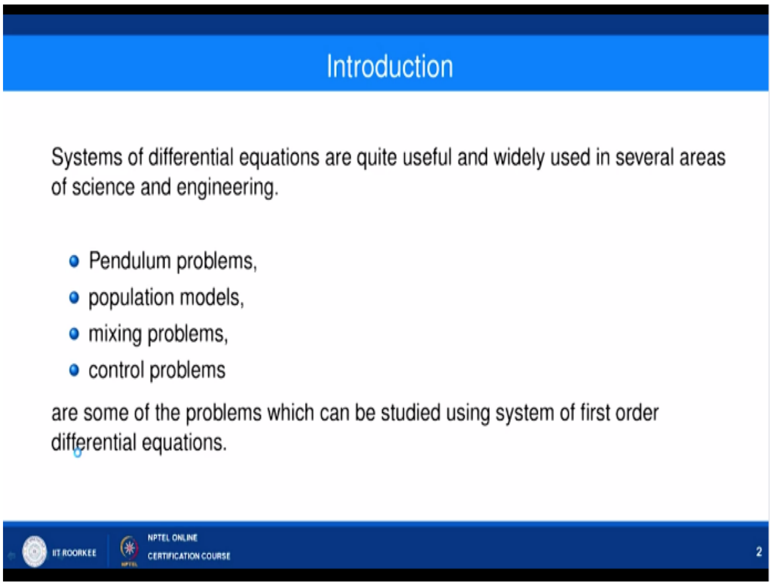
Dynamical Systems and Control
Prof. D. N. Pandey
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture - 05
Linear Systems – I

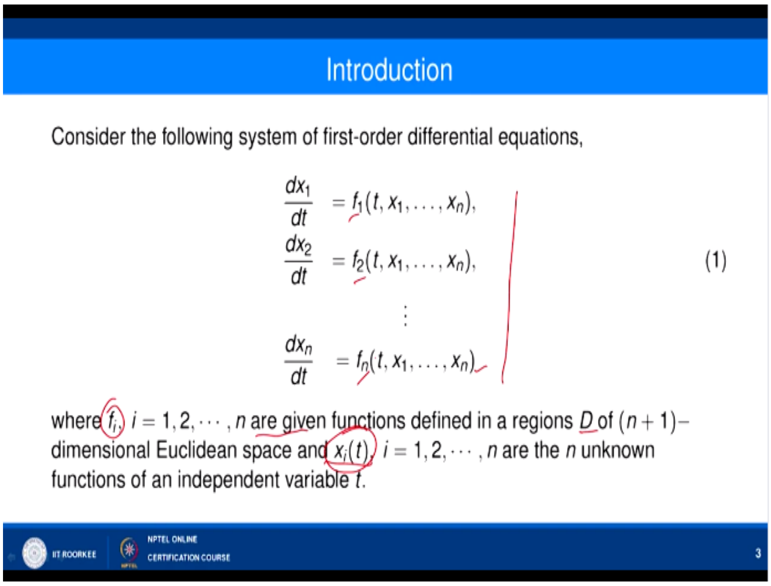
Hello friends. Welcome to this lecture. In this lecture, we will start our discussion on linear system and its properties, existing uniqueness and some properties related to linear system. So first let us see that the system of differential equations are quite useful and widely used in several areas of science and engineering.

(Refer Slide Time: 00:49)



And they are some common which we have already discussed that is some problem related to pendulum problem, population models, mixing problem; mixing means we have tank mixing problem, fluid mixing problem, control problems. These are some of the problems which can be studied using system of first-order differential equations.

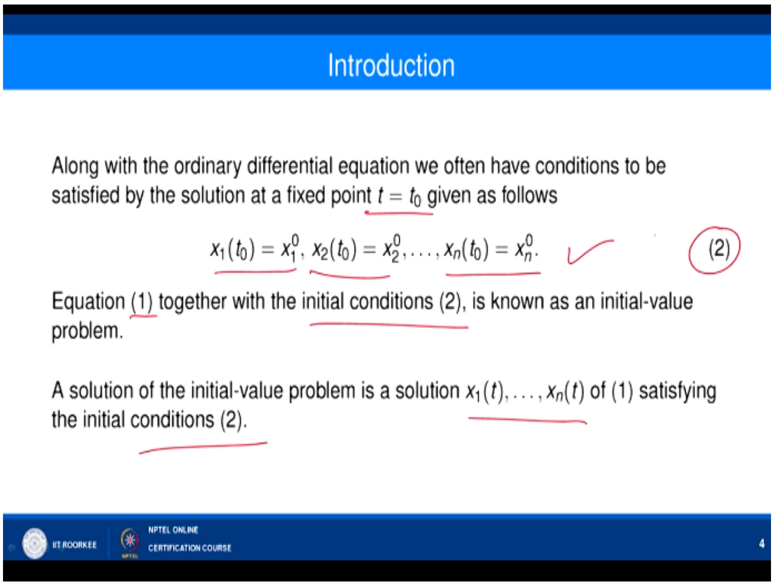
(Refer Slide Time: 01:18)



