

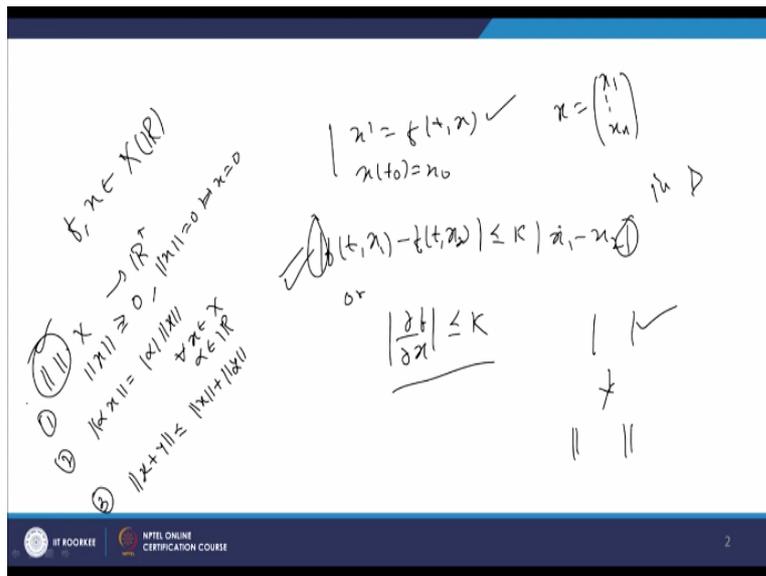
Dynamical Systems and Control
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Lecture - 06
Linear Systems – II

Hello friends. Welcome to this lecture. In this lecture, we will continue our study of linear system. So if you recall in previous lecture, we have discussed the general system that is $\dot{x} = f(t, x)$ and we have considered certain examples and discussed the first draft of existence and uniqueness theorem for $\dot{x} = f(t, x)$. Now you want to discuss the existence and uniqueness theorem for linear system.

And for that we need to define the notation of distance function or we can say that if we have a system like $\dot{x} = f(t, x)$ and $x(t_0) = x_0$.

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The only difference here that x represent a vector now that it is $n \times 1$ vector. So earlier it is just a scalar-valued function but now it is a vector-valued function. So if you remember in case of a scalar-valued function, if f satisfies the ellipses condition in some kind of domain D or you can say that the partial derivative of f with respect to dependent variable is bounded, then this system the initial-value problem has a unique solution.

And we have obtained the unique solution using the Picard iteration. Now in a similar case, we can define the existence and uniqueness theorem for vector-valued function. The only

thing we have to consider is the notation that we are using here that in place of modulus, now we have to use the function which is known as norm function. So norm function is also similar to a distance function.

Here we can define the norm of X is a function from say X cross X to say \mathbb{R}^+ . So here X is the space where all these functions are involving. So all these f, x all these are belonging to the vector function X which is defined as X over \mathbb{R} . So it means that if we take say norm of X then it is giving you say nonnegative value, the norm of X is ≥ 0 and sorry here this norm is defined only for X to \mathbb{R}^+ not for X .

So here this is little bit, so it is defined only for function from X to \mathbb{R}^+ such that norm of X is nonnegative quantity and norm of X is $= 0$ if and only if $x=0$ and norm of αx is $=$ modulus of α times norm of x which is true for all x belongs to capital X and α belongs to say scalar field that is \mathbb{R} here. So this is one property, this is another property, this third property is the norm of $x+y$ is \leq norm of x + norm of y .

This property is commonly known as triangle inequality and if a function satisfies these 3 properties, then function is denoted as norm function. So here it is just a simple generalization of this absolute function here. So now to define existence and uniqueness theorem, we need to consider some function which can be treated as norm function on the space of functions from which we are taking the functions x .

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Definition 7

The sequence of matrices $\{A^{(k)}\}$ converges to the matrix A if and only if the sequence of real numbers $\{|A^{(k)} - A|\}$ has limit zero, and in this case, we write

$$\lim_{k \rightarrow \infty} A^{(k)} = A$$

By definition of norm, $A^{(k)} \rightarrow A$ if and only if the sequence $\{a_{ij}^{(k)}\}$ of complex numbers, representing the element in the i th row and j th column in the matrices $\{A^{(k)}\}$, converging to the element a_{ij} of the matrix A as $k \rightarrow \infty$ for each of the n^2 elements $(i, j = 1, 2, \dots, n)$.

$\|A^{(k)} - A\| \rightarrow 0$
 $\|b_n - b\| \rightarrow 0$ as $n \rightarrow \infty$


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So here let us move and first we define the following definition that the sequence of matrices A_k converges to the matrix A if and only if the sequence of real numbers norm of $A_k - A$ has limit 0 and in this case we write $\lim_{k \rightarrow \infty} A_k = A$. So here once we know the norm, then we need to know what do you mean by that f_n is converging to f right. So for that we have the following definition that f_n converging to f means the norm $f_n - f$ is tending to 0 as n tending to infinity.

So f_n converging to f means it is defined in the sense of norm that is norm of $f_n - f$ is tending to 0 as n tending to infinity. Now how we define the same thing in terms of matrix, so it means that when we say that the matrix A_k is converging to given matrix A in the sense of norm. So here we can say that the sequence of matrices A_k converges to the matrix A if and only if the sequence of a real number.

