Advanced Engineering Mathematics
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Lecture – 26
Bilinear Transformations

Hello friends, welcome to my lecture on bilinear transformations. In many applications involving boundary value problems which are associated with Laplace equation we need to find conformal mapping which maps a disc on to the half plane v greater than 0 where the circular boundary of the disc is mapped on to the boundary of the half plane, v = 0 means the real axis of the w plane. So an important class of conformal mappings that maps circles to lines.

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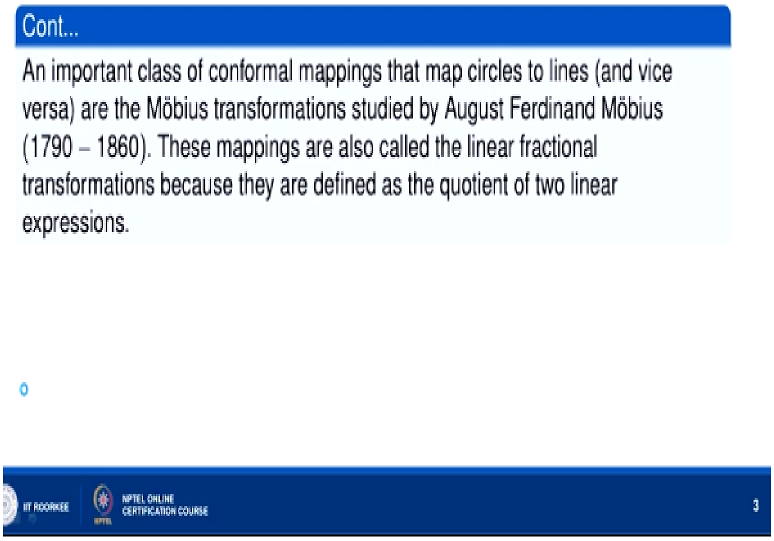

And vise versa are the Mobius transformations is studied by August Ferdinand Mobius during the period 1790 to 1860, these mappings are also called the linear fractional transformations because they are defined as the quotient of 2 linear expressions. Now let us see how we define a Mobius transformation.

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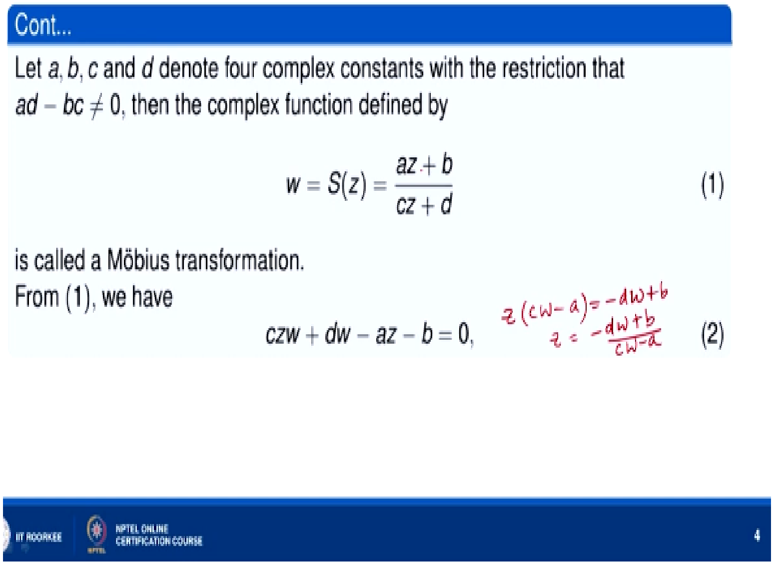

So let us say a, b, c, d be 4 complex constants where the restriction on the complex constants is that a * d – b * c is not = 0, then the complex function defined by w = Sz = az + b/cz + d is called a Mobius transformation. Now this transformation can also be expressed as czw + dw – az – b = 0. When you look at this equation, you notice that this equation is linear both in z as well as in w. So this transformation is also called as a bilinear transformation. Now one more thing we can notice from here, from this equation czw + dw – az – b = 0.

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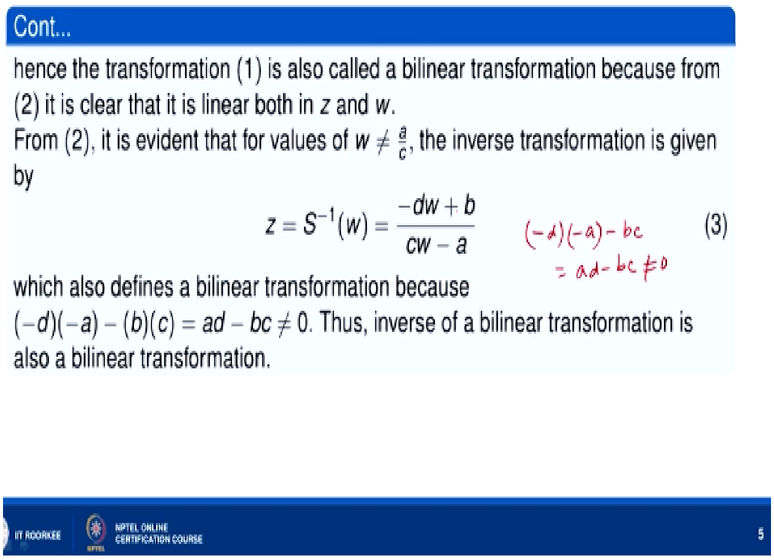

That is to say z * if you collect the coefficient of z here, z is multiplied by if you collect the coefficient here, z times cw – a = -dw + b. We can write this equation as z times cw – a = -dw + b, therefore if w is not = a/c, we may write it as z = -dw + b/cw –a. So if w is not = a/c okay then we can write the inverse transformation that is z = S inverse w which is given by –dw + b/cw –a.

Now here if you notice that if you take the coefficients of –d and –a, -b * c, you see here –d * -a –b * c, this is = ad – bc okay and ad – bc we have assumed to be nonzero okay. So z = -dw + b/cw –a is also a bilinear transformation okay. You can see how it is bilinear transformation because here we said that w = az + b/cz + d where ad okay a * d – b * c is not = 0, here also we notice that –d * -a, -b * c is nonzero.

So it also defines a bilinear transformation and thus we can say that, the inverse of a bilinear transformation is also a bilinear transformation.

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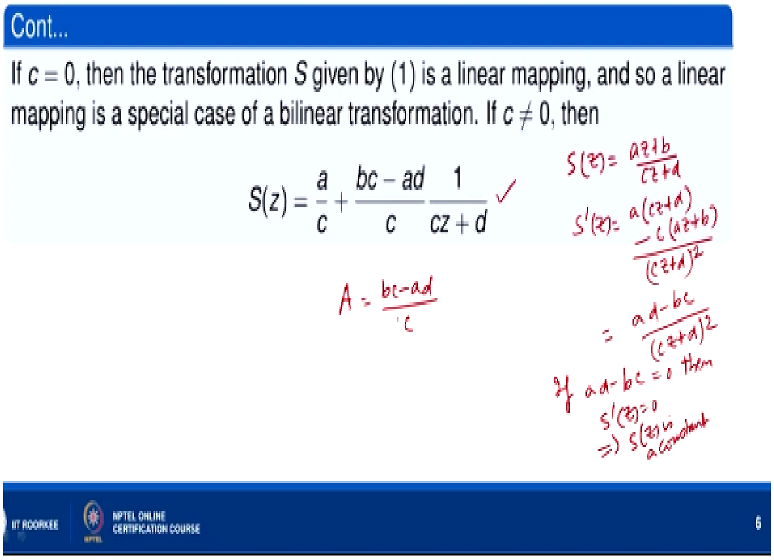

Now you can see that if c = 0 then the transformation given by 1 okay, this is the transformation given by 1, if c = 0, w will be = z + b/d. Now if c = 0 okay, then neither a, nor d can be 0 okay. Here you notice that if c = 0, then w = az + b/d okay. Since ad – bc is nonzero, so if c = 0 then neither a, nor d can be 0, okay. So if c = 0 then neither a, nor d can be 0 and therefore w = az + b/d is a linear transformation.