Now in general we can define the first-order differential equation system as follows. So consider the following system of first-order differential equation, dx1/dt=f1 t, x1 to xn, dx2/dt=f2 t, x1 to xn and so on dxn/dt=fn t, x1, xn here. Now here these fi's are i=1 to n are given functions defined in a region D of n+1 dimensional Euclidean space and these xi t, i from 1 to n are the n unknown functions of an independent variable t.

So our aim is like to find out these xi's such that it satisfy the equation number 1 for given f1, f2 and fn.

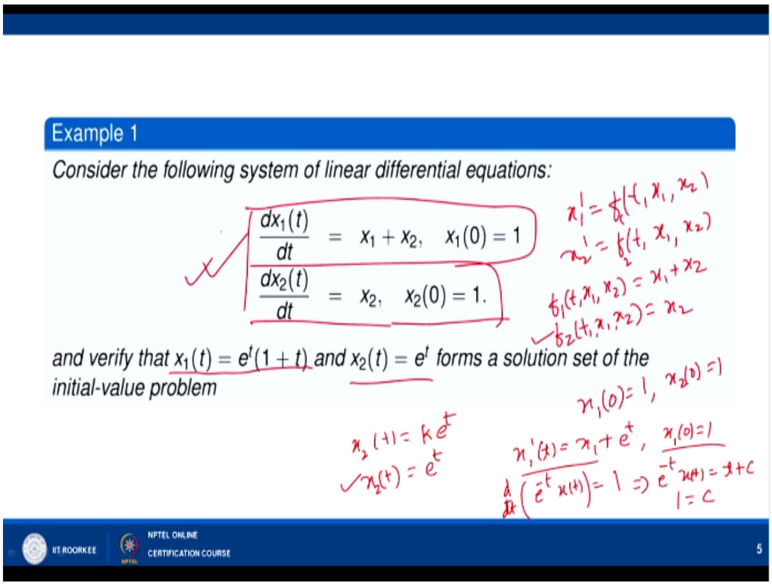
(Refer Slide Time: 02:18)



So most of the time along with the ordinary differential equation, we often have conditions to be satisfied by the solution at a fixed point say t=t0 and which can be written like this that x1 t0 is=x1 0, x2 t0=x2 0 and so on xn t0=xn 0. So equation 1 means the differential equation 1 along with the conditions which is defined at t=t0 generally called as initial conditions. So equation 1 together with the initial condition 2 is known as an initial-value problem.

And in a (()) (03:03) language a solution of the initial-value problem is the solution x1 to xn t of 1 satisfying the initial condition 2. So it means that roughly I can say that the solution of the initial-value problem is n unknowns x1 to xn says that it satisfy the equation number 1 along with the condition given in 2 that is x1 t0=x1 0, x2 t0=x2 0 and so on.

(Refer Slide Time: 03:31)



So now let us consider some example to which we try to understand the concept of system of differential equations. So consider the following system of linear differential equations, dx1/dt=x1+x2, x1 0=1 and dx2 t/dt=x2, x2 0=1 here. So here it is similar to your problem x1 dash=f t, x1, x2 and x2 dash=f t sorry f1 and f2 x1 and x2. So here your f1 of t, x1 and x2 is given by x1+x2 and f2 t, x1 and x2 is given by say x2 here.