Now what do you mean by sequence of a real number? Here if you consider since if you apply norm on A_k , then it will give you some real value. So it means that here this norm of A_k is a real value. So it means that this norm of $A_k - A$ is also a real value. So it means that for every k this value gives you some real value, so it means that corresponding to this sequence of matrices, we have a sequence of real numbers.

So we can say that A_k converges to matrix A means this sequence of real number which we obtain using the norm function this will also converge to 0. So we say that this A_k converges to A provided that this norm of $A_k - A$ has a limit 0 or in notation we can write down that $\lim_{k \rightarrow \infty} A_k = A$. So this notation should be clear in the sense of norm.

So by definition of norm, this sequence A_k will converge to A if and only if the sequence of each term that is a_{ij}^k of complex number representing the element in the i th row and j th column in the matrices A_k converging to the element a_{ij} of the matrix A . So it means that sequence of matrix will converge to the element function A provided that the corresponding ij element of the sequence will converge to the ij th element of the limit function.

So it means that this A_k converge to A provided that a_{ij}^k will converge to a_{ij} right for each of the n square element that is ij running from 1 to n here. So it means that A_k converging to A means in the sense of norm that $\lim_{k \rightarrow \infty} A_k = A$.

and how we can understand in terms of element that each element i th element of the sequence will converge to the i th element of the limit function that is the matrix A .

So that is what how we say understand in the sense of norm. Now a matrix function is a function which maps each point t of an interval I to $1n$, only one $n \times n$ matrix A_t . So here this we have defined in terms of matrices, sequence of matrices.

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A matrix function is a function which maps each point t of an interval I to one and only one $n \times n$ matrix $A(t)$.

we call a matrix function $A(t)$ to be continuous, differentiable, or integrable on an interval I if each of its n^2 elements $a_{ij}(t)$ are continuous, differentiable, or integrable respectively on I .

Often, we need to use the following important inequality

$$\left| \int_c^d A(t)b(t)dt \right| \leq \int_c^d |A(t)||b(t)|dt \quad (10)$$

for $c < d$, assuming, for example, that $A(t)$ and $b(t)$ are continuous on $c \leq t \leq d$.

Now if we have sequence of matrices function, it means that now we are talking about these kind of sequence $A_k t$. So here for each t , this $A_k t$ will give you a scalar value. So this $A_k t$ is considered to be a sequence of matrix-valued function. So a matrix function is a function which maps each point t of an interval to one and only one $n \times n$ matrix A_t . Now we call a matrix function A_t to be continuous, differentiable or integrable depending on its i th element that is $a_{ij} t$ are continuous, differentiable or integrable.

So a matrix is said to be continuous, differentiable or integrable provided that each of its term that is a_{ij} must be continuous, differentiable or integrable. So it means that A_t has a property means its component has the same property. Now very often we use the following very important inequality that is norm of c to d , $\| \int_c^d A_t b_t dt \| \leq \int_c^d \| A_t \| \| b_t \| dt$. Here d is some value which is bigger than c assuming that A_t and b_t are continuous function.

Or we can say that in simple sense that for which this integrals are defined. So if we assume that $A_t b_t$ are continuous function, then these integrals are defined and we can talk about this inequality. So this inequality 10 is quite important and this can be proved once we know a

particular norm but this inequality is true in any of the norm here. So most common type of norms which we are going to use in this particular course are given as norm 1 and norm 2.

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Most common norms are $\|A\|_1$ and $\|A\|_2$, defined as

$$\|A\|_1 = \sum_{i,j=1}^n |a_{ij}|, \quad \|A\|_2 = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2}.$$

As all norms are equivalent, so we drop the subscript and denote the norm of the matrix A by $\|A\|$ only.

We can easily check that the matrix norm satisfies the following basic properties:

- (i) $\|A + B\| \leq \|A\| + \|B\|$
- (ii) $\|AB\| \leq \|A\| \cdot \|B\|$ H.W
- (iii) $\|Ab\| \leq \|A\| \cdot \|b\|$

for all matrices $A_{n \times n}, B_{n \times n}$ of complex(real) numbers and vectors $b_{n \times 1} = (b_i)$, here $\|b\| = \sum_{i=1}^n |b_i|$.

And norm 1 of A is defined as summation, i, j from 1 to n modulus of a_{ij} and this is known as one norm and two norm is defined as this norm of A with two norm is given as under root of i, j from 1 to n modulus of a_{ij} square. So this is known as L1 norm, this is known as L2 norm or most of the time it is known as Euclidean norm and here we are using one important property that in finite dimension spaces all the norms are equivalent.

So it means that depending on the requirement, we can use any of the norms and the property which we prove using any of these kinds of norms are say independent of the norm we have used. So it means that as all norms are equivalent, so we drop the subscript and denote the norm of the matrix A by norm of A only and we can easily check that the matrix norm satisfies the following basic properties.

I am not proving in this particular course but you can check that these inequalities are true. So norm of $A+B$ is \leq norm of A + norm of B . Second, norm of AB is \leq norm of A * norm of B and norm of Ab is \leq norm of A dot norm of b and please remember that since we are asking to prove these 3 properties.

