, okay. Now so what we will do? We will write Xz as sigma k=1 to p z inverse*Akz/z-zk, okay.

And then apply the inverse Z-transform. z inverse of z/z-rk=rk to the power n by using this formula, okay. Now let us apply the delay property. The delay property says that if Z-transform of xn=Xz, then Z-transform of xn-k, nx-k=z to the power -kXz, okay. So inverse Z-transform when you take, inverse Z-transform of Xz, okay. This will give you the sequence xn. So this will be sigma k=1 to p inverse Z-transform Ak we can write, then inverse Z-transform of z inverse*z/z-zk. Let us write z/z-zk here z-rk, okay.

So this will be sigma k=1 to pAk*z, now we apply this property. You see, so this is that z inverse, if you take the inverse Z-transform, this will imply that xn-k=z inverse z to the power -kXz, okay. So take k=1, so this will be z/z-rk has transformed, inverse Z-transform has rk to the power n. And when we multiply by z to the power -1 here, n is replaced by n-1 here. So we have rk to the power n-1, okay. So using this delay property, this one, okay, Z-transform of z inverse*z/z-rk=, so rk to the power n-1, okay.

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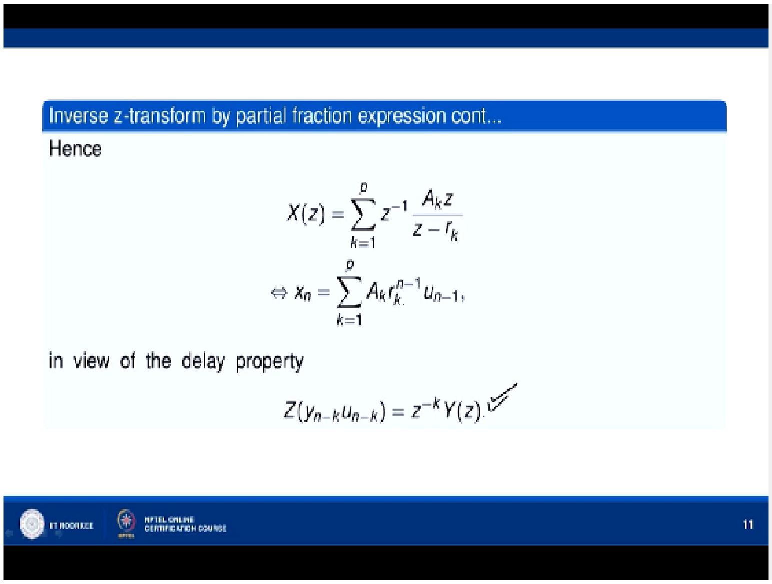

So we get here Akrk to the power n-1yn-1 because here you have, you can multiply this un also. So un-1, okay. This is xn-k*un, okay. So this will become actually un-k sorry. So this will be rk to the power n-1yn-1 and k=1 to p in view of this property. So this property we write here, the property that I have written here xn-kun-k=z inverse, yes, so this property we apply.

So z of yn-kun-k z to the power -kYz, using this property, this inverse Z-transform of this will give you xn=sigma k=1 to pAk and z inverse of z/z-k, okay, will give you rk to the power n-1un-1. So this is how we can find the Z-transform.

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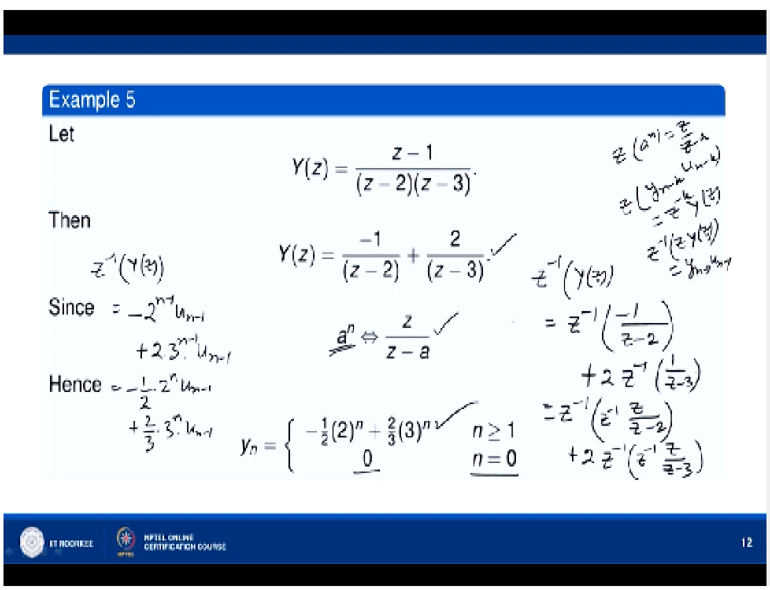