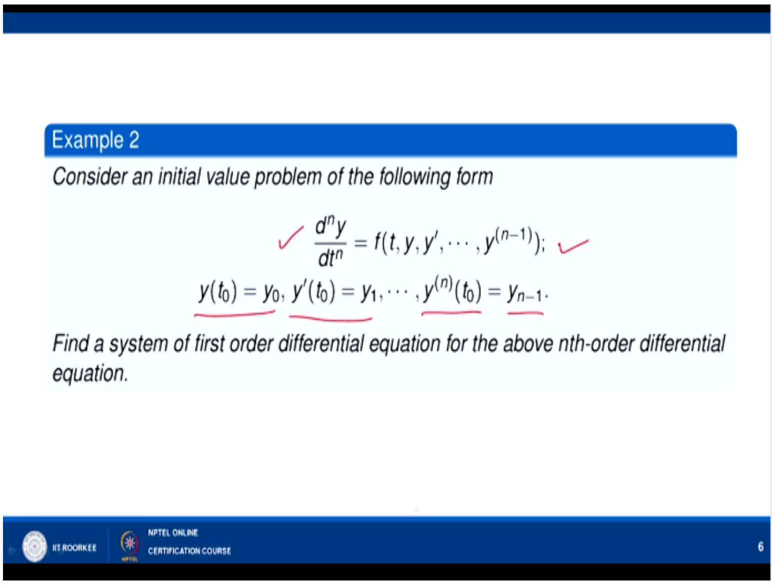
An initial condition is given at the point 0 that is x1 0=1 and x2 0=1. Now here if you look at this system is defined and if you look at the second system that dx2/dt is=x2, so it means that the function f2 is not depending on the variable x1 and so we can easily solve this system this linear equation dx2/dt=x2 provided that x2 0=0 and you can find out that x2 t is=constant times e to power t and the condition x2 0=1 will fix the value k and it is given as e to power t.

So x2 t is given as e to power t and once your x2 t is given to you, you can use the first equation that is dx1/dt=x1+x2 and you can find out the value of x1 also. So here we can say that x1 dash t is=x1+x2 is given as et and x1 0=1. If you look at this is nothing but a linear differential equation, scalar linear differential equation and you can simply that is you can write it d/dt as e to power –t x of t is=1.

What you can do? You just multiply by e to power –t which is the integrating factor of this part x1 dash t-x1 and we can simplify and we can write this as e to power –t x of t=t+some constant let us say call c and the condition is given that x1 0=1 if you use the condition then it is nothing but 1=c, so it means that c is fixed now. So we can simplify this and we can write down that x1 t=e to power t times 1+t and x2 t=e to power t is the solution of this system of linear differential equation.

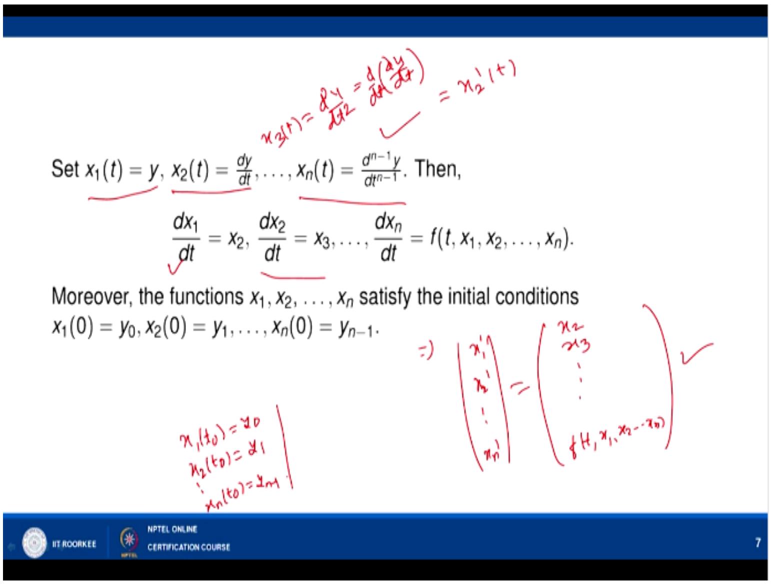
So here we can say that the given system of linear differential equation is having a solution x1 t=e to power t 1+t and x2 t=e to power t and we can call these as that x1 and x2 forms a solution set of the initial-value problem given as this.

(Refer Slide Time: 06:42)



Now coming to example number 2 which is very important in the sense that it gives you the way that any differential equation of nth order can be reduced to a system of first-order differential equation. So here the differential equation is given as this, the dny/dtn=f t, y, y dash up to yn-1 along with the initial condition that is y t0=y0, y1 dash t0=y1 and yn t0=yn-1. So this is the nth order differential equation and we want to convert this to a system of first-order differential equation and let us see how we can do this.