So it is a liner mapping okay and linear mapping thus we can say is a special case of a bilinear transformation. Now further more let us notice that if c is not = 0, then we can write Sz = this form, we have Sz = az + b/cz + d okay. So if c is not = 0 then Sz can also be expressed in this form a/c + bc – ad/c * 1/cz + d. Now here you can also see that if ad – bc = 0, okay, if ad – bc = 0 then you see that Sz = a/c. Sz becomes a constant okay.

So we need to put the condition on a, b, c, d that ad – bc must be nonzero. This can also be seen otherwise we have Sz = az + b if you find S prime z, S prime z will be = derivative of the numerator that is a * cz + d – derivative of the denominator that is c * az + b/cz + d whole square, okay. So this is = acz, acz will cancel, we get ad – bc/cz + d whole square. So if ad – bc = 0 then S prime z = 0 for all z and so it means that Sz is a constant.

Okay so to avoid Sz being a constant we put the condition on a, b, c, d that ad –bc is nonzero. So if your c is not zero, we can put Sz in this form. Now from here we notice that if you take a = bc – ad/c, this a you define bc – ad/c = a okay.

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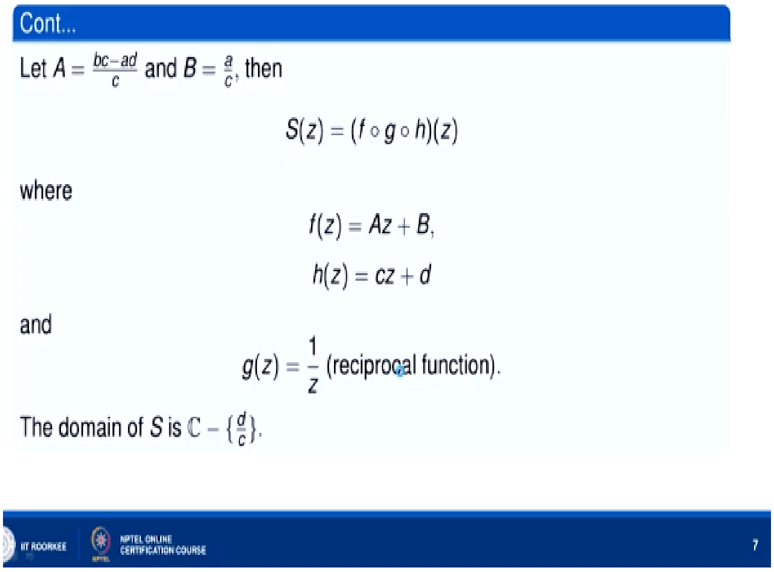

And B = a/c, this a/c we define as B, then Sz is a composite mapping, it is composition of f, g and h, fo, go hz, where fz = az + b, hz = cz + d and gz = 1/z which is known as a reciprocal function. The domain of c is the set of all complex numbers C – d/c okay, d/c is not there because it makes Sz to be infinity. So Sz, the bilinear transformation is a composition of 2 liner functions, this one and this one and one reciprocal function.

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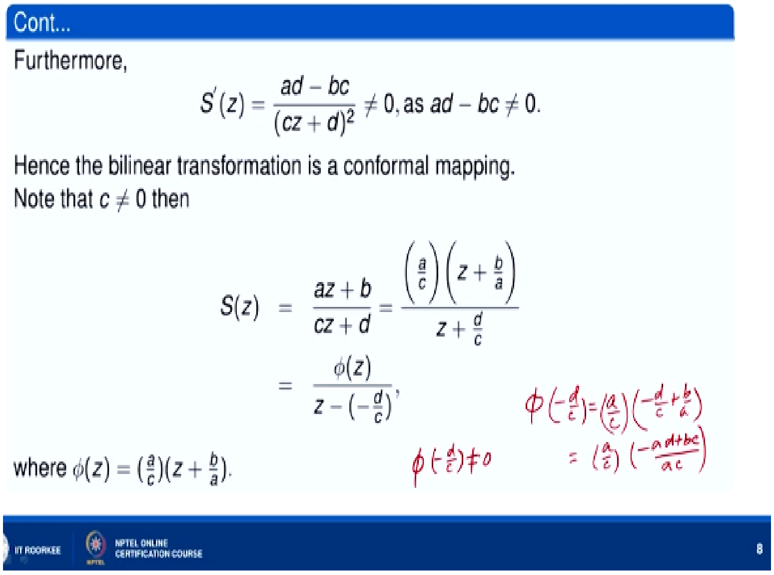

Now as I said earlier S prime z = ad – bc/cz + d whole square okay, so this is not 0 because ad – bc is not 0 okay and since the derivative of Sz is nonzero okay over all points z in the domain of s, therefore the bilinear transformation defined by Sz is a conformal mapping. Now further note that if c is not = 0, then Sz can be written as az + b/cz + d = a/c * z + b/a / z +d/c.

Now let us denote a/c * z + b/a by the function phi z. Then phi z is a/c * z + b/a, it is a linear function, okay, so it is an analytic function and more over that, we noticed that phi – d/c, if you put – d/c for z in phi, then phi – d/c = a/c * -d/c + b/a, which is = a/c * -ad + bc, okay, so we assuming that ad – bc is nonzero therefore phi z is a linear function and phi – d/c is not equal to 0.

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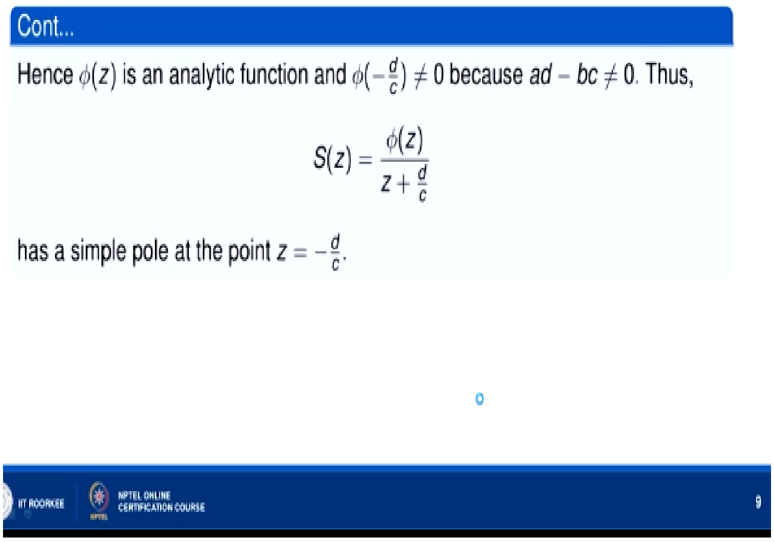

Okay so phi z is an analytic function and phi – d/c is not = 0 because of the fact that ad – bc is nonzero, thus we can say that the bilinear transformation Sz = phi z/z + c as a simple pole at z = -d/c.