So I am assuming that capital A and capital B are square matrix and small b is a $n \times 1$ size vector and norm of b is given as summation $i=1$ to n modulus of b_i . So this is the norm of b is

nothing but absolute sum of its component basically. So we are using this, so this giving you as a homework.

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Existence and Uniqueness for Linear Systems

Consider the linear system

$$y' = A(t)y + F(t) \quad (11)$$

and let the matrix $A(t)$ and the vector $F(t)$ are continuous on an interval I . Then the vector function $f(t, y) = A(t)y + F(t)$ is continuous for (t, y) in D , where D is the strip $\{(t, y) : t \in I, |y| < \infty\}$, and $\frac{\partial f}{\partial y_k} = \text{col} \begin{pmatrix} a_{1k}(t) \\ a_{2k}(t) \\ \vdots \\ a_{nk}(t) \end{pmatrix}$, $(k = 1, 2, \dots, n)$;

where col stands for column, hence $\frac{\partial f}{\partial y_k}$ are also continuous in D for $k = 1, 2, \dots, n$.

Handwritten notes: $X(t)$, $Y(t)$, $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$, $a_{21}x_1 + \dots + a_{2n}x_n$, $a_{n1}x_1 + \dots + a_{nn}x_n$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, $2t = \frac{d^2x}{dt^2}$


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Now come to again the same question that is existence and uniqueness for linear systems. So consider the linear system that is $y' = Ay + F$ of t . Now here I just want to give you one thing that here for this entire course most of the time I use X as a dependent variable and t as an independent variable and we write X_t as same function depending on t and again we can use this notation also in place of X_t , I may write Y_t as the same meaning that is Y is a dependent function depending on the variable t .

So try to understand (12:51) that here $y' = Ay + Ft$ means y is a dependent variable and independent variable is t . So now consider the linear system $y' = Ay + F$ of t and let this matrix function A_t and the vector function F_t are continuous on interval I . Then, the vector function f_t, y that is $A_t y + F_t$ is a continuous function for t, y in D where D is the strip this that is the set of all t, y such that t belongs to the interval I and modulus of y is $< \infty$.

So in this interval your $F_t y$ which is defined as $A_t y + F_t$ is a continuous function and if you look at the partial derivative with respect to this component of y , this we can find out $\frac{df}{dy_k}$ is the columns k th columns of matrix A_t that is $a_{1k} t, a_{2k} t, \dots, a_{nk} t$. So it means that the k th partial derivative that is $\frac{df}{dy_k}$ is given by the column vector $a_{1k} t, a_{2k} t$ up to $a_{nk} t$.

Now since we have already assumed that the matrix function A_t is a continuous function, it means that its components are continuous. So it means that each component is a continuous function. So we can say that $\frac{df}{dy_k}$ is also a continuous function and here this column means this is column. So it means that $\frac{df}{dy_k}$ are also continuous in domain D for k is running from 1 to n . So how we can obtain this let us try to understand.

Here we have $a_{11}y_1 + a_{12}y_2$ and so on $a_{1n}y_n$ and so on $a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n$. Now we want to find out the partial derivative of f with respect to y_k . So here if you look at if you differentiate f_1 with respect to y_k what you will get, you will get a_{1k} . If you look at f_2 , f_2 here I am assuming that $f = f_1$ to f_n . So it means that f you find out, so $\frac{df}{dy_k}$ means it is $\frac{df_1}{dy_k}$ and so on $\frac{df_n}{dy_k}$ right.

So it means that $\frac{df_1}{dy_k}$ will be a_{1k} , $\frac{df_2}{dy_k}$ will be a_{2k} that is written here and $\frac{df_n}{dy_k}$ is given as a_{nk} . So $\frac{df}{dy_k}$ you can obtain from this and we can say that since A_t is continuous, so $\frac{df}{dy_k}$ is also continuous and we also assume that this F_t is continuous so $f(t, y)$ this $f(t, y)$ which we denote as $A_t y + F_t$ is continuous, not only this is continuous, the partial derivative is also continuous.

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Thus (11) has a unique solution $\phi(t)$ passing through any given point (t_0, η) with t_0 in I ; and this solution exists on an interval containing the point t_0 .

For the present case of D , this means that the solution exists on the whole interval I ; the norm of the solution ϕ , is bounded by a constant independent on t and such a bound is called a **priori bound**.

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So this implies that the system (11) which is denoted as $\dot{y} = A_t y + F_t$ has a unique solution $\phi(t)$ passing through any given point (t_0, η) where t_0 is in the interval I where we are assuming that A_t and F_t are continuous functions and this solution exists on an interval containing the point t_0 and for the present case of D this means that the solution exists on the

whole interval I the norm of the solution ϕ is bounded by a constant independent on t and such a bound is called a priori bound.

So it means that we say that if in this case if we assume that this A_t and F_t are continuous for all values of t , then the solution exists for the entire space until unless the solution remains bounded. So if solution remains bounded, then we simply say that solution exists for all values of t or we can say that given any point t_0 , η we have a solution passing through the point t_0 , η . Now let us prove in a proper say vigorous manner whatever we have stated.