Now let us say for example Yz=z-1/z-2 z-3. When you break it into partial fraction, you get Yz=-

1/z-2+2/z-3. And you know that a to the power goes to z/z-a when we take the Z-transform, okay. So by the inverse Z-transform of Yz will be z inverse of Yz as we have just now done the, by using the delay property. So z inverse of -1/z-2, okay. +2*z inverse of 1/z-3, okay. So we can write it as z inverse of, - we can write outside, okay.

Then z inverse*z/z-2. and here we have 2*z inverse z inverse of z/z-3. Now Z-transform of a to the power n is z/z-a, okay. Then Z-transform of yn-kun-k=z to the power -kYz. Let us apply this property. So then k=1 here, okay. So z inverse of z*Yz will be equal to yn-1un-1, okay. So here 2 to the power n is the, Z-transform of 2 to the power n is z/z-2. So we will get this as z inverse/z will be equal to -2 to the power n-1un-1+2*3 to the power n-1*un-1, okay.

So this is nothing but, you can write it as -1/2*2 to the power nun-1+2/3*3 to the power nun-1, okay. Now un-1 is 0 when n=0. So at n=0, yn is 0 and when n is greater than or equal to 1, un-1=1, so you get this answer, okay.

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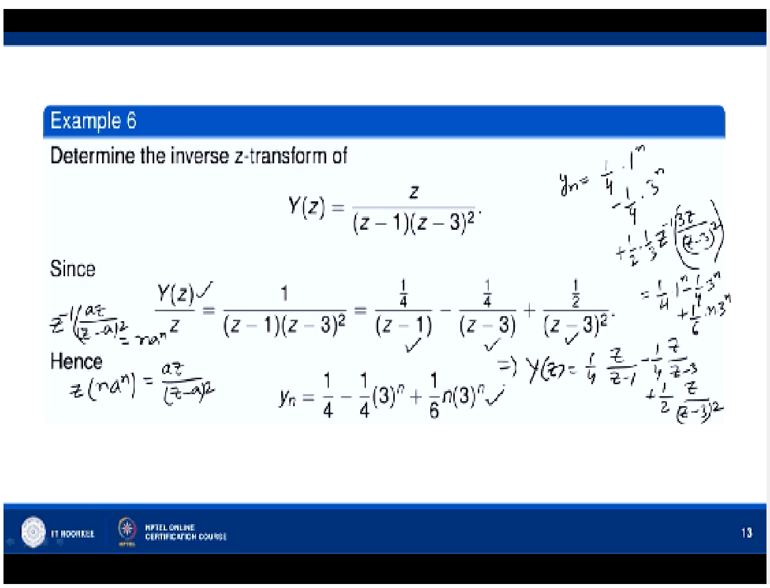

Now determine the inverse Z-transform of z/z-1 z-3 whole square. Now what we do here? We can write Yz/z, okay. Yz/z, there is another technique where we write the partial fractions of Yz/z. Yz/z will be equal to 1/z-1 z-3 whole square. When you break it into partial fractions, you get this, this and these are the partial fractions. Now then after we have got the partial fractions, we multiply z both sides.

So this will give you Yz=z/1/4z/z-1 -1/4z/z-3 and then 1/2z/z-3 whole square, okay. So inverse Z-transform will give you yn sequence. yn will be equal to 1/4 1 to the power n, okay. Then -1/4 3 to the power n and then here, now let us see, let us recall that Z-transform of na to the power n is az/z-a whole square. So we have to use this one, okay. So inverse Z-transform of az/z-a whole square=na to the power n, okay.