(Refer Slide Time: 07:32)

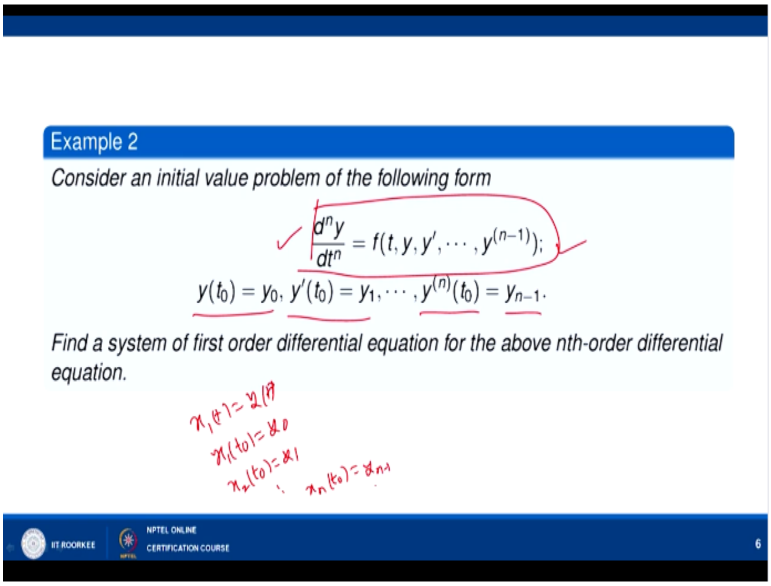


So for converting it let us assume some n variables and these n variables we can say that we can identify that this is nth order differential equation, so we have to choose n number of variables and let us choose n number of variables like x1 t=y, x2 t=dy/dt and xn t=dn-1/dt n-1. Now using the definition of these variables, we can simplify that x1 t is=y1 and x2 t=dy/dt, so if you combine this we can say that x2 t is=dx1/dt.

So first differential equation we can write it as dx1/dt=x2 and similarly we can look at x3 t, x3 t is=d2y/dt square and we can simply say that this is nothing but d/dt of dy/dt and dy/dt is x2, so we can say that your x3 is x2 dash t. So we can write dx2/dt=x3 and so on in this way we can say that dxn/dt is=f t, x1, x2 and so on. So it means that here we can write down our equation like x1 dash, x2 dash up to xn dash.

And this I can write it here as say x2, x3 and so on up to f t, x1, x2 and so on. So it means that this is the differential equation system of a sort of differential equation.

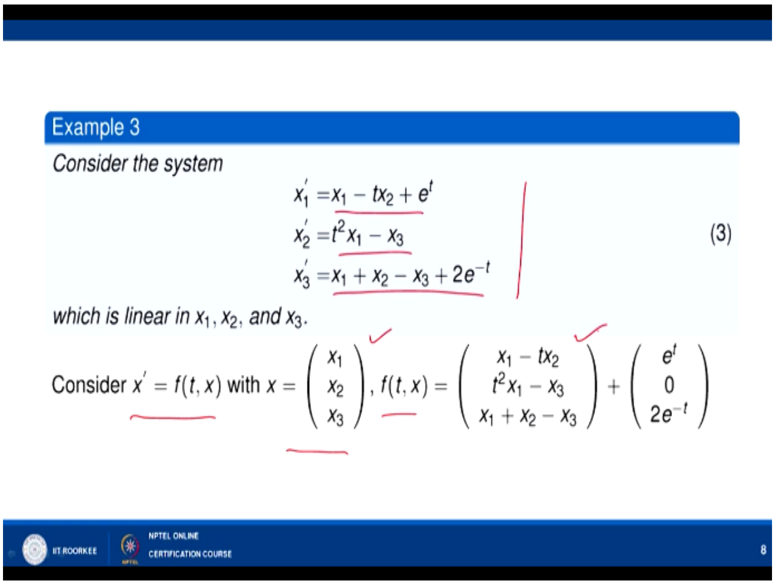
(Refer Slide Time: 09:11)



And initial condition also we can find out, so initial conditions are given as y t0=y0, y dash t0=y1 and yn t0=yn-1. So since we have assumed that x1 t is=y, so we can say that y of t, so we can say that x1 of t0 is=y0. Similarly, we can say that x2 t0=y1 and so on xn t0 is given as yn-1. So it means that along with the system of differential equation, we can get our initial condition as x1 t0=y0 and x2 t0=y1 and up to xn t0=yn-1.