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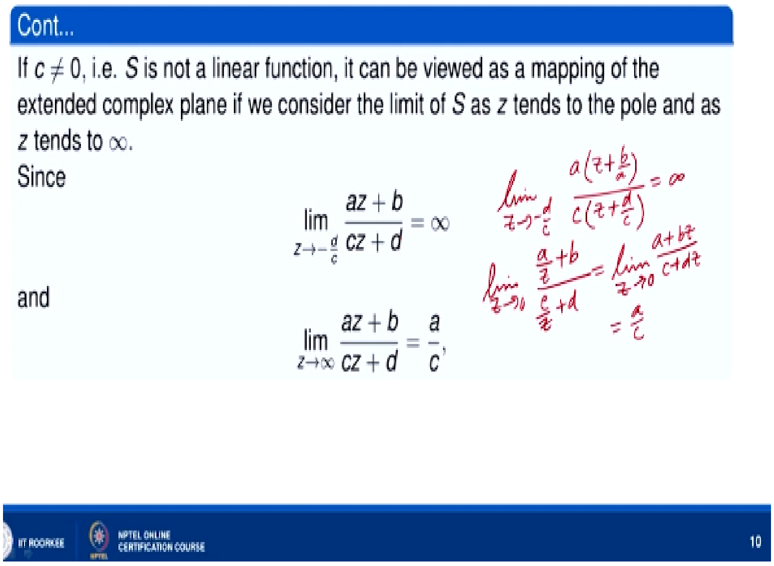

Now if c is not = 0 that is S is not a linear function, we have seen that when c becomes 0, a and d are not 0. So S = a/d * z + b/d okay and S then becomes a linear function. So if C is not = 0, that is we are assuming that S is not a linear function it can be viewed as a mapping of the extended complex plane if we consider the limit of S as z tends to the pole and as z tends to infinity. So when z tends to –d/c what we notice was you see it is limit z tends to –d/c, az + b/cz + d.

So we can write a times z + b/a /c times z + d/c, now when z tends to –d/c, the denominator goes to 0, therefore Sz goes to infinity and here when z goes to infinity in order show that limit of az + b/cz + d as z goes to infinity we write it as limit z tends to 0, replace z/1/z, so a/z + b/c/z + d or we write it as limit z tends to 0, a + b * z/c + d * z, when z tends to 0 it goes to a/c. So to determine the limit as z tends to infinity we replace z/1/z in Sz and then take the limit as z tends to 0.

So limit z tends to infinity is a/c, limit z tends to –d/c, limit becomes infinity when z tends to –d/c. So this was we can say that the function S maps the extended complex z plane to extended complex w plane.

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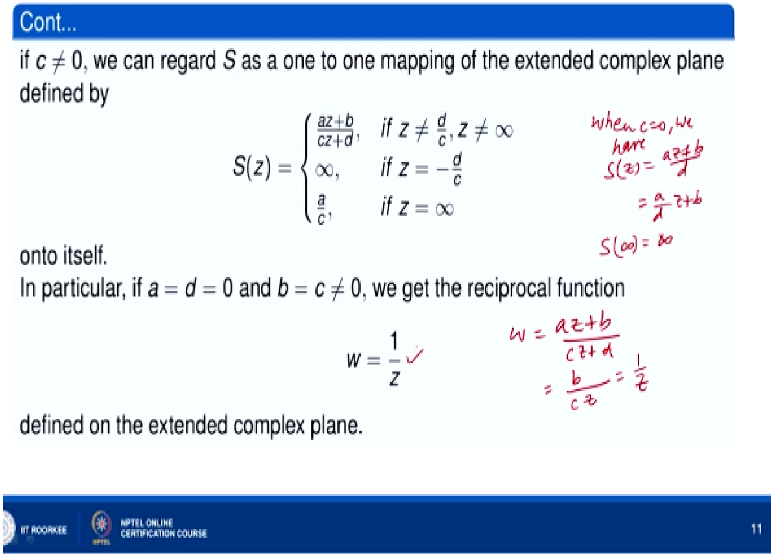

So if c is not = 0 we then regard S as a 1 to 1 mapping of the extended complex plane defined by Sz = az + b/cz + d if z is not = d/c z is not = infinity Sz = infinity if z = -d/c and Sz = a/c if z = infinity on to itself okay. So Sz maps extended complex z plane * extended complex w plane when c is not = 0 and it is given by this, when c = 0 we have Sz = az + b/d okay. So a/d * z + b, then S infinity = infinity.

Infinity is of the 2 complex planes z and w correspond to each other okay. So Sz maps extended complex z plane * extended complex w plane. Now in particular if a = d = 0, okay of a = d = 0, b = c not = 0 let us see w = az + b/cz + d okay, here if you notice that if you take a = d = 0, so this becomes, so if in particular if a = d = 0, we get b/cz okay b and c are equal, so this is 1/z okay. So when a and b, d are 0 and b = c not = 0 then we get w = 1/z defined on the extended complex z plane.

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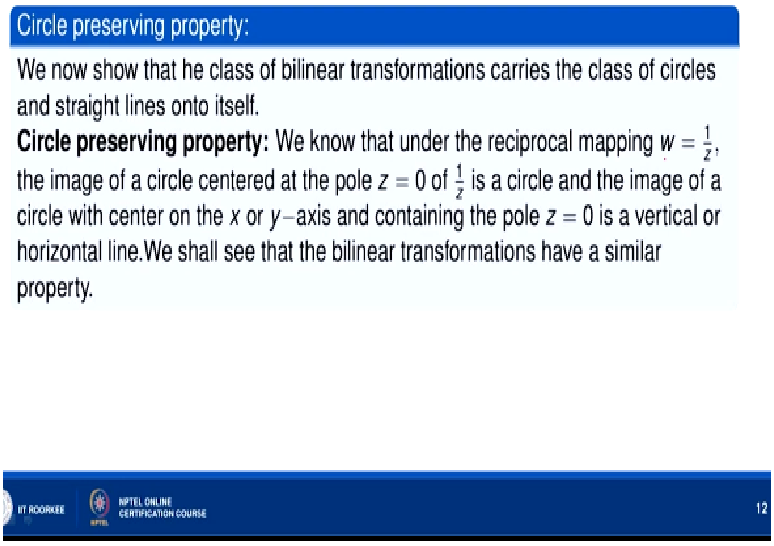

Now let us show that that class of bilinear transformation carries the class of circles and straight lines onto itself. First we show the circle preserving property. We know that under the reciprocal mapping w = 1/z, the image of a circle centered at the pole z = 0 of 1/z is a circle okay and the image of a circle with centre on the x or y axis and containing the pole z = 0 is the vertical or horizontal line. So we are going to see that this property is also possessed by bilinear transformations.

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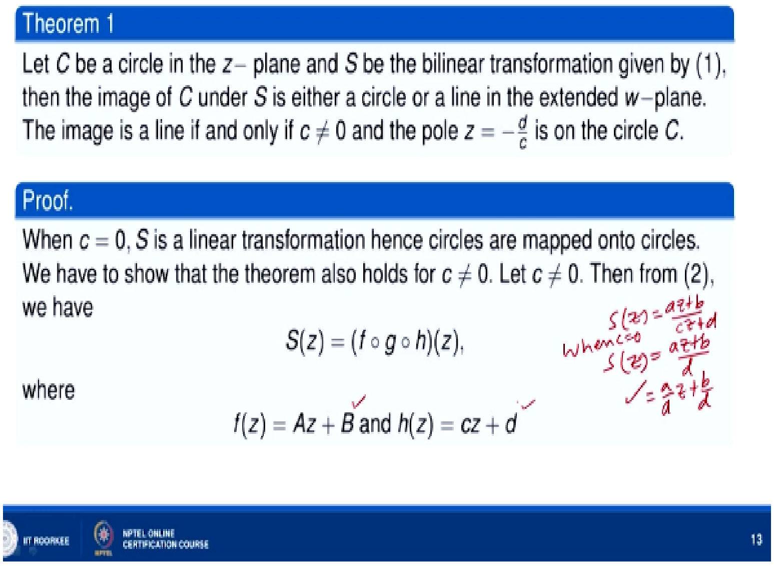

Let C be a circle in the z plane and S be the bilinear transformation given by the equation 1 that is Sz = Az + B/Cz + D, then the mapping of the circle C under S is either a circle or a line in the extended w plane, the image is a line if and only if C is not = 0 and the pole z = 1d/c of the bilinear transformation is on the circle C. So let us first look at the case when c = 0. When c = 0 then we get Sz = az + b/d okay.