Now here we have z/z-3 whole square, okay. We have to take the inverse Z-transform of this. So we write 1/2*1/3 and here we can make it 3z/z-3 whole square. So when you bring it to this form, inverse Z-transform of this, okay, we can find. We write it in the standard form. To do that, we get 1/2 1/3 and then z inverse of this, okay. So this inverse Z-transform of this, then will be equal to n*3 to the power n. So we have 1/4 1 to the power n-1/4 3 to the power n, then 1/6 n times 3 to the power n, okay. So we get this, 1/4-1/4 3 to the power n+1/6n3 to the power n.

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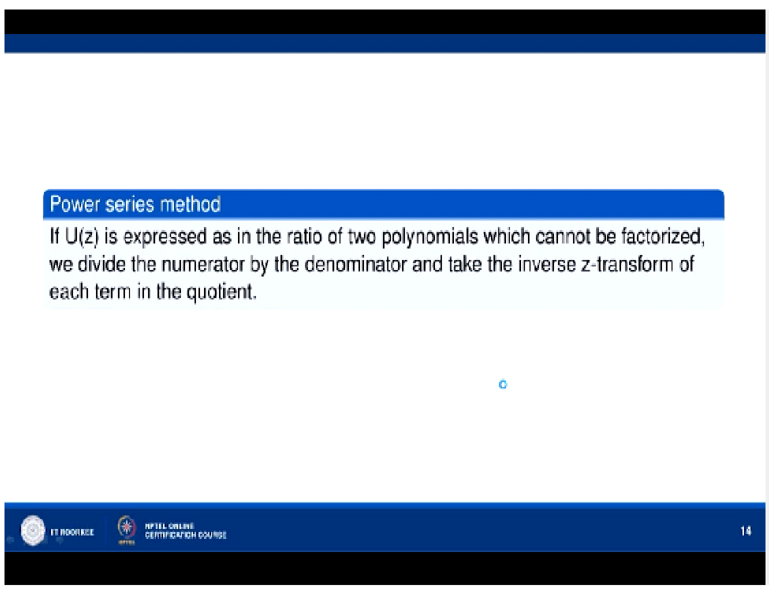

Now if Uz is expressed; in the power series method, we expressed Uz in the form of a power series and then take the inverse Z-transform. Now let us consider this.

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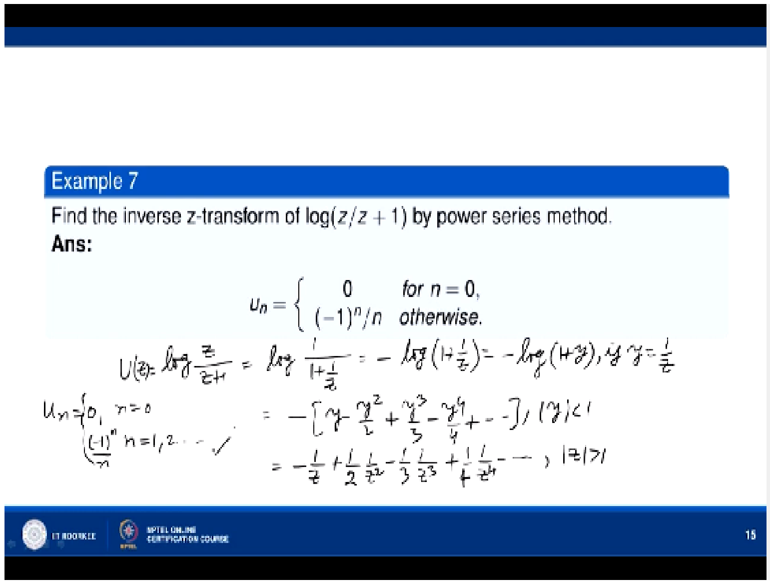

log z/z+1 and we find the inverse Z-transform of this by power series method. So we can write log z/z+1 as; we can write log z/z+1=-log 1+y, if y we take as 1/z. Now we know that log of 1+y can be expanded in a power series if mod of y<1. So we can write this as -y-y square/2+y cube/3-y4/4 and so on provided mod of y<1, okay. Now let us replace y1/z, so we get -1/z+1/2*1/z square-1/3*1/z cube, okay, like this, 1/4 1/z to the power 4 and so on, okay.

And where mod of z>1, okay. So here we can see that the un, okay, let us take this as Uz. So un=0 when n=0, okay. And when n=1, 2, 3 and so on, it is -1 to the power n/n, okay. If you take n=1, u1 is -1, u2 is 1/2, u3 is -1/3, u4 is 1/4 and so on. So this is how we can find the inverse Z-transform by expanding log z/z+1 in the form of a power series.

