

Advanced Engineering Mathematics
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Lecture - 15
Taylor Series

Hello friends, welcome to my lecture on Taylor Series.

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Suppose

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad |z - z_0| < R. \quad (1)$$

We know that a power series with a non-zero radius of convergence represents an analytic function within its circle of convergence. Now the question arises, if we are given an analytic function f that is analytic in some domain D , can we represent it by a power series?



Suppose $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ for $|z - z_0| < R$, that means the power series converges in the region $|z - z_0| < R$. Now we know that the power series with the non-zero radius of convergence represents an analytic function within its circle of convergence. So the question now the natural question is, if we are given an analytic function f in some domain D , then can we represent it by a power series? So our next theorem answers this question.

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Theorem 1

Let $f(z)$ be analytic in a domain D and let $z = z_0$ be any point in D . Then $f(z)$ can be uniquely represented by the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n. \quad (2)$$


This representation is valid in the largest open disc with center at z_0 contained in D .

Proof

Let C be a circle with center z_0 lying entirely within the domain D . For any point $z \in C$, by Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds, \quad (3)$$



This called Taylor theorem. Let $f(z)$ be analytic in a domain D and let $z=z_0$ be in any point in D . So let us consider any domain D , okay and let us take a point z_0 in the domain D , okay. Then $f(z)$ can be uniquely represented by the power series $f(z)=\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$, this is $f^{(n)}(z)$ means n th derivative of $f(z)$ at $z=z_0$ over n factorial * $z-z_0$ to the power n .

Now this representation of $f(z)$ is valid in the largest open disc the center at z_0 , contained in D , that is the radius of convergence or the radius of the disc will be the distance of the point z_0 from the nearest point on the boundary of D . Let C be a circle center at z_0 and lying entirely within the domain D . So let us consider circle the center at z_0 , okay, so let us consider a circle say this circle is C which center $f(z_0)$ lying entirely within the domain D .

Now we consider any point z belonging to C , belonging to the interior of C , okay. So for any point z belonging to the interior of C by Cauchy integral formula $f(z)=\frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds$, okay. Any; the value of the function f at any point interior to the circle C can be found by Cauchy integral formula and it is given by $f(z)=\frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds$.

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Proof cont...

then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z_0} \frac{1}{1-\frac{z-z_0}{s-z_0}} ds$$

$\frac{1}{1-q} = \frac{1}{(1-q)-(1-q)q}$
 $= \frac{1}{(1-q)} \left\{ 1 + \frac{z-z_0}{s-z_0} \right\}$

From the geometric progression, we have

$$\frac{1}{1-\frac{z-z_0}{s-z_0}} = 1 + \frac{z-z_0}{s-z_0} + \left(\frac{z-z_0}{s-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{s-z_0}\right)^n + \frac{(z-z_0)^{n+1}}{(s-z)(s-z_0)^{n+1}}$$

and hence (3) becomes

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z_0} ds + \frac{(z-z_0)}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^2} ds + \frac{(z-z_0)^2}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^3} ds + \dots + \frac{(z-z_0)^n}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^{n+1}} ds + \frac{(z-z_0)^{n+1}}{2\pi i} \int_C \frac{f(s)}{(s-z)(s-z_0)^{n+1}} ds$$

Then, $f(z)$ can be expressed as $1/2\pi i$ integral over C $fs/s-z_0$ $1/1-z-z_0/s-z_0$. So $1/s-z$, okay. I can write it as $1/s-z_0-z-z_0$, okay. Then I can write it as $1/s-z_0$ $1-z-z_0/s-z_0$, okay. So $fs/s-z$ is written as $fs/s-z_0 * 1/1-z-z_0/s-z_0$ ds. Now, we know that $1+q+q$ square and so on q to the power $n+1/1-q$ okay is equal to $1/1-q$. We know that $1+q+q$ square and so on q to the power $n-1$, this is equal to $1-q$ to the power n upon, if you take here n up to n if you take $1+q+q$ square and so on q to the power n if you take then this there are $n+1$ so $1-q$ to the power $n+1$ upon $1-q$.