So this is the system of differential equation along with the initial condition defined like this. So it means that this example gives us the idea that every nth order differential equation, ordinary differential equation can be reduced to a system of first-order differential equation in this way in the way which we have just discussed.

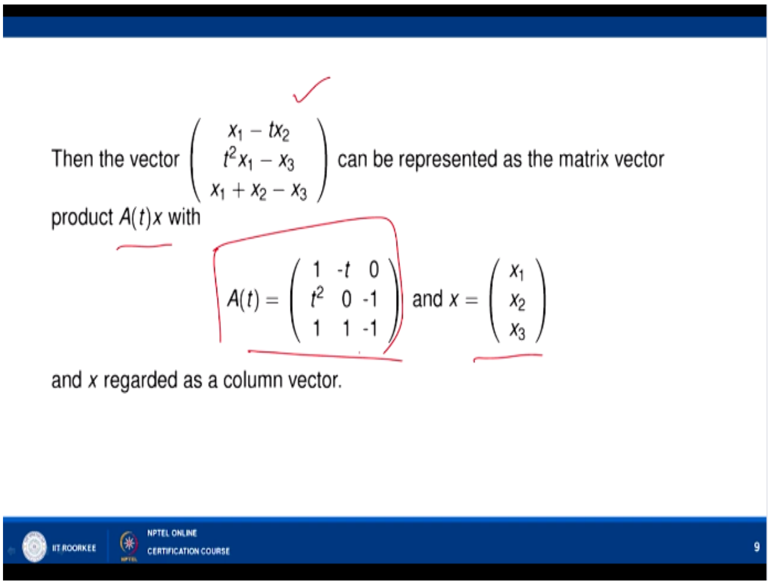
(Refer Slide Time: 10:17)



Now moving onto example 3 and here we consider the system x1 dash=x1-tx2+e to power t, x2 dash=t square x1-x3 and x3 dash=x1+x2-x3+2e to power –t. Now we want to form the system of first-order differential equation from this and here we can say that each system here this is linear in terms of x1, x2 and x3. So we can form this equation number 3 as follows, x dash=f t, x, we can look at the equation number 3 as x dash=f t, x where x is defined as a vector x1, x2, x3.

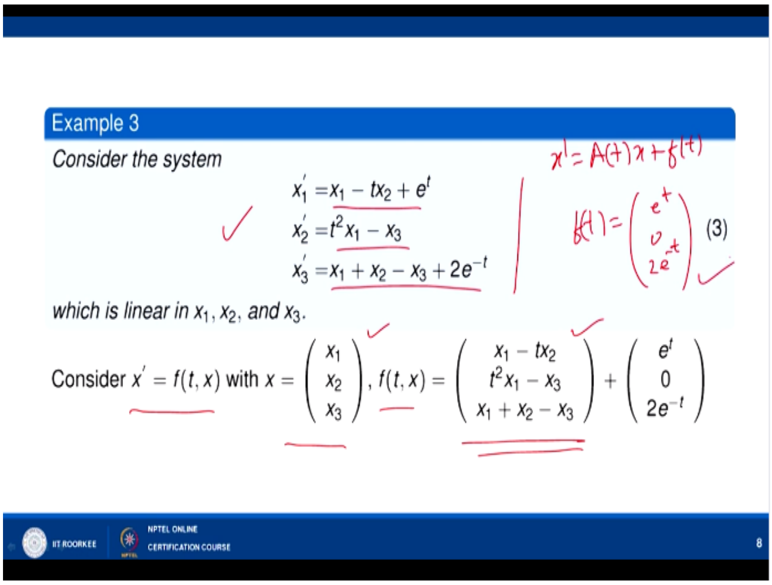
And f t, x we can define as x1-tx2 t square x1-x3 x1+x2-x3+e to power t, 0, 2e to power –t. So it means that we can write this equation number 3 as x dash=f t, x where x is a vector given like this and f t, x is a vector given like this.