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Theorem 8
 If $A(t), F(t)$ are continuous on some interval $a \leq t \leq b$, if $a \leq t_0 \leq b$, and if $|\eta| < \infty$, then the system (11) has a unique solution $\phi(t)$ satisfying the initial condition $\phi(t_0) = \eta$ and existing on the interval $a \leq t \leq b$.

Proof Let $\phi(t)$ be the unique solution satisfying $\phi(t_0) = \eta$, existing on for t on an interval J . To show that this solution exists on the whole interval $a \leq t \leq b$, it suffices to show that $\|\phi(t)\|$ is bounded by a constant independent of t . For t in J , substitution of ϕ into (11) gives

$$\phi'(t) = A(t)\phi(t) + F(t)$$

Integration gives

$$\phi(t) - \phi(t_0) = \int_{t_0}^t A(s)\phi(s)ds + \int_{t_0}^t F(s)ds$$

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That if A_t and F_t are continuous on some interval closed interval a to b , if $a \leq t_0 \leq b$ means this t lying in closed interval a, b and t_0 is also lying in that closed interval and if norm of η is $< \infty$ then the system 11 has a unique solution ϕt satisfying the initial condition $\phi t_0 = \eta$ and existing in the entire interval a to b here. So there is a small difference between this case and the scalar-valued case.

In a scalar-valued, the solution exists in a small neighborhood of t_0 but here solution exists for the entire interval a to b here and this is happening due to the linearity of A_t that $f t, y$ is linear with respect to y , the dependent variable. So let us prove this result and let ϕt be the unique solution satisfying $\phi t_0 = \eta$ existing on for t on an interval J , some interval J . So to show that this solution exists on the whole interval that is closed interval a to b , it is sufficient to show that this norm of ϕt is bounded by a constant independent of t .

So let us take t in J , here we can denote this J as the closed interval a, b and substitute of ϕ into equation number 11, we can have $\phi'(t) = A(t)\phi(t) + F(t)$ and we can integrate directly and we can have this as $\phi(t) - \phi(t_0) = \int_{t_0}^t A(s)\phi(s) ds + \int_{t_0}^t F(s) ds$. So just we have written the differential equation and we simply integrate it.

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By using the initial condition and taking norms, we obtain

$$|\phi(t)| \leq |\eta| + \left| \int_{t_0}^t A(s)\phi(s) ds \right| + \left| \int_{t_0}^t F(s) ds \right|$$

We continue with the argument for $t \geq t_0$; using properties of the norm and the inequality (10), we have

$$|\phi(t)| \leq |\eta| + \int_{t_0}^t |A(s)||\phi(s)| ds + \int_{t_0}^t |F(s)| ds$$

Since

$$|\eta| + \int_{t_0}^t |F(s)| ds \leq |\eta| + \max_{a \leq t \leq b} |F(t)|(t - t_0) \leq |\eta| + \max_{a \leq t \leq b} |F(t)|(b - a) = K_1.$$

Handwritten notes on the slide:
 $\|A+B\| \leq \|A\| + \|B\|$
 $\left\| \int_{t_0}^t A(s)\phi(s) ds \right\| \leq \int_{t_0}^t \|A(s)\| \|\phi(s)\| ds$
 $\leq \int_{t_0}^t \|A(s)\| \|\eta\| ds$

And when we integrate we have here we are assuming that the initial condition $\phi(t_0)$ is given as $\phi(t_0) = \eta$. So using the initial condition, we can write norm of $\phi(t)$ is \leq norm of $\eta +$ norm of $\int_{t_0}^t A(s)\phi(s) ds +$ norm of $\int_{t_0}^t F(s) ds$. So here we are using this thing that norm of $A+B$ is \leq norm of $A +$ norm of B . So this property we are using. So once we have this, we continue with the argument for $t \geq t_0$.

So using the properties of the norm, we can write the norm of $\phi(t) \leq$ norm of $\eta +$ here we use this property that $\int_{t_0}^t$ norm of this $A(s)\phi(s) ds$ is \leq $\int_{t_0}^t$ norm of $A(s)$ norm of $\phi(s)$ and ds . So we are using this property in this second term and we can write this as $\int_{t_0}^t$ norm of $A(s) \times$ norm of $\phi(s) ds +$ norm of $\int_{t_0}^t F(s) ds$. Now we want to find out the bound for $\phi(t)$ which is independent of say this t .

So here we can write we just evaluated this term and this term, so norm of η is finite quantity because it is a fixed number. Now we have to look at the integral $\int_{t_0}^t F(s) ds$. So this I can write this as since F is a continuous function on the closed interval a to b , then maximum will exist and we can take out that maximum part out. So maximum F from a to b $F(t)$. Now in integral we have $\int_{t_0}^t ds$ which I am writing as $t - t_0$ and since $t \geq t_0$, so I can write it here.

So it means that if this since this is finite quantity, we can replace this quantity norm of $\eta + \text{maximum of } F_t \text{ over the interval } a \text{ to } b$ *this $t-t_0$ is bounded by $b-a$ and we can denote this whole quantity as K_1 quantity. So K_1 is a constant which is not depending on t .