So I can write it as $1+q+q$ square and so on and q to the power $n+q$ to the power $n+1$ upon $1-q=1/1-q$ so this is equal to q to the power n here, okay. So from this formula we are writing $1+q+q$ square and so on q to the power $n+q$ to the power $n+1/1-q=1/1-q$ using from this formula, okay. So this is q , okay. This is q , so $1/1-q=1+q+q$ square q to the power n and then q to the power $n+1$ so q to the power $n+1$ means $z-z_0$ over $s-z_0$ to the power $n+1$, okay and then $1-q$.

So q to the power $n/1-q$, so we have q to the power $n+1$ upon $1-q$ means $z-z_0$ over $s-z_0$ raise to the power $n+1$ and 1 over $1-q$, $q=z-z_0$ upon $s-z_0$. So what it is, $z-z_0$ to the power $n+1 * s-z_0/s-z_0$ to the power $n+1$, okay $s-z_0$ to the power $n+1 * s-z$. So this $s-z_0$ will cancel with $s-z_0$ here and we will get $z-z_0$ to the power $n+1$ upon $s-z * s-z_0$ to the power n , okay. So this is this term. Okay. This term is q to the power $n+1/1-q$, q to the power $n+1/1-q$ where $q=z-z_0$ over $s-z_0$, okay. So this is how we have written this equation.

Now when you put this value okay value of $1/(1-z-z_0/s-z_0)$ using this equation this equation gives us $f(z) = 1/2\pi i \int_C f(s)/s-z_0$. This term corresponds to this one, okay. Then $z-z_0$ over $2\pi i$, $z-z_0$ over $2\pi i$ integral over $C f(s)/s-z_0$ whole square ds then $z-z_0$ whole square $/2\pi i$ integral over $C f(s) ds/s-z_0$ to the power 3 and so on. Now the terms corresponding to this, so $z-z_0$ to the power $n/2\pi i$ integral over $C f(s) ds/s-z_0$ to the power $n+1$ and then the last term $z-z_0$ to the power $n+1/2\pi i$ integral over $C f(s)/s-z_0$ to the power $n+1 ds$, okay.

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Proof cont...

Using Cauchy's integral formula for derivatives, we can rewrite the above equation as

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + R_n(z), \quad (4)$$

where

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \int_C \frac{f(s)}{(s-z)(s-z_0)^{n+1}} ds$$

The equation (4) is called the Taylor's formula with remainder R_n . Now we shall show that $R_n(z) \rightarrow 0$, as $n \rightarrow \infty$. Since f is analytic in D , $|f(z)|$ has a maximum value, say M on the contour C . In addition, since z is inside C , $|z-z_0| < R$, and hence $|s-z| = |s-z_0-(z-z_0)| \geq |s-z_0| - |z-z_0| = R-d$, where $d = |z-z_0|$.

Handwritten notes:
 $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(s) ds}{(s-z_0)^{n+1}}$
 $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(s) ds}{s-z_0}$
 $M = \max_{z \in C} |f(z)|$

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Now using Cauchy integral theorem formula for derivatives we have the Cauchy integral formula, let us recall that, $f^{(n)}(z_0) = n \text{ factorial} / 2\pi i \int_C f(s) ds / (s-z_0)^{n+1}$ where n takes values 1,2,3 and so on. And $f(z_0) = 1/2\pi i \int_C f(s) ds / (s-z_0)$, okay. So using those formulas okay, the first term okay first term this one is $f(z_0)$, this is f' prime z_0 over 1 factorial, this f'' double prime z_0 over 2 factorial, this is $f^{(n)}(z_0)$ over n factorial, okay.

So we can write this equation as $f(z) = f(z_0) + f'$ prime z_0 over 1 factorial $z-z_0$, f'' double prime z_0 over 2 factorial $z-z_0$ square. Then n th derivative of $f(z)$ at z_0 over n factorial $z-z_0$ to the power $n + R_n(z)$. $R_n(z)$ is the remainder of the series after these $n+1$ terms, so $R_n(z)$ is $z-z_0$ to the power $n+1$ this expression. This expression is $R_n(z)$. So this is $z-z_0$ to the power $n+1$ over $2\pi i$ integral over $C f(s) ds / (s-z_0)^{n+1}$.

Now this equation, this equation is called the Taylor's formula with remainder $R_n(z)$, okay. Now we shall show that, this $R_n(z)$ the remainder of the series after $n+1$ terms goes to 0 as n goes to infinity. Since, now we are given that $f(z)$ is analytic in the domain D so since f is analytic in the domain D and the circle C lies within D okay, so $f(z)$ is analytic on C and therefore, mod of $f(z)$ assumes its maximum value, let us take that maximum value to be M on the contour C okay.