So a/d * z + b/d. We know that when c becomes 0 a and d cannot be 0, so Sz is a linear function okay and under a linear function we know that only the size of the figure may be distorted, the basic shape of the figure does not change. So circles will be mapped onto circles under this linear map. Now let us show that the theorem also holds for the case when c is not = 0.

For the case c = 0, we have already resolved that due to the function Sz being a linear function circles are mapped into circles. So let us consider c not = 0 when c is not = 0 then we know that since Sz can be written as f o g o h z, the composition of f g and h, where g and h, f and g are linear functions, fz = az + b, hz = cz + d.

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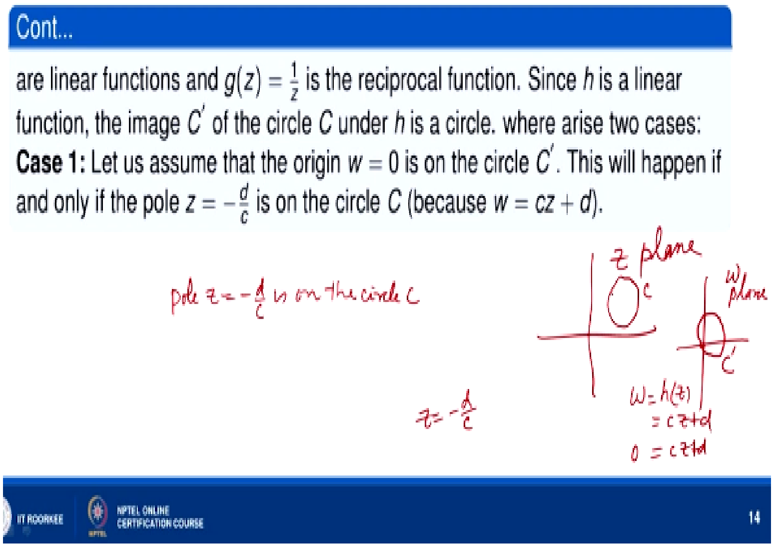

And are linear functions and gz is 1/2 which is the reciprocal function, so since h is a linear function okay first of all let us see when you see the effect of S on z okay.

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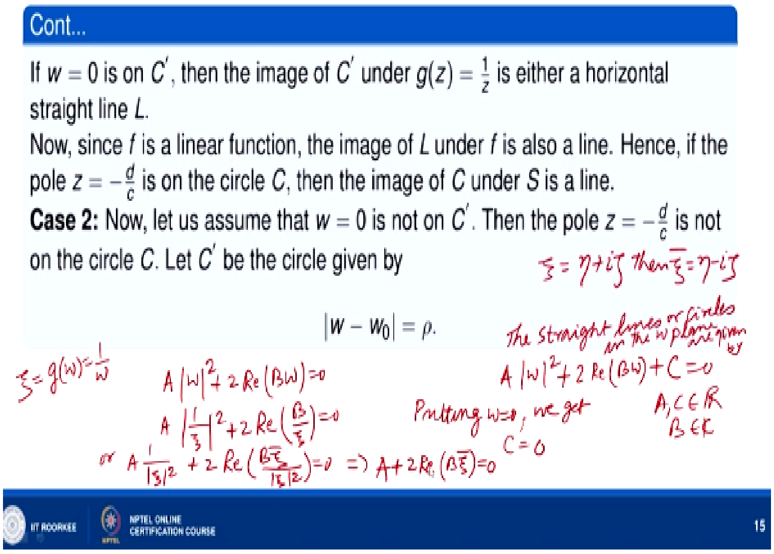

You have to see the effect of h on z, because Sz is composition of f, g and h. So first we see the effect of h mapping on the complex number z. So what we see is that since h is a linear function, the image C dash of the circle C under h is a circle okay. Now there arise 2 cases, case 1 let us assume that the origin of w = 0 on the circle C dash. So let us say this is your z plane, and this is your w plane.

We have a circle here c okay, in z plane okay, and hz is, this is w = hz let us say okay, hz we know is az + b okay and we have seen that under a linear function circles are mapped into circles, so the circle C maps into a circle C dash. Now circle C dash may pass through the origin w = 0, it may not pass through the origin w = 0. So let us first consider the case where C dash passes through the origin okay.

This is your C dash okay, circle C dash. So first let us consider that the origin w =0 lies on the circle C dash, this will happen if and only if the pole z = -d/c is on the circle c. You can see here w = 0, w = 0 implies AZ + B = hz was = cz + d. So when w = 0, cz + d = 0 which will give you z = -d/c and we know that z = - d/c is the pole of the bilinear transformation. So this means that pole z = -d/c is on the circle C okay.

Because w plane, in the w plane C dash is the image of C and the point w = 0 okay, corresponds to the point z = -d/c. So z = -d/c lies on the circle C okay. Now if w = 0 is on C dash okay, then the image of C dash under gz = 1/z is a straight line, okay, so this we can see very easily, the circles are straight lines in the w plane is given by this equation. Let us say some real constant a times mod of w square + 2 times real part of B * w + C = 0 okay.

The circles are straight lines in the, straight lines or circles in the w plane are given by this equation in complex, are given by a mod of w square + 2 times real part of Bw + C where a and C are real constant, A, C belong to R and B is a complex constant, B belongs to C okay. So if w = 0 lies on this circle, w = 0 lies on C dash then what do we get here, putting w = 0 we get C = 0 okay.

So C = 0 means now the equation reduces to A times mod of w square + 2 times real part of B W = 0 okay. Now let us see the image of this circle okay, image of this circle under the transformation gz = 1/z okay, so let us say that xi = gw, gw = 1/w okay. So let us say from w plane we are now going to xi plane okay, so xi = 1/w, so we put w as 1/xi here. So a times 1/mod of xi square + 2 times real part of B/xi = 0.

Or we can say that this is a times 1/mod of xi square + 2 times real part of xi conjugate divided by xi xi conjugate is mod of xi square = 0. Now we can multiply this equation by mod of xi square, mod of xi square is a real quantity, so it will come out of the real part. So we can get A + 2 times real part of B xi conjugate = 0 okay and A is a real constant we know 2 times real part of B xi conjugate = 0.

If xi = suppose we write xi = say eta + i zeta let us say okay, zeta s xi l eta + xi a eta + i times zeta then xi conjugate will be eta – i zeta okay and B if you take as alpha + i beta then what do we get. Let me clarify this.