(Refer Slide Time: 11:22)



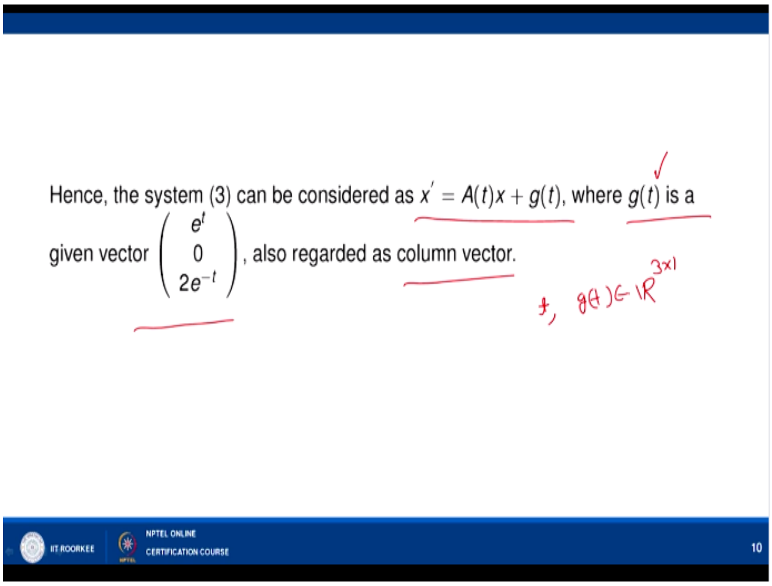
So it means that this vector the vector given here can be further simplified and you can write down this vector x1-tx2 t square x1-x3 and x1+x2-x3 as At x where At is given as 1, –t, 0; t square, 0, -1; 1, 1, -1 and x is given like this.

(Refer Slide Time: 11:48)



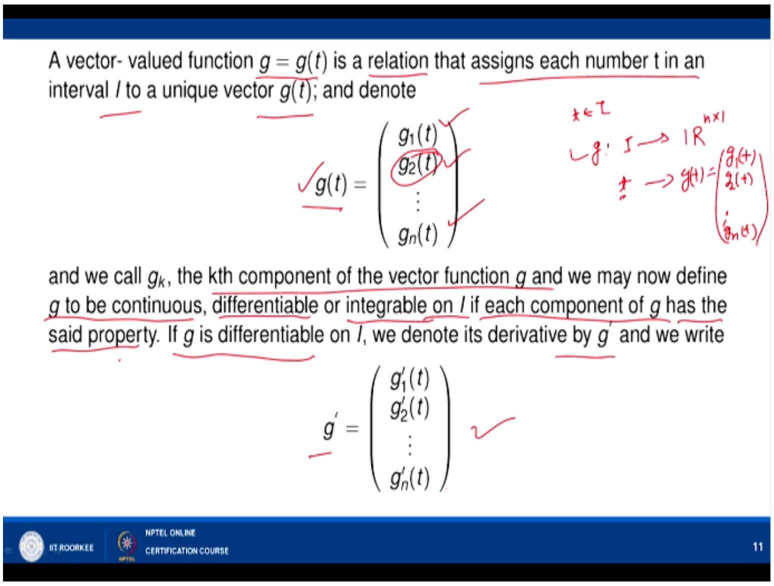
So it means that this system I can write out equation number 3, I can write down this as x dash=At x+say f of t like this where f of t is given by e to power t, 0, 2e to power –t. So it means that the system given as equation number 3 can be reduced to x dash=At x+ft where ft is given like this and At is defined as follows and here your x is regarded as column vector.

(Refer Slide Time: 12:27)



And the system 3 as we have pointed out is written as x dash=At x+gt where gt is a given vector e to power t, 0, 2e to power –t and it is also regarded as a column vector. Now if you look at gt is what basically, here gt is a vector-valued function, so it means what, that for each t this e to power t gives you a real value and 2e to power –t gives you a real value. So it means that for each t gt belongs to R3 x 1. So it means that for each t, gt gives you a vector in 3 x 1, so gt gives you a 3 x 1 vector for each t.

(Refer Slide Time: 13:17)