So mod of $f(z)$ say $M = \text{maximum value of mod of } f(z) \text{ but } z \text{ belongs to } C$, okay. So further more z is inside C , you can see z is inside C so mod of $z - z_0 < R$, okay. This z is inside C . And the radius of C is R , okay. So mod of $z - z_0 < R$, okay. R is the radius of the circle. So in addition, since z is inside C mod of $z - z_0 < R$ and so mod of $s - z$ must be $\geq R - D$. Now let us see how we get this. $S - Z = S - Z_0 - z - z_0$, okay.

So mod of $s - z = \text{mod of this, which is } \geq \text{mod of } s - z_0 - \text{mod of } z - z_0$. Okay. So D is your mod of $z - z_0$. Okay. And mod of $s - z_0$, mod of $s - z_0 = R$ because s varies along the circle C , okay. This S is the variable of integration, okay and the integration is along the curve C , okay. So; and the radius of the circle is R so mod of $s - z_0$, z_0 is the center of the circle so mod of $s - z_0 = R$, so this is $R - D$, okay. D is the distance of z from the center z_0 .

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Proof cont...

Now,

$$|R_n(z)| \leq \frac{MR}{(R-d)} \left(\frac{d}{R}\right)^{n+1}$$

$|R_n(z)| = \frac{|z-z_0|^{n+1}}{2\pi} \left| \int_C \frac{f(s) ds}{(s-z)(s-z_0)^{n+1}} \right|$
 $\leq \frac{d^{n+1}}{2\pi} \frac{M}{(R-d)R^{n+1}}$

Since, $d < R$, $\left(\frac{d}{R}\right)^{n+1} \rightarrow 0$ as $n \rightarrow \infty$, we conclude that $|R_n(z)| \rightarrow 0$ as $n \rightarrow \infty$. Hence $R_n(z) \rightarrow 0$ as $n \rightarrow \infty$ for all z inside C . Now, by the identity theorem of the power series, the representation of $f(z)$ in the form (2) is unique in the sense that (2) is the only power series with center at z_0 which represents the given function $f(z)$. Since $f(z)$ is analytic in D , the radius R will be at least as large as the distance of z_0 from the nearest point on the boundary of D . If it is larger, then the series may no longer represent $f(z)$ at all point of D which lie in the interior of the circle of convergence.

$\left| \frac{f(s)}{(s-z)(s-z_0)^{n+1}} \right| \leq \frac{M}{(R-d)R^{n+1}}$


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Now, so mod of $R_n(z)$, okay mod of $R_n(z)$ let us find. Mod of $R_n(z) = \text{mod of } z - z_0 \text{ to the power } n+1/2\pi$ and then mod of integral over C $f(s) ds / s - z^* s - z_0$ raise to the power $n+1$, okay. Yes, so let

us see integrant mod of $f(s)/s-z^*s-z_0$ to the power $n+1$. This \leq mod of $f(s) \leq m$ because m is the maximum value of the mod of $f(z)$ along the curve c . And mod of $s-z$ is $\geq R-Z$. So this $\leq m/R-d$ and mod of $s-z_0 = R$, so R to the power $n+1$, okay.

So this integrant $\leq m/R-d * R$ to the power $n+1$. And the length of the curve $c = 2\pi R$, so this \leq mod of $z-z_0$, mod of $z-z_0$ is D , okay so D to the power $n+1/2\pi * 2\pi R$ which is the length of the circle C , okay $* m/R-d*R$ to the power $n+1$. Okay. So this cancels with this and what we get is $m*R$ so this is $m*R/R-d$ okay then d/R to the power $n+1$. Now $D < R$. Okay, D is distance between z and z_0 , okay.

z lies to the interior of C and z_0 is the center of C . So definitely R is bigger than mod of $z-z_0$, okay. That is $D < R$ and so $d/R < 1$ and therefore, d/R to the power $n+1$ goes to 0 as n goes to infinity, and we thus conclude that, mod of $R_n(z)$ goes to 0s and goes to infinity. And hence, $R_n(z)$ goes to 0 as n goes to infinity for all z which lie inside C . Now, by let us apply the identity theorem of the power series. So by the identity theorem of the power series the representation of $f(z)$ in the form 2, okay.