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and letting

$$K_2 = \max_{a \leq t \leq b} |A(t)|, \checkmark$$

We have

$$\int_{t_0}^t |A(s)| |\phi(s)| ds \leq \max_{a \leq t \leq b} |A(t)| \left(\int_{t_0}^t |\phi(s)| ds \right) = K_2 \int_{t_0}^t |\phi(s)| ds$$

this inequality can be written as

$$\checkmark |\phi(t)| \leq K_1 + K_2 \int_{t_0}^t |\phi(s)| ds \checkmark$$

where K_1 and K_2 are non negative constants and are independent of t , but that they do depend on $A(t)$, $F(t)$, a , and b .

And we can also let K_2 in a similar manner because A_t is also assumed to be a continuous function that closed interval. So continuous function defined on closed interval will assume its maximum value. So assuming K_2 as that maximum value of A_t , we can simplify t_0 to t norm of A_s *norm of ϕ s ds as maximum t from a to b A_t , this we can take it out and t_0 to t ϕ of s ds .

So this value is replaced by K_2 and we can write K_2 times t_0 to t ϕ s ds . So if we use these bounds then we can rewrite our inequality as norm of ϕ $t \leq K_1 + K_2$ times t_0 to t norm of ϕ s ds . Here K_1 and K_2 are nonnegative constant and are independent of t , nonnegative is followed from this thing that it is maximum values of nonnegative value. So it has to be nonnegative.

So norm of ϕ t is $\leq K_1 + K_2$ t_0 to t norm of ϕ s ds and K_1 and K_2 are nonnegative constants and independent of t but of course this K_1 and K_2 are depending on this matrix A_t , F_t and the initial and final point a and b here. Now here to find out the bound of norm of ϕ t , we use the Gronwall inequality and here in Gronwall inequality this K_1 is nonnegative, K_2 is nonnegative.

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By Gronwall inequality, we obtain

$$|\phi(t)| \leq K_1 \exp(K_2 |t - t_0|) \leq K_1 \exp(K_2(b - a))$$

This shows that $|\phi(t)|$ is bounded by a constant. Therefore, $\phi(t)$ exists on the entire interval $a \leq t \leq b$, and this completes the proof.

We also note that if $A(t)$ and $F(t)$ are continuous on $-\infty < t < \infty$, the above arguments apply to every **finite** subinterval; of course, in this case the solution $\phi(t)$ need not remain bounded as $t \rightarrow \pm\infty$. The same remark applies if $A(t)$ and $F(t)$ are continuous on $a < t < b$, but not necessarily on $a \leq t \leq b$.

So this can be simplified to this thing, the norm of $\phi(t)$ is $\leq K_1$ times exponential of K_2 modulus of $t-t_0$ and this modulus of $t-t_0$ is further bounded by $b-a$. So we can say that norm of $\phi(t)$ is $\leq K_1$ times exponential of K_2 times $b-a$ and you can see that here $\phi(t)$ is bounded by a constant. Therefore, $\phi(t)$ exists on the entire interval a to b and this completes the proof and so what we have proved here that if $A(t)$ and $F(t)$ are continuous function on the closed interval a to b , then the solution exists for the entire interval a to b .

And we have shown that this is happening because that the solution is bounded by a constant, solution remains finite throughout the interval. So now we also note that if $A(t)$ and $F(t)$ are continuous on $-\infty$ to ∞ , the above argument whatever we have presented apply to every finite subinterval whether the $A(t)$ is open, whether it is open and closed, all these things we can assume.

The only thing we may not get is that your solution may not remain bounded as t tending to $+\infty$. So it means that solution will exist for the entire interval but we do not have the guarantee that $\phi(t)$ need not remain bounded. So it means that same remark applies if $A(t)$ and $F(t)$ are continuous on open interval a to b but not necessarily on closed interval a to b .

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Corollary 9

If $A(t)$, $F(t)$ are continuous on an interval I , closed or open, finite or infinite, and if $t_0 \in I$, $|\eta| < \infty$, then the equation (11) has a unique solution $\phi(t)$ satisfying the initial condition $\phi(t_0) = \eta$ and existing on I .

$$\begin{cases} X' = A(t)X + F(t) \\ X(t_0) = \eta \end{cases} \quad (I)$$

So now we have a following corollary that if A_t and F_t are continuous on an interval I whether it is closed or open, finite or infinite and if t_0 belongs to that interval I and the eta norm of eta is finite, then the equation 11 has a unique solution. What is equation 11? $X' = A_t X + F_t$ of t and $x(t_0) = \eta$. So this equation number 11 has a unique solution satisfying the initial condition that is $\phi(t_0) = \eta$ and existing on the interval I . So this is the corollary of the previous result here.

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For autonomous systems, following existence and uniqueness theorem may be stated.

Theorem 1

(Existence-uniqueness theorem) There exists one, and only one, solution of the initial-value problem

$$\frac{dx}{dt} = Ax, \quad x(t_0) = x^0 = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix}. \quad (12)$$

Moreover, this solution exists for $-\infty < t < \infty$.

Now moving on now if we have A_t independent of t so that is A_t is a constant function, then we call the corresponding system as autonomous system. Then, so from the previous theorem we can write down the following existence and uniqueness theorem may be stated that they exists one and only one solution of the initial-value problem $dx/dt = Ax$ and $x(t_0) = x_0$ where x_0 is given by x_1^0 to x_n^0 .