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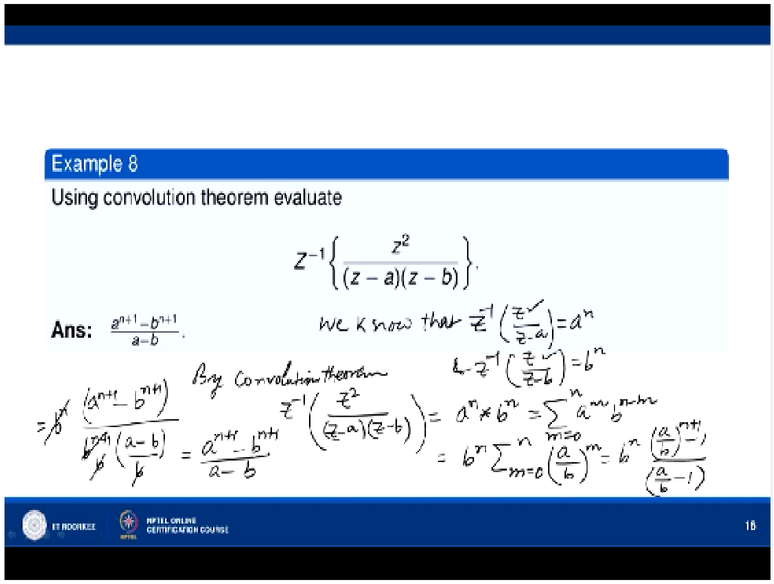

Now we have in the previous lecture, discussed the convolution theorem. We can use convolution theorem here to determine the inverse Z-transform. So we know that z inverse of z/z-a, this is equal to a to the power n. And z inverse of z/z-b is b to the power n, okay. So then by convolution theorem, z inverse of z square/z-a z-b, product of this and this, let this be Uz, this be Vz. So z inverse of Uz*Vz=convolution of unvn.

So a to the power n convolution with b to the power n, okay. And by definition of convolution of 2 sequences, we get this as sigma m=0 to n a to the power m*b to the power n-m, okay. So this is equal to b to the power n can be written outside and we get sigma m=0 to n a/b raise to the power m, okay. So this is the geometric series. So we can write b to the power n, then we have a/b raise to the power n+1-1/a/b-1, okay.

So this will be equal to, if we calculate this, this will be equal to b to the power n a to the power n+1-b to the power n+1/b to the power n+1*z-b/b, okay. So this cancels with this, you get b here and b cancels with this b. So we get a to the power n+1-b to the power n+1/a-b, okay.

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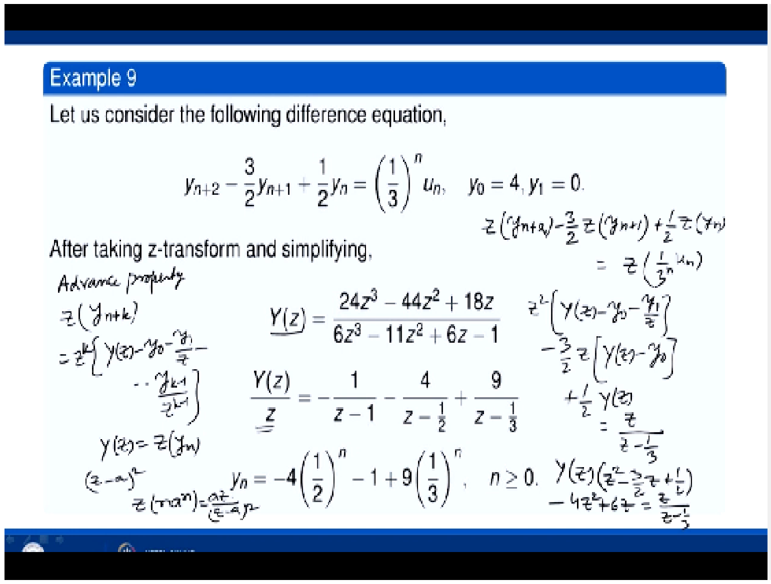

Now let us find the solution of the difference equation. So we have the difference equation yn+2-3/2yn+1+1/2yn=1/3 to the power nun, un is the unit step function, y0=4, y1=0. So let us take Z-transform of the, this given equation. So Z-transform of yn+2-3/2, we are using linearity property when we are writing the Z-transform of the left side, okay. So 3/2 of yn+1+1/2z of yn=Z-transform of 1/3 to the power nun, okay.