In this form 2, is unique in the sense that 2 is the only power series with center at z_0 which represent the given function $f(z)$. Since $f(z)$ is analytic in D the radius R will be at least as large as the distance of z_0 from the nearest point and the boundary of D . We can increase the radius of this circle C as long as I mean we do not cross the boundary of D . Okay, that means this R can be can utmost be the distance of z_0 from the nearest point on the boundary of D .

If we take it to be more than that, more than the distance of z_0 from the; if we cross the boundary of D then the series may no longer represent $f(z)$ at all points of D which lie in the interior of the circle of convergence.

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Note

Every analytic function has derivative of all orders and can be represented by power series of the form (2). This is not true in general for real functions; for example, $f(x) = e^{-\frac{1}{x^2}}$, $x \neq 0$ and $f(0) = 0$ has derivatives of all orders at $x = 0$ but cannot be represented by a power series about $x = 0$.

Theorem 2

Every power series with a non-zero radius of convergence is the Taylor series of the function represented by that power series.

Proof

Let the power series

$$\sum_{n=0}^{\infty} c_n(z - z_0)^n$$

have a non-zero radius of convergence R .

Handwritten notes:
 $f'(0) = 0$
 $f''(0) = 0$
 $f^{(n)}(0) = 0, \forall n \geq 1$
 Power series about $z=0$
 $\Rightarrow f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots = 0$

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Every analytic function has derivatives of all orders. And we know that every analytic function we have seen, by the Cauchy integral formula for higher order derivatives, that every analytic function has derivatives of all orders and now we have seen that it can be represented by a power series with the form 2, okay, power series of the form 2. Now this is not true in general for real function.

For example, if you consider $f(x) = e^{-1/x^2}$ when x is not 0 and $f(0)=0$ then we can see that $f'(0) = 0$, $f''(0) = 0$, $f'''(0) = 0$, equal to 0 and so on. $f^{(n)}(0) = 0$ for all n , okay. So $f(x) = e^{-1/x^2}$; it for $-1/x^2$ when x is not 0 and $f(0)=0$ has derivatives of all orders at $x=0$ but it cannot be represented by the power series about $x=0$. If you write the power series about $x=0$ what it will be?

Power series, about $x=0$ will be $f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$ and so on, okay. Now we know that here, $f(0)=0$, $f'(0)=0$, $f''(0)=0$ so power series becomes 0, okay. So it definitely does not represent $f(x) = e^{-1/x^2}$. So this function cannot be represented by power series about $x=0$. So this result holds in the case of complex analytic functions only.

Now every power series with the non-zero radius of convergence now if you to have any power series whose radius of convergence is non-zero then it is the Taylor series of the function

represented by that power series. Let us prove this, so let the power series $\sum_{n=0}^{\infty} C_n(z-z_0)^n$ have a non-zero radius of convergence, say given by R .

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Proof cont...

Then it represents some analytic function $f(z)$ in the disk $|z - z_0| < R$, i.e.

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

Hence,

$$f'(z) = c_1 + 2c_2(z - z_0) + \dots$$

and more generally

$$f^n(z) = n!c_n + (n+1)!c_{n+1}(z - z_0) + \dots$$

all these series converge in the disk $|z - z_0| < R$ and represent analytic functions. Hence these functions are continuous at $z = z_0$.

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Then it represents some analytic function we know, say, let us take that analytic function to be $f(z)$ and say the region of convergence is $|z - z_0| < R$ then $f(z) = \sum_{n=0}^{\infty} C_n(z - z_0)^n$. Now we can term by term differentiate this series, so $f'(z)$ is equal to $C_1 + 2C_2(z - z_0) + \dots$ and so on. And more generally n th derivative of $f(z)$ is $n!$ $C_n + (n+1)!$ $C_{n+1}(z - z_0) + \dots$ and so on.

Now all these series converge and have the same radius of convergence R , that means they all converge in the region $|z - z_0| < R$ and represent analytic functions. Hence, these functions are continuous at $z = z_0$. And therefore, $f(z) = c_0$, okay $f(z) = c_0$, $f'(z_0)$ will be equal to c_1 and $f^n(z_0)$ will be equal to $n!$ C_n , okay.

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Proof cont...

Thus we obtain

$$f(z_0) = c_0, \frac{f'(z_0)}{1!} = c_1, \dots, \frac{f^{(n)}(z_0)}{n!} = c_n,$$

Hence

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

which is the Taylor series of the function $f(z)$.