And this solution exists for the all values of t from –infinity to infinity that we have already seen in the case of non-autonomous system that if At is continuous for all t then solution will exists for the entire interval and as a consequence of the previous theorem, we can write down the existence and uniqueness theorem for autonomous case as well. So far we have discussed the say conditions by which we can guarantee the solution.

And in fact guarantee the existence and uniqueness of the solution. Now we have given the conditions for linear system. Now once we know that this solution exists we want to know the structure of the solution set, it means that how many solutions we have. Once we have a solution, we want to know whether we have one solution, two solution, three solution and how many solutions we have and how we can obtain that solution.

So far we have done only the existence and uniqueness part. Now we have to discuss whether we have how the solution look like and how to find out the solution. So we start with the simplest linear system that is linear homogenous system that is $x' = Ax$.

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Once we know that a solution exists, we want to know the structure of solutions of the linear system. So we start with the linear homogenous linear system $x' = A(t)x$.

$$\frac{dx}{dt} = A(t)x, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

and $A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}.$ (13)

Here we assume that the matrix $A(t)_{n \times n}$ is a continuous matrix over an interval $I = [a, b]$.

Handwritten notes:
 $x(t) \Rightarrow C x(t)$
 $x(t) \Rightarrow x(t) \cdot y(t)$
 $A(t) = A_{ij}(t)$
 $x' = Ax \quad (Cx)' = Cx' = CAx = ACx$

So first we deal with this linear homogenous system and with the help of this we try to say something about the solution set. So consider the linear system $dx/dt=Ax$ where x is a vector x_1 to x_n and At is a matrix of terms $a_{ij} t$, so matrix At is given as this. Here we assume that matrix At is a continuous matrix over an interval I which is closed interval a, b. So we have already discussed that this system has a unique solution which exists for the entire interval a, b.

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Remark 1
Since the system (13) is a linear system so we can easily check that if $x(t)$ and $y(t)$ are any two solutions of (13). Then

- 1 $cx(t)$ is a solution, where c is a constant, and
- 2 $x(t) + y(t)$ is a solution,

Let V denotes the vector space of all the solutions of the linear system (13), we want to know the dimension of V and a basis of V .

Definition 10
A set of vectors u_1, u_2, \dots, u_m is linearly dependent if there exist scalars $c_i, i = 1, \dots, m$ not all zero, such that the linear combination $\sum_{i=1}^m c_i u_i = 0$.

Handwritten notes: $x' = Ax$
 $0 \in$

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Now first remark we want to consider is that since the system is linear system we can easily check that if $x(t)$ and $y(t)$ are any two solutions of (13), then $cx(t)$ is also a solution. So it means that if we have say $x(t)$ is a solution then we want to claim that c of $x(t)$ will also be a solution and also we want to claim that if $x(t)$ and $y(t)$ are two solutions, then the $x(t)+y(t)$ will also be a solution so how we can prove?

This is very easy to prove, $x(t)$ is a solution means that $x' = Ax$. Now we want to claim that cx is also a solution, so we can write down, we want to show that cx will also satisfy the same differential equation. So you find out derivative of this. So cx , so this is cx dash, so this x dash is A of x and it is linear, so I can write it x . So here we can say that the cx will also satisfy the same homogenous system, homogenous linear differential equation number 13.

So it means that cx will also be a solution. Similarly, we can prove that $x(t)+y(t)$ is also a solution and if we can prove these two conditions, then we can say that the set of the solution of $x' = Ax$ forms a vector space. If you look at 0 is already belongs to this because for any homogenous system 0 solution is a trivial solution which always exists. So corresponding to linear homogenous system 0 is always there.

And if we have one nontrivial solution that is x , then constant multiple of x will also be a solution and if we have two solutions x and y then the addition will also serve as a solution. So it means that we can say that the solution set of homogenous linear system forms a vector

space. So let V denote the vector space of all the solutions of the linear system (13) and we want to find out the dimension of V and a basis of V .

So once it is a vector space, we want to find out the dimension and the basis, basically the dimension and a particular basis of the vector space V . So to define basis and dimension, we need to know what do we mean by say linearly independent and dependent vector. So we define a set of vectors u_1 to u_m is said to be linearly dependent if there exist scalars say c_1 to c_m not all 0 such that the linear combination summation $i=1$ to m $c_i u_i=0$.

So in this case, when we have linear combination $\sum c_i u_i=0$ with some of the constants are nonzero then we call vector c_1 to u_m are linearly dependent and if this linear combination is 0 implies that all the constants are 0, we call u_i 's are linearly independent vectors.

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Definition 11
A set of vectors u_1, u_2, \dots, u_m is linearly independent if it is not linearly dependent.

Theorem 2
If the matrix $A(t)_{n \times n}$ is continuous on an interval I , then on the interval I , the solutions of the system

$$x' = A(t)x \quad (14)$$

forms a vector space of dimension n .

To show that the dimension of vector space is n , we have to construct a basis for vector space consisting of n linearly independent vectors in V i.e. of n linearly independent solutions of (14) on I . Let t_0 be any point of I and let e_1, e_2, \dots, e_n be any n linearly independent points in Euclidean n -space E_n .