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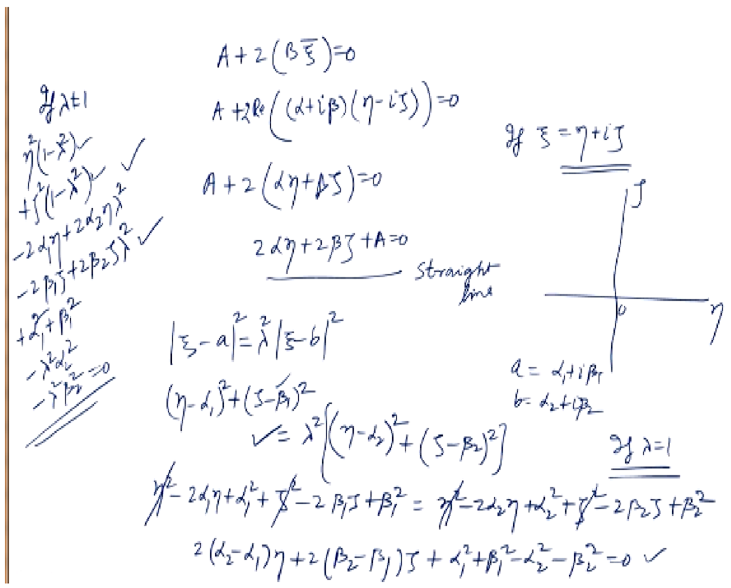

This is A + twice B times xi conjugate = 0, so A + 2 times B is a complex, I can write as alpha + i beta, xi conjugate is we have taken xi as eta – xi conjugate is eta – i zeta, so eta – i zeta = 0 okay and this is the real part. So A + 2 times, now we can take real part here. So real part is alpha eta and then – i square beta zeta, so this means that + beta zeta = 0. So what do we get 2 alpha eta + 2 beta zeta + a = 0 okay.

So if xi = eta – eta + i zeta okay, that means in the xi plane, in the xi plane this is your eta, this is your zeta axis, in the xi plane, the equation comes out to be 2 alpha eta + 2 eta zeta + a okay = 0 where alpha beta A are real numbers. So this represents a straight line and therefore we can say that the image of C dash under gz = 1/z is a straight line says we denoted by L okay.

Now since f is a linear function okay, now we have seen the effect of h mapping and g mapping, now we have to see the finally effect of h f mapping, so now since f is a linear function the image of this straight line L under f is also line, hence if the pole z = -dyc lies on the circle c then the image of c under S is a line. Now let us take the case 2, let us assume that w = 0 is not there on C dash okay.

Then under the mapping h okay, hz = let us recall hz = cz + d okay and circle c is mapping into circle c dash under the mapping hz okay. So if z w = 0 is not on c dash okay, then w = 0 corresponds to z = -d/c okay. So if zw = 0 is not on C dash, then z = -d/c is also not there on the circle C and z = -d/c we know that it is the pole of the bilinear transformation. So if z w = 0 is not on C dash, then the pole z = -d/c is not on the circle C.

Now let us say that the equation of the circle C dash in the w plane, w plane is this, w we have defined as w = hz here okay. So in the w plane let us say the equation of C dash be given by mod of w – w0 = rho. Now since w = 0 is not on C dash, remember w = 0 is not on C dash, so this mean that mod of w0 is not = rho okay, w = 0, w =0 you put here then we get mod of w0 = rho. So if w = 0 is not on C dash then mod of w0 is not = rho. So that implies from here.

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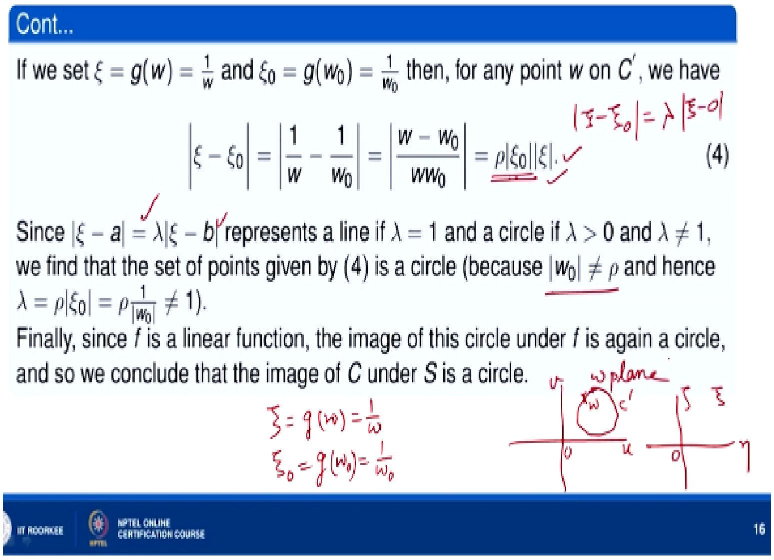

Now let us set xi = gw, so from w plane now we are going to xi plane okay. So this is your w plane, in the w plane we have the circle C dash okay, which does not pass through w = 0 and from here we are going to xi plane okay, xi = you can say eta, zeta okay. So here we are setting gw = xi and gw = 1/w.

So what do we get xi0 = let us say gw0 okay, the point w0 which is the center of the circle C dash w0 is the center of the circle C dash, suppose it is mapped into xi 0 then xi 0 = 1/w 0 okay. Now take any point w on C dash, take any point w on C dash, then let us look at the value of mod of xi – xi0. Mod of xi – xi0 = mod of 1/w – 1/w 0 which is mod of w – w0/w w0 okay.

Now this is = rho times mod of xi 0 * mod of xi is C because mod of w – w0 is rho, mod of w – w0 = rho, 1/w, mod of 1/w is mod of xi and mod of 1/w0 is mod of xi0. So this is = rho times mod of xi0 * mod of xi, now let us look at the equation mod of xi – a – lambda times mod of xi – b. Now it is known that mod of xi – a = lambda times mod of xi – b represents a line if lambda = 1 and it represents a circle if lambda is > 0 not = 1.

This is very simple, we can easily prove this, so let us suppose we have mod of xi – a = lambda times mod of xi – b okay, xi = eta + xi zeta and a we can take as say alpha 1 + i beta 1 and b we can take as alpha 2 + i beta 2, then mod of xi – a = lambda times mod of xi – b let us square both sides. So we get mod of xi – a square = eta – alpha 1 whole square + zeta – beta 1 whole square = lambda square times eta – alpha 2 whole square + zeta – beta 2 whole square.

Now if lambda = 1 okay, of lambda = 1 then what will happen, this will be you can open this, this is eta square – 2 alpha 1 eta + alpha 1 square + zeta square – 2 beta 1 zeta + beta 1 square = eta square – 2 alpha 2 eta + alpha 2 square + zeta square – 2 beta 2 zeta + beta 2 square okay. So when lambda = 1 you can see the square terms eta square, zeta square, eta square, zeta square cancel.

And what we have we have the equation of a straight line. You can collect the coefficients of alpha eta and zeta, okay, so we get 2 times alpha 2 – alpha 1 * eta okay and then we get 2 times beta 2 – beta 1 * zeta and then we get alpha 1 + beta 1 square – alpha 2 square – beta 2 square = 0, which is the equation of a straight line in the eta zeta plane okay. So if lambda = 1 mod of xi – a = mod of xi – b represents a straight line okay.

But if lambda is not = 1 what will happen, then eta square, zeta square term will not cancel, so if lambda is not = 1 okay, then this equation implies that we will have eta square times 1 – lambda square okay, eta square times 1- lambda square then we have -2 alpha 1 okay, and then we will have zeta square here, here we will have lambda square zeta square, so we will have lambda square 1-lambda square.

This is zeta square, eta square * 1- lambda square + zeta square * 1-lambda square we have – 2 alpha 1 eta and here we will get – 2 alpha 2 lambda square eta okay. So we will get that as + 2 alpha 2 eta lambda square okay and then we can have here similarly -2 beta 1 zeta here from her -2 beta 1 zeta and here we will get -+ 2 beta 2 zeta * lambda square okay and then we will have alpha 1 square, beta 1 square, lambda square alpha 2 square with negative sign.