Or we can say, $c_0 = f(z_0)$, $c_1 = f'(z_0)$, $c_n = \frac{f^{(n)}(z_0)}{n!}$, and thus, we get $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$ which is the Taylor series of the function $f(z)$, okay. So this prove the theorem, that is any power series with a non-zero radius of convergence is the Taylor series of that function, Taylor series of the function represented by that power series. Okay, now let us consider the function $f(z) = \frac{1}{1-z}$, okay.

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Example 3

The Taylor series of

$$f(z) = \frac{1}{1-z} \quad \checkmark \text{ analytic for all } z \text{ except } z=1$$

is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

$f(z)$ is singular at $z = 1$ which lies on the circle of convergence.

So, Taylor series with center at $z=0$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} z^n$$

$f'(z) = \frac{1}{(1-z)^2} \cdot (-1) = \frac{1}{(1-z)^2}$
 $f''(z) = \frac{2}{(1-z)^3} \cdot (-1) = \frac{2!}{(1-z)^3}$
 $f'''(z) = \frac{2 \cdot 3}{(1-z)^4} \cdot (-1) = \frac{3!}{(1-z)^4}$
 In general $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$
 $\Rightarrow f^{(n)}(0) = n!, n=1,2,3,\dots$

So here, $1/1-z$ we can see this function is analytic for all z except $z=1$. Let us find it Taylor series expansion about $z=0$. So that means the $f(0)$ we have to find, $f(0)=1$ and if you find f' prime z , f' prime $z = -1/1-z$ whole square * -1 so $1/1-z$ whole square okay, f'' double prime z let us find. It is

$2 \cdot 3! \cdot (1-z)^{-4}$. If you find f triple prime z it will come out to be $2 \cdot 3! \cdot (1-z)^{-4}$, so this means 3 factorial divided by $(1-z)^4$. Okay.

In general, so this can be written as 2 factorial divided by $(1-z)^3$ and here it can be written as 1 factorial divided by $(1-z)^2$. In general, $f^{(n)}(z)$ okay is equal to n factorial divided by $(1-z)^{n+1}$. So this is valid for $n=1,2,3$ and so on, okay. Now this implies that $f^{(n)}(0) = n$ factorial, okay $n=1,2,3$ and so on, right. So the Taylor series, the center at $z=0$ is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ which is equal to $\sum_{n=0}^{\infty} z^n$.

$f^{(n)}(0)$ is n factorial, so n factorial divided by n factorial z^n , we get $\sum_{n=0}^{\infty} z^n$. Okay. So the Taylor series of the function $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. And we know that this series converges in the region $|z| < 1$, okay and the sum of the series is $\frac{1}{1-z}$. So $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$.

Now, we can see that if you take $z=1$, okay if we take $z=1$ then $z=1$ is a singularity of the function $f(z)$; singularity of the function means the functions $f(z)$ (23:45) to be analytic at $z=1$ or $f(z)$ is not analytic at $z=1$, so $f(z)$ is singular at $z=1$ and $z=1$ lies on the circle of convergence, here. This is your, this $|z|=1$, okay and $z=1$ lies here, okay. So this series converges for all z which lie to the interior of $|z|=1$.

So $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$; this is; so $\sum_{n=0}^{\infty} z^n$ is the Maclaurin series of $\frac{1}{1-z}$ and the region of convergence is $|z| < 1$. Now if you take the series $\frac{1}{1+z}$, okay.

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Suppose $f(z) = \frac{1}{1+z}$

$$= \frac{1}{1-(-z)} = \frac{1}{1-\zeta}, \quad \zeta = -z \checkmark$$

$$\frac{1}{1-\zeta} = \sum_{n=0}^{\infty} \zeta^n, \quad |\zeta| < 1 \text{ in the } \zeta\text{-plane}$$

Then

$$\checkmark \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n, \quad |z| < 1$$

$$= \sum_{n=0}^{\infty} (-z)^n, \quad |z| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1 \checkmark$$