So it means that a set of vectors u_1 to u_m is linearly independent if it is not linearly dependent. It means that if you form a say linear combination of $\sum a_i u_i=0$, then the only condition we must get is that $a_i=0$ for each $i=1, 2$ and m . So if we have this condition holds, then we say that vectors u_1 to u_m are linearly independent.

Now next theorem we want to show that if the matrix A_t is continuous on an interval I , then on the interval I , the solution of the system $x' = A_t x$ forms a vector space of dimension n . So far we have proved that it forms a vector space, what we want to prove is that the dimension is n . So to show that the dimension of vector space is n , we have to construct a basis. A basis is basically what?

Is the set of linearly independent vectors which span the entire vector space? Our claim is that the vector space is of dimension n means we have to find out n linearly independent vectors which spans the entire vector space. So it means that for vector space consistent of n linearly independent vectors in V that is of n linearly independent solutions of 14 on interval I .

So here we want to find out that corresponding to this homogenous system we have n linearly independent solution it means that ϕ_1 to ϕ_n are n linearly independent solution of this linear homogenous system. So this is our required thing. So let t_0 be any point in interval I and let e_1 to e_n be any n linearly independent points in Euclidean space R^n , E_n basically here I am denoting it as.

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By existence and uniqueness theorem the system (14) possesses n solutions z_1, z_2, \dots, z_n , each of which exists on the entire interval I , and each solutions satisfies the initial condition

$$z_j(t_0) = e_j \quad (j = 1, 2, \dots, n). \quad (15)$$

We first show that solutions z_1, z_2, \dots, z_n are linearly independent on I . Suppose they are not. Then there exists constant a_1, a_2, \dots, a_n not all zero, such that

$$a_1 z_1(t) + a_2 z_2(t) + \dots + a_n z_n(t) = 0 \quad \forall t \in I. \quad (16)$$

In particular, putting $t = t_0$, and using the initial conditions 15, we have

$$a_1 e_1 + a_2 e_2 + \dots + a_n e_n = 0, \quad \checkmark \Rightarrow a_i = 0 \quad \forall i=1, \dots, n$$

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So by existence and uniqueness theorem, the system 14 has n solutions called z_1 to z_n each of which exists on the entire interval I and each solution satisfy the initial condition that $z_j(t_0) = e_j$.

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Definition 11

A set of vectors u_1, u_2, \dots, u_m is linearly independent if it is not linearly dependent.

$$\sum a_i u_i = 0 \Rightarrow a_i = 0 \quad \forall i=1, 2, \dots, m$$

Theorem 2

If the matrix $A(t)_{n \times n}$ is continuous on an interval I , then on the interval I , the solutions of the system

$$x' = A(t)x$$

forms a vector space of dimension n .

$$\begin{aligned} & n \quad \varphi_1, \dots, \varphi_n \\ & x = A(t)x \quad (14) \\ & x(t_0) = e_j, \quad \forall j, z_j \end{aligned}$$

To show that the dimension of vector space is n , we have to construct a basis for vector space consisting of n linearly independent vectors in V i.e. of n linearly independent solutions of (14) on I . Let t_0 be any point of I and let e_1, e_2, \dots, e_n be any n linearly independent points in Euclidean n -space E_n .

So what we are actually doing, we first find out say n linearly independent vectors in \mathbb{R}^n that Euclidean space \mathbb{R}^n . Then, corresponding to these n linearly independent vectors, you consider this initial-value problem that is $x' = Ax$ with $x(t_0) = e_j$. So for each j , we have a solution because that solution is guaranteed from the theorem of existence and uniqueness theorem that for each j we have a solution.

Call that solution as z_j , so that solution we call this as z_j . So for j from 1 to n we have n solution that is z_1 to z_n are solution of the initial-value problem which satisfy the following condition that $z_j(t_0) = e_j$ for j from 1 to n . Now we want to show that the solutions are linearly independent and for that let us assume that they are not linearly independent and they exist constant a_1 to a_n not all 0s and such that linear combination of $a_i z_i$ $i=1$ to n is $=0$.

So and when you write it $a_1 z_1(t) + a_2 z_2(t) + \dots + a_n z_n(t) = 0$ for all t belongs to I . Now this is important right. So it means that this is true for all t and a_1 to a_n 's are not all 0. So in particular if it is true for all t , it is true for $t=t_0$ also. When you put $t=t_0$ in equation number 16, what you will get? $a_1 z_1(t_0) + a_2 z_2(t_0) + \dots + a_n z_n(t_0) = 0$. Now we already know $z_1(t_0)$ is e_1 , $z_2(t_0) = e_2$ and so on.

So $a_1 e_1 + a_2 e_2 + \dots + a_n e_n = 0$ and we know that since e_1 to e_n 's are linearly independent vectors of Euclidean space then this forces us to take $a_i = 0$ for all $i=1$ to n . So it means that we have assumed in beginning that a_i 's are not all 0 but here we get that a_i has to be 0 because e_1 to e_n 's are n linearly independent vectors in Euclidean space. So it means that here we have a contradiction.

It means that the assumption which we have assumed earlier that z_1 to z_n are not linearly independent is not correct and hence we can say that z_1 to z_n are linearly independent solutions. So once this linearly independent essay is over, now we want to show that this z_1 to z_n forms a spanning set for the vector space.