And –lambda square beta 2 square. Now you can look at this equation okay, in this equation you see that the coefficient of eta square and zeta square are same 1- lambda square, 1-lambda square and lambda is not = 1, so the coefficients are not 0 and therefore this equation represents a circle okay. So mod of xi – a = lambda times mod of xi – b represents a straight line when lambda = 1 and it represents a circle when lambda is > 0 but not = 1.

Okay so we see that this equation represents a line if lambda = 1 and a circle if lambda is > 0, but not = 1, now we find that the set of points given by this equation, mod of xi – z0 = rho times mod of xi 0 * mod of xi 0 * mod of xi is a circle y because mod of w0 is not = rho, we have seen earlier okay w = 0 does not lie on the circle C dash, so mod of w is not = mod of w 0 is not = rho okay.

And so lambda, lambda = rho xi 0 okay, let us put this lambda rho xi 0 as lambda. So lambda = rho mod of xi0 = rho times mod of xi0 is 1/w0 okay. So rho * 1/w0, this is not = 1 okay because of this. So rho * 1/mod of w0 is not = 1, that means lambda is not = 1, so this is not = 1. So mod of xi – xi0 = some lambda times mod of xi. Now this can be compared with this equation, you can see mod of xi – xi0 = this is lambda, mod of xi – 0 okay.

So taking xi0 = a okay, lambda = rho mod of xi0 and b = 0. We can compare this equation, this equation with this equation okay. So what do you notice that lambda is not = 1 here, therefore this equation represents a circle okay. So the set of points given by this equation is circle, so this means that the circle C dash when does not pass through the origin maps into a circle in the xi plane.

Now finally since f is the linear function, the image of the circle in the xi plane under f is again a circle and so we conclude that the image of c under the bilinear transformation S is a circle. So this proves the theorem.

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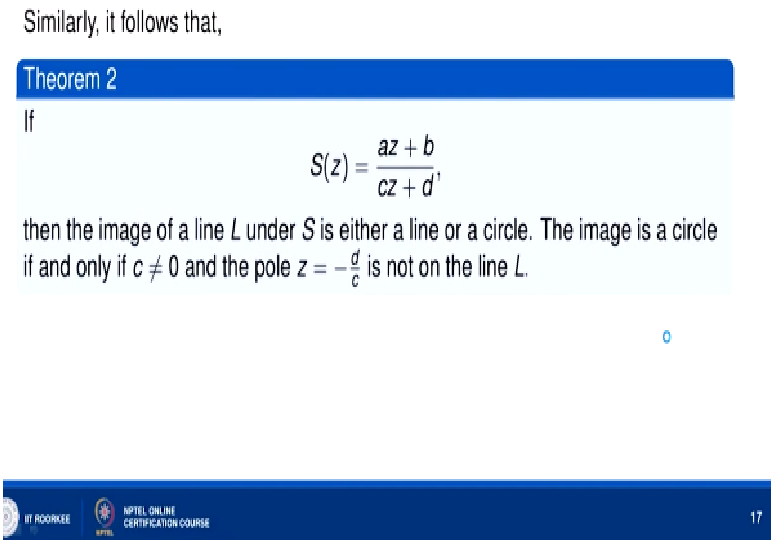

Similarly, we have the other theorem which says that again if Sz is bilinear transformation, then the image of a line L, now we will take the case of a line, earlier in the theorem when we took the case of a circle in the z plane we took a circle and we saw it is image under S, whether it is line or circle. Now here we will take the line in the z plane and we will see what happens under the bilinear transformation S.

What is the image of the line L under S okay? So if Sz = az + b/cz + d then the image of a line L under S is either a line or a circle. Now the image is a circle if and only if c is not = 0 and the pole z = -d/c is not on the line L. Now let us see how it happens.

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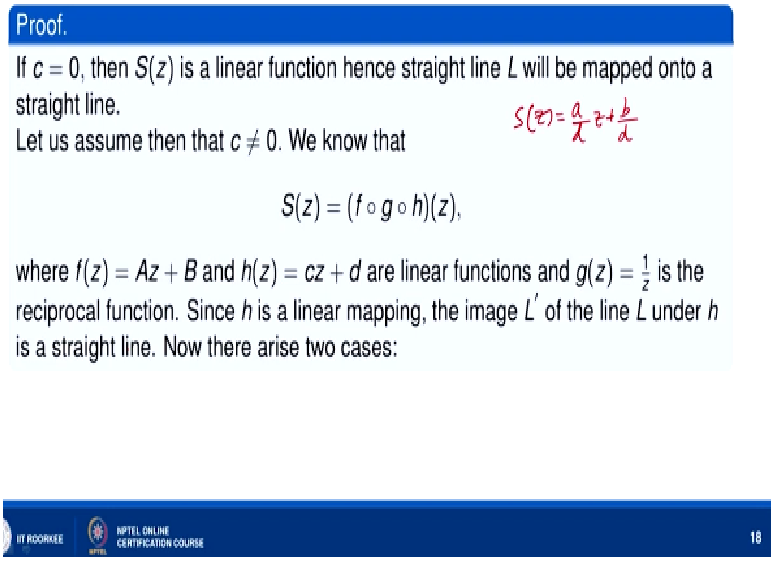

So if c = 0 then we noticed earlier Sz = a/d * z + b/d okay, which is the linear function. So linear function preserves the shape of the, so if it is straight line, it will remain straight line under the mapping Sz okay. So the straight line L will be mapped into a straight line. Now let us assume that the c is not = 0, when c is not = 0, we recall that Sz can be expressed as the composition of f, g and h, f o g o h.

Where fz and hz are linear functions okay and gz = 1/z is the reciprocal function. Now since h is a linear function, hz = cz + d the image L dash of the line L under h is a straight line. Now again there arise 2 cases like in theorem 1 okay.

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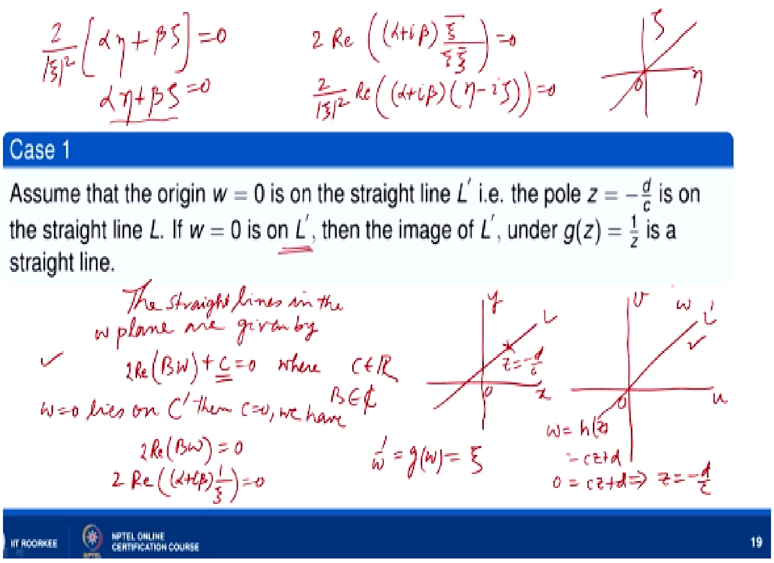

So first case is that the line passes through the origin okay, the straight line L dash in the w plane okay, this is your z plane, we have taken any straight line L okay, in the z plane okay. Now it is mapped into straight line L dash in the w plane, under the mapping hz, w = hz and hz = cz + d right okay. So the line L dash may or may not pass through the origin w = 0. So let us consider the case when it passes through the origin okay, this is your L dash.