$f(z) = \frac{1}{1+z^2}$ Let us take $\zeta = z^2$

$$= \frac{1}{1-\zeta} = \sum_{n=0}^{\infty} (-1)^n \zeta^n, \quad |\zeta| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n (z^2)^n, \quad |z^2| < 1 \quad \text{or} \quad |z| < 1$$

$$f(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \quad |z| < 1$$

$$\Rightarrow |z|^2 < 1 \Rightarrow |z| < 1$$

Suppose, let consider now $f(z)=1/1+z$ then the Taylor series for this function $f(z)$ can be found from the Taylor series of $f(z)=1/1-z$. How we can do that? This can be written as $1/1-(-z)$, okay. So this is equal to $1/1-zeta$ where $zeta$ is equal to $-z$, okay. Now the series; the function $1/1-zeta$, okay in the $zeta$ plane this is $\sigma n=0$ to infinity. This is $zeta$ to the power n , okay where $\text{mod of } zeta < 1$, okay in the $zeta$ plane. In the $zeta$ plane, $1/1-zeta = \sigma n=0$ to infinity $zeta$ to the power n where $\text{mod of } zeta < 1$.

So using the transformation $zeta = -z$ now. Then, $1/1+z = \sigma n=0$ to infinity $\text{mod of } -z$ to the power n and $\text{mod of } zeta = \text{mod of } -z < 1$, okay. So I can say that this is summation $n=0$ to infinity $-z$ to the power n $\text{mod of } z < 1$. So this is also equal to $\sigma n=0$ to infinity -1 to the power n z to the power n $\text{mod of } z < 1$. So $1/1+z$ okay, the Maclaurin series expansion of $1/1+z$ can be found out from the Taylor series expansion of the function $f(z)=1/1-z$ by making use of the transformation the $zeta = -z$.

Now if you take one more function say $f(z)=1/1+z^2$. Okay, $1/1+z^2$. Okay, then here also let us take $zeta$ to be equal to z^2 , okay. Let us take $zeta$ to be equal to z^2 then $f(z)=1/1+zeta$, okay which is equal to; by making use of this expression, okay. This $\sigma n=0$ to infinity -1 to the power n $zeta$ to the power n where $\text{mod of } zeta < 1$. We have just now seen $1/1+z$ can be written as $\sigma n=0$ to infinity -1 to the power n z to the power n $\text{mod of } z < 1$.

So using this we can write $1/(1+\zeta) = \sum_{n=0}^{\infty} -1 \text{ to the power } n \zeta \text{ to the power } n$ mod of $\zeta < 1$. Now let us use $\zeta = z^2$. So we shall write $\sum_{n=0}^{\infty} -1 \text{ to the power } n \text{ and } z^2 \text{ to the power } n$ and then mod of $z^2 < 1$ okay. Or we can say $f(z) = 1/(1+z^2)$ is given by $\sum_{n=0}^{\infty} -1 \text{ to the power } n z \text{ to the power } 2n$ okay and mod of $z < 1$ because of mod $z^2 < 1$ means mod of $z^2 = \text{mod of } z * \text{mod of } z < 1$ which implies that. Mod of $z^2 < 1$ and which implies mod of $z < 1$. Okay.

So the Maclaurin series expansion of $1/(1+z^2)$ can also be found without determining the values of the derivatives of $f(z)$ at $z=0$. We can make use of the Maclaurin series expansion of $1/(1+z)$ for this purpose and use the transformation $\zeta = z^2$. So when we want to find the Taylor series expansion of a given function we use alternate methods to determine the Taylor series expansion of the function of z . Usually we do not have to find the derivatives of $f(z)$ at $z=0$. We can make use of the Taylor series for the function $1/(1-z)$.

In the region mod of $z < 1$ or $1/(1+z)$; in the region mod of $z < 1$ and the given function is then broken into by rest terms by using the partial fractions and then we use the known Taylor series expansions for the function $f(z) = 1/(1-z)$ and $1/(1+z)$. So with that we confined the Taylor series expansion of a given function without determining the derivatives of the function at various points, because it is; there are infinitely many terms, we can only find the derivatives for a finite number of terms.

But by using the expansion $1/(1-z)$ and $1/(1+z)$ we can write the entire series, Taylor series for the given function. So in the next lecture, we shall discuss Laurent series. And in the case of Laurent series, because here in the case of the function Taylor series the function $f(z)$ is assume to be analytic in the neighbourhood of the point z_0 in some neighbourhood of the point z_0 , but practically in applications we always do not have this situation.

We have to expand a function around those points where the function is not analytic. So in that; in those cases, we make use of the Laurent series, so that we will discuss in our next lecture. Thank you very much for your attention.