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which is impossible unless a_1, a_2, \dots, a_n are all zero because it contradicts the assumed linear independence of e_1, e_2, \dots, e_n .

Now we show that these n linearly independent solutions of 14 span vector space V . Compute the value of the solution ψ at t_0 and let $\psi(t_0) = e$. Since the constant vectors e_1, e_2, \dots, e_n are linearly independent in Euclidean n -space E_n , they form a basis for E_n and there exists unique constants c_1, c_2, \dots, c_n such that the constant vector e can be written as

$$e = c_1 e_1 + c_2 e_2 + \dots + c_n e_n.$$

$X' = A(t)X$
 $X(t_0) = e \in E_n$
 $\psi(t)$



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So we saw that these n linearly independent solution of 14 span the whole vector space V . So compute the value of the solution ψ at t_0 and let $\psi(t_0) = e$. So let us assume that there exists a solution ψ of the initial-value problem, you have $x' = Ax$ and with the initial condition that this x of $t_0 =$ some say let us call this as e . So let us say that solution is given by ψ of t that is $\psi(t)$ satisfy this differential equation as and $\psi(t_0) = e$.

Now we want to claim that this $\psi(t)$ can be written as linear combination of $z_i(t)$, for that we simply say that since the constant e_1 to e_n are linearly independent vectors in Euclidean space E_n . So it means that in particular I can write e which is a member in Euclidean space E_n and e_1 to e_n forms a basis of E_n , so it means that this e can be written as linear combination of e_i 's as follows that $e = c_1 e_1 + c_2 e_2 + \dots + c_n e_n$.

So it means that I can find out these constant c_1 to c_n such that e can be written as summation $c_i e_i$.

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Now consider the vector

$$z(t) = c_1 z_1(t) + c_2 z_2(t) + \dots + c_n z_n(t).$$

Clearly, $z(t)$ is a solution of (14) on I . Moreover the initial value of z is

$$z(t_0) = c_1 e_1 + c_2 e_2 + \dots + c_n e_n = e.$$

Therefore $z(t)$ and $\psi(t)$ are both solution of (14) on I with $z(t_0) = \psi(t_0) = e$.
 Therefore by uniqueness part of existence and uniqueness theorem $z(t) = \psi(t)$ for every $t \in I$ and the solution is expressed as the unique linear combination

$$\psi(t) = c_1 z_1(t) + c_2 z_2(t) + \dots + c_n z_n(t) \text{ for every } t \text{ on } I.$$

$x' = A(t)x$

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Now using this constant our claim is you consider a vector $z(t) = c_1 z_1(t) + c_2 z_2(t) + \dots + c_n z_n(t)$. Now look at this vector $z(t)$ since z_1 to z_n each is a solution of the system $x' = A(t)x$, so it means that $z(t)$ is also a solution of this differential equation. So it means that $z(t)$ is a solution of (14) on interval I . Now not only it is a solution, the value of $z(t_0)$ we can calculate this as $c_1 z_1(t_0) + c_2 z_2(t_0) + \dots + c_n z_n(t_0)$.

And we can already say that $z_1(t_0) = e_1$ and $z_2(t_0) = e_2$ and so on, so $z(t_0)$ we can calculate that it is coming out to be e . So it means that $z(t)$ which is given by this linear combination is a solution which satisfy the initial condition $z(t_0) = e$. So it means that $z(t)$ and we have already assumed that the $\psi(t)$ both have the solution of (14) that is $x' = A(t)x$ and both satisfy the same initial condition that is $z(t_0) = \psi(t_0) = e$.

But this cannot happen because we know that we have a unique solution of any initial-value problem. So it means that by existence and uniqueness part $z(t)$ has to be identically $\psi(t)$. So if $z(t)$ is identically $\psi(t)$, then I can replace this $z(t)$ here by $\psi(t)$. It means that $\psi(t)$ can be written as linear combination of $z_i(t)$ as $c_1 z_1(t) + c_2 z_2(t) + \dots + c_n z_n(t)$. So it means that any solution of $x' = A(t)x$ can be written as linear combination of the $z_i(t)$.

So it means that these $z_i(t)$ are n linearly independent vectors and also forms a spanning set for the solution set of $x' = A(t)x$ and hence we can say that the solution set of $x' = A(t)x$ which we denote as V contains 0 vectors, forms a vector space and its dimension is given as n . So it means that the solution space is a vector space of dimension n that is we have discussed here.

So with this theorem what we have proved here that the solution set of homogenous linear system of differential equation forms a vector space and the dimension is n . So this we have done by constructing a basis of n element which is linearly independent and forms a spanning set. Now here I just assume that you can find out these n linearly independent Eigen vectors with the properties.

Now we really want to know how to find out these n linearly independent solution of $\dot{x} = Ax$ and once we know that n linearly independent solutions of this then any solution of this can be written in terms of vector. So I stop here our discussion and will continue the study of the system of linear homogenous equation in next class and in next class we try to find out the solution of linear homogenous system in a particular.

So far we have done that there exists a solution and the solution set will form a vector space but we have not given a method how to find out a given solution which satisfy the given initial condition, so that we will continue in next lecture. So here we stop. Thank you very much for listening us. Thank you.