So you assume that the origin w = 0 is on the straight line L dash, this means that now w = 0 lies here okay. So this means that w = 0 will give you cz + d = 0, this means that z = -d/c okay. So z = -d/c okay will lie on L and we know that z = -d/c is the pole of the bilinear transformation. So if w = 0 is on the straight line L dash, then z = -d/c lies on the straight line L. If w = 0 is on L dash then the image of L dash under gz = 1/z is a straight line.

Now let us see how we get this. So as we have seen earlier, the totality of straight lines are circles. The circles and are straight lines and straight lines are given by in the w plane they are given by this equation, w plane are given by a times mod of w square + 2 times BW okay + c = 0, where A, C belong to R and B belongs to C okay. Now here we have assumed that w = 0 lies on L dash okay.

So w = 0 let us put in this, we are going to see under the mapping gz = 1/z what happens to the image of L dash okay. So w = 0 lies on C dash okay this C dash okay. So then this C = 0 okay. This means that we have a times mod of w square + 2 times b * real part, this should be real part here, bw = 0 okay. So if w = 0 lies on L dash then the image of, now we are taking it as a line okay, L dash.

So L dash if it is a line then a must be 0 okay. So this term should not be there okay. So we have this equation. This equation of L line okay, equation of line, the straight lines in the w plane. The straight lines in the w plane are given by this equation okay because if a is nonzero then that will represent a circle. So 2 times real part of BW + C =0, now C = 0 so what do we get, we will get this as twice real part of BW =0 okay.

So now let us consider gw = xi and gw = 1/w. So from w plane we are now going to from this plane we are now going to xi plane, so 2 times real part of b let us again take as alpha + i beta and then we will have w, w = 1/xi, so this is = 0 okay. Now what do we do, let us multiply by xi conjugate, so this will be let me write here okay. So twice real part of alpha + i beta * xi conjugate/ xi xi conjugate = 0 okay.

So this will be xi xi conjugate is mod of xi square okay, so it will come out of the real part. So 2 times mod of xi square and we have real part of alpha + i beta * eta – i zeta, this is you xi plane, eta zeta okay. So now you can see the real part if you take it will give you 2 times mod of xi square times we will have alpha eta okay, + beta zeta, okay this is real part, alpha eta + beta zeta = 0.

So alpha eta + beta zeta = 0 means, okay, the line passes through the origin in the zeta plane okay, alpha eta + beta zeta = 0 means the line passes through the origin in the, so straight line, again we get a straight line which passes through the origin okay. So when you take this straight line in the z plane under the mapping xz it maps in to the straight line L dash, L dash if it passes through the origin then it is image under gz = 1/z is again a straight line which passes though the origin. Okay so this is what we get.

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Furthermore, because f is a linear function, the image of this line under f is also a line. Thus, if the pole $z = \frac{-d}{c}$ is on the line L, then the image of L under S is a line.

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Now further more now, so now we have to see the effect of f. So further if f is, since f is a linear function, the image of the line in the xi plane under f gives is again a line, so if the pole z = -d/c lies on the line L then the image of L under S is a line.

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Case 2

Assume that the point $w = 0$ is not on L', i.e. the pole $z = \frac{-d}{c}$ is not on the line L. Let L' be the line given by

$$Bu + Cv + D = 0$$

where $w = u + iv$.

$w = h(z) = cz + d$

$w = 0 \Leftrightarrow z = -\frac{d}{c}$

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Now let us assume that w = 0 does not lie on L dash, if w = 0 does not lie on L dash, then w = hz = cz + d will imply that w = 0, implies and implied by z = -d/c. So when w = 0 does not lie on L dash, then z = -d/c also does not lie on the line L. Now L dash let us say be given by the line L dash in the w planes, suppose it is given by Bu + Cv + D = 0. This is the equation of the straight line L dash in the w plane you can see here we have taken the constant D.

Because this line does not pass through the origin okay. So let us see what happens to this under the mapping gz = 1/z.

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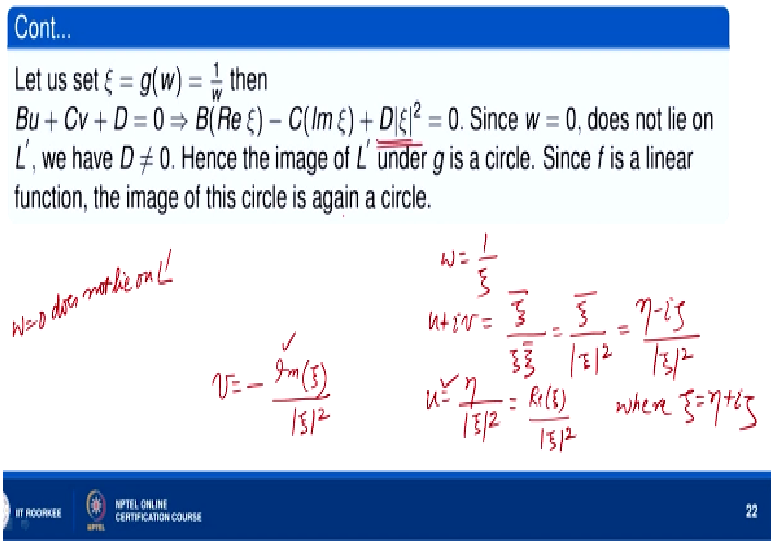

So let us say xi = gw = 1/w then Bu + Cv + D = 0 gives you we have xi = 1/w, so w = 1/xi okay. So u + iv you can express as xi conjugate divided by xi xi conjugate. So xi conjugate divided by mod of xi square. So u = if xi is eta + i zeta then it is eta – i zeta, eta + i zeta okay. So then u = eta/mod of xi square which is real part of xi/mod of xi square and v = - imaginary part of xi/mod of xi square. So let us put these values of u and v in this equation.

So we get v times real part of zi/mod of zi square, I have multiplied the equation by mod of xi square and then –c times imaginary part of xi + d times mod of xi square = 0. Now since w = 0 does not lie on L dash okay, so what will happen, w = 0, this does not lie on L dash. Okay so under the mapping xi = 1/w what happens, d cannot be 0, because d = 0 means u and b are 0s, d = 0 means, see when w =0 lies on L dash, u and b both will be 0.

U and b both will be 0 so d will be 0 okay. So w = 0 does not lie on L dash means D cannot be 0 and hence what do we have this coefficient of mod of xi square okay. This D is nonzero and therefore be real part of xi – c imaginary part of xi + D times mod of xi square, mod of xi square is eta square + zeta square. So eta square + zeta square term will be there in this equation in xi eta.

So this equation represents a circle. So the image of L dash under z is a circle. Now since f is the linear function the image of this circle under f is again a circle. So this is the proof of theorem 2.