Now let us use the advanced property. Advanced property we discussed in the previous lecture. It says that z of yn+k=z to the power k*Yz-y0-y1/z and so on yk-1/z to the power k-1, okay. So here Yz is the Z-transform of yn, Yz=Z-transform of yn sequence. So here when you write the Z-transform of yn+2, let us use this advanced property. So take n=2, so we get z square, inside the bracket, we get Yz-y0 and then we get k=2, so we get y1/z, okay.

Then -3/2, we get z of yn+1. Again we use the advanced property. So we get here z*Yz-y0, okay, +1/2z of yn we take as Yz. Right hand side, un=1 for all n greater than or equal to 0. So I need the Z-transform of 1/3 to the power n which is z/z-1/3, okay. Now y0=4, y1=0, okay. So putting the values of y0 and y1 and simplifying this equation, we will get the value of Yz, okay.

So the coefficient of Yz is Yz*z square-3/2z+1/2, okay. y0 we put as 4, okay, so we get -4z square. Here y0=4, so we get - - +6z. So -4z square+6z, okay. This term is 0 because y1 is 0. And then it is equal to z/z-1/3, okay. Now we can simplify this and we get Yz as 24z cube-44z square+18z/6z cube-11z square+6z-1, okay. Then as we have discussed earlier, we write Yz/z, okay.

Because 6z cube-11z square+6z-1 has simple roots z=1, z=1/2, z=1/3. So writing Yz/z, we will; this Yz/z we will then factorize, okay. Yz/z will be 24z square-44z+18/6z cube-11z square+6z-1. So you divide by 6 in the numerator and denominator and the coefficient of the denominator, here coefficient of z cube we make as unity, okay. So after you make the coefficient of z cube unity here, its factors will be z-1, z-1/2, z-1/3.

And the corresponding partial fractions are -1/z-1, -4/z-1/2+9/z-1/3. Now you multiply by z, okay. So Yz=-z/z-1-4z/z-1/2+9z/z-1/3 and then you take the inverse Z-transform. You will get yn. yn=-, inverse Z-transform of -z/z-1, that is we have -1. Then -4*inverse Z-transform of z/z-1/2 which is -4*1/2 to the power n. And then +9*inverse Z-transform of z/z-1/3 which is 9*1/3 to the power n.

So this is how we solve this difference equation. When we want to solve the difference equation, we take the Z-transform of the difference equation. We left hand side, when we write the Z-transforms, we use the linearity property and the advanced property. The advanced property and of course, sometimes we need delay property also. So we use the advanced property or the delay property to get the Z-transform of the difference equation.

Then we solve this for Yz, okay. Yz is a quotient of 2 polynomials, okay. When we have Yz as a coefficient of 2 polynomials, we factorize the denominator, see its factors. If the factors are linear, okay, we write Yz/z, that is we divide Yz/z and then Yz/z expression is factorized into simple factors. After we have got the fractions of Yz/z, then we multiply by z to get the value of Yz.

And then once Yz is obtained, we take the inverse Z-transform, okay keeping in mind that inverse Z-transform of z/z-a is a to the power n, will lead you to the solution of the given difference equation. Now if in the denominator, you have the factor occurring more than once, okay, say if it occurs more than once, say it occurs twice, okay, so then you have z-a whole square in the denominator.

Suppose you get z-a whole square in the denominator, okay. Then we will need to use that formula Z-transform of na to the power n=az/z-a whole square. We will need to use this formula. So here also we will write Yz/z and then we will write the partial fractions. We multiply by z. When we will multiply by z, the partial fraction corresponding to the z-a square factor will have z/z-a whole square, okay.

So inverse transform will be easy to find. It will be na to the power n/a, that is n*a to the power n-1, okay. If z-a occurs thrice, then we will need to use Z-transform of n square a to the power n because when you take the Z-transform of n square a to the power n, there you get z-a whole cube in the denominator. So that is how we find the inverse Z-transform once we have solved for the given difference equation for Yz and we get the solution of the difference equation. So thank you very much for your attention.