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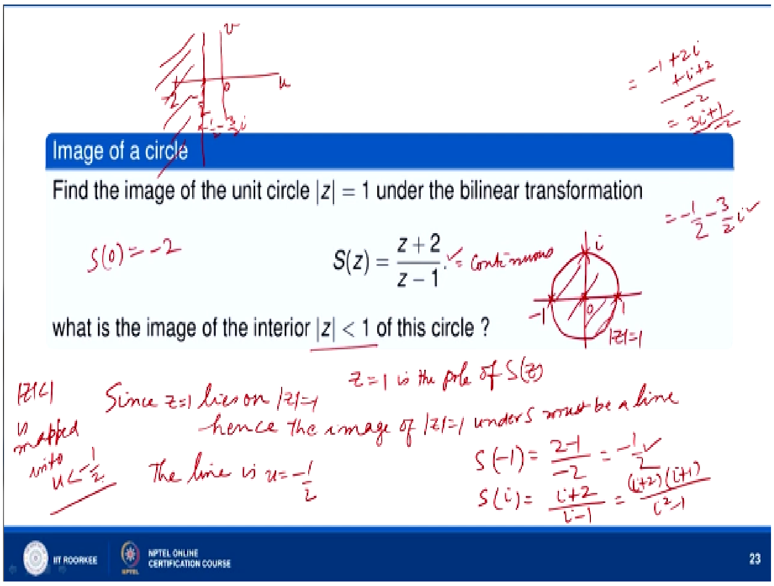

Now let us see how we find the image of the unit circle mod of z = 1 under the bilinear transformation Sz = z + 2/z – 1. So let us see this is our circle, mod of z =1 okay, now we see that z =1 is the pole of Sz, okay, Sz = 1 is the pole of the bilinear transformation Sz and z = 1 lies on mod z = 1. So let us go to theorem 1. So it says that let c be a circle in the z plane. We have unit circle in the z plane as is the bilinear transformation given by 1, then the image of C under S is either a circle or a line in the extended w plane.

The image is a line if and only if c is not = 0 and the pole is on the circle c okay, we have seen that the pole z = 1 lies on the circle c therefore image must be a line. So in this example first conclusion from theorem 1 is that okay, since z = 1 lies on mod of z = 1 okay, hence the image of mod of z = 1 under S must be a line. Now we need 2 points to draw the line okay. So let us take 2 points on the circle here and let us find their image.

Then we will be able to draw this line. So let us take one point as say -1 okay, one point as -1 on this circle, another point let us take as i, then S – 1 we can see S – 1 is 2 – 1/-2 okay. So this is -1/2 okay and Si if you find S i is i + 2/i -1. So I can write it as i + 2 * i + 1/i square – 1 and this gives you how much, this is = i square, so – 1 + 2i + i + 2/-2, so what we get here 3i +1/-2. So this is = -1/2, -3/2 i okay.

So we have got the image of 2 points on the circle in the z plane. So the image of t2 points on the circle is this one. One is -1/2, the other is -1/2, -3/2. So let us draw in the w plane okay, this figure. So see -1 goes to -1/2, this is -1/2 here. So this is ub let us say, -1/2 here and 1/2 –

3/2 is here okay. So -1/2, -3/2 i, so this is the line, that is u = -1/2, the line is u = -1/2 that is the line parallel to the imaginary axis in the w plane.

Now let us take the image of the interior mod z < 1, so we know that Sz = mod of z + 2/z – 1 is a rational function of z and therefore it is the continuous function of z at each point of it is domain okay. So let us take the point z = 0 okay, z = 0 it lies in mod of z <1 and under the mapping Sz let us see where does it go okay. So S0 comes out to be -2 okay. So the interior there is a point we have taken as a test point in the interior of mod z, interior of the circle at origin we have taken and it goes to -2 and -2 lies here okay.

That is in the left side of the line u = -1/2, therefore the region inside the circle mod z = 1 maps to the lower left half of the line u = -1/2 okay, so mod z <1 is mapped into u < -1/2 in this region okay. This is the example 1.

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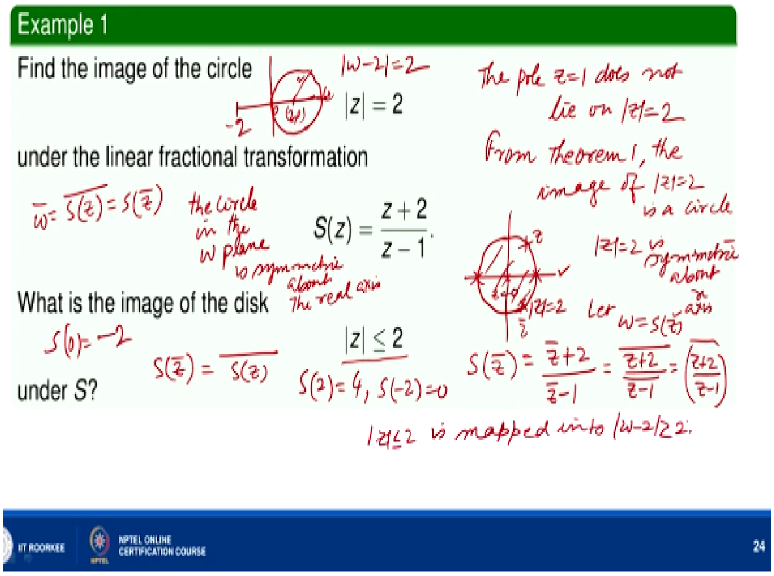

Now let us see the example 2, find the image of the circle mod z = 2 under the same bilinear transformation we are taking okay. Now here let us notice that the pole z = 1 does not lie on mod z = 2. So what does the theorem say, theorem says that let C be a circle in the z plane, now we have the circle mod z = 2 as the bilinear transformation, then the image of C under S is either circle or a line in the extended w plane.

The line if and only if she is nonzero and the pole at – d/c is on the circle. So pole is not on the circle, therefore the line is not the image, the image of the circle is the circle. So we notice from the theorem that the image of mod of z = 2 is a circle okay. So from theorem 1, the image of mod z = 2 is a circle. Okay now let us draw the circle mod z = 2. So this is mod z = 2. We can see that it is symmetric about the real axis okay.

Also about the imaginary axis, so mod z = 2 is symmetric about x axis. This means that if z is a point on the circle okay, then z conjugate is also point on the circle okay. If z is a point on the circle mod z = 2 then z conjugate is also a point on the circle mod z = 2. Now let us say that let w be = Sz okay, so suppose z goes to w in the w plane okay, then let us see where does this z conjugate go okay.

So Sz conjugate let us see what is Sz conjugate, Sz conjugate is z conjugate + 2/z conjugate -1, okay and this I can write as z + 2 conjugate/z-1 conjugate and this can also be written as z + 2/ z – 1 whole conjugate. So and this is what, this is conjugate of Sz. So Sz conjugate = conjugate of Sz. Now what does it mean okay. Sz = w okay. So what do you notice? w conjugate = Sz conjugate = S of z conjugate.

So if z goes to w then z conjugate goes to w conjugate okay. This means that if z w lies on the circle then w conjugate also lies on the circle, so the image of mod of z = 2 in the w plane must be symmetric about the real axis. So the circle in the w plane is symmetric about the real axis, okay, now let us take 2 points on the circle mod z = 2. Let us take z = 2 and z = -2 and see their images under Sz okay.

So S2 you can see S2 = 4, okay and S-2 = 0 okay, this means that the circle in the w plane is symmetric about the real axis okay and the point 2 and -2 go to 0 and 4 which also lie on the real axis, this means that 0 and 4 because of the symmetry must be the ends of a diameter of that circle okay. So we can say that circle is like this 0 and 4, okay, so this is 0, this is 4 okay. So that means 2, 0 must be the center of the circle and radius must be 2 okay.

So we have the equation of the circle as mod of w – 2 = 2 okay. Now let us ask what is the image of the interior of this circle okay, mod z = 2, so suppose you take again z = 0 here, z = 0 maps to S0 = -2 okay, S0 = -2 and -2 does not lie, I mean it lies outside this circle, -2 is here okay. So this means that interior of the circle is mapped on to the exterior of mod of w – 2 = 2. So mod z <= 2 is mapped into the mod of w -2 >= 2 okay.

So that is the conclusion which we can draw, so this is mapped into mod of $w - 2 <= 2$, okay, so with this I would like to, we came to the end of this lecture, thank you very much for your attention.