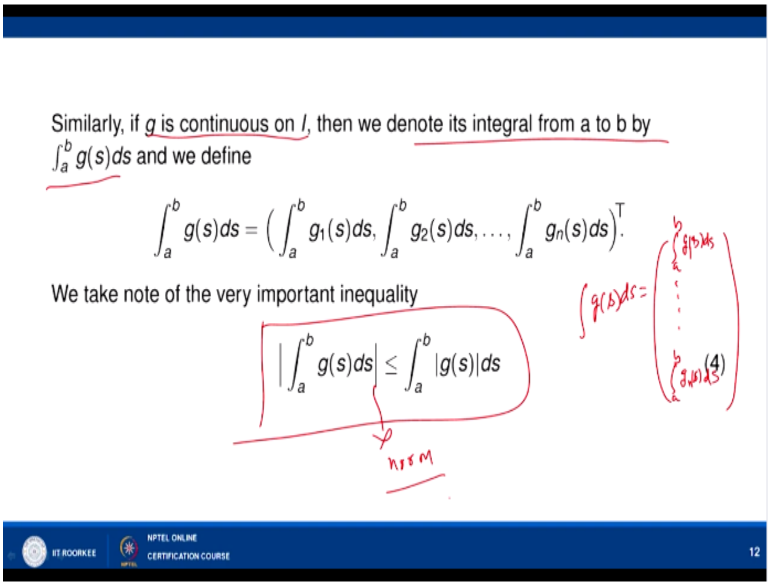
So in general we can define like this. A vector-valued function g=gt is a relation that assign each number t in an interval I to a unique vector gt. So it means that it is a map from say an interval which belongs to so t belongs to I, so it means I can write g as from I to say Rn x 1 here. So for given t, you can map to gt which is denoted as g1 t, g2 t and so on gn t.

So basically here vector-valued function g is defined as a map which is a map defined from I to Rn x 1, for given any t it gives you a vector defined as gt like g1 t, g2 t and gn t and we call like its components are the kth component of the vector function g and we may define the continuity of these vector-valued function continuity, differentiability and integrability of these vector-valued function in terms of its component.

So we can say that g is continuous, differentiable or integrable on an interval I if each component of g means g1 t, g2 t, and gn t has the said property. It means that if g is said to be continuous, if each component is continuous, g is said to be differentiable. If each component is differentiable, g is said to be integral if each component is integrable. Now we can define if g is differentiable, we can denote the derivative of g by g dash and it can be given as g1 dash t, g2 dash t and gn dash t.

So it means that the property continuity, differentiability and integrability is given in terms of component and we can find out the derivative of vector-valued function g in terms of its component and it is given by this that g dash=g1 dash t g2 dash t up to gn dash t.

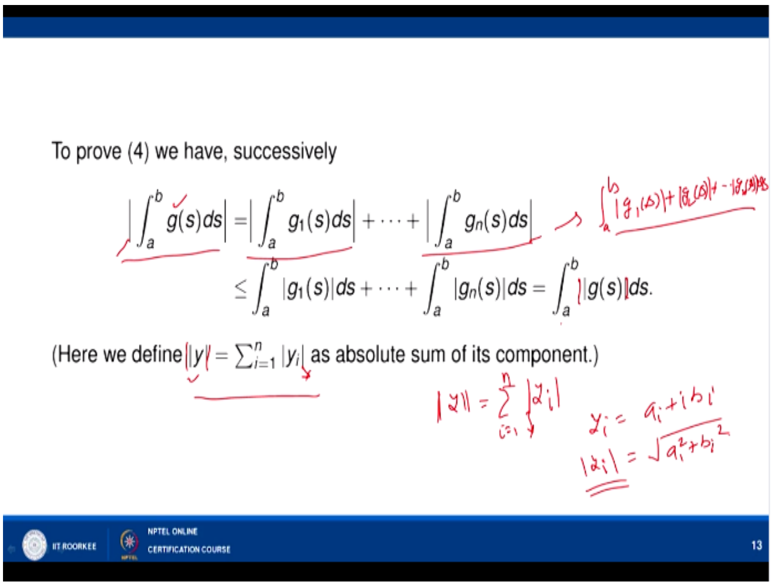
(Refer Slide Time: 15:32)



And in a similar way, we can define that if g is continuous on interval I, then we denote its integral from a to b by a to b gs ds and we can define a to b gs ds as basically it is also a vector. In the first component, it is a to b g1 s ds, a to b g2 s ds and a to b gn s ds. So it means that integral of g of s ds is basically an integral of its each component. That is how we define the integral of a vector-valued function.

And most of the time, we use this kind of important inequality that is norm of a to b gs ds is<=a to b norm of gs ds. Now here this represent the norm of a vector-valued function and here in this we will talk about this norm little bit later and we will see what kind of function is defined as a norm of the function.

(Refer Slide Time: 16:45)



Now to prove this inequality, we can simply write the norm of a to b gs ds and here norm we are taking like this that norm of y is nothing but absolute sum of its component that is norm of y is given as i=1 to n modulus of yi. Now here it is simply modulus and here this represents the norm of y and to differentiate between the modulus and norm many at times we use this notation that is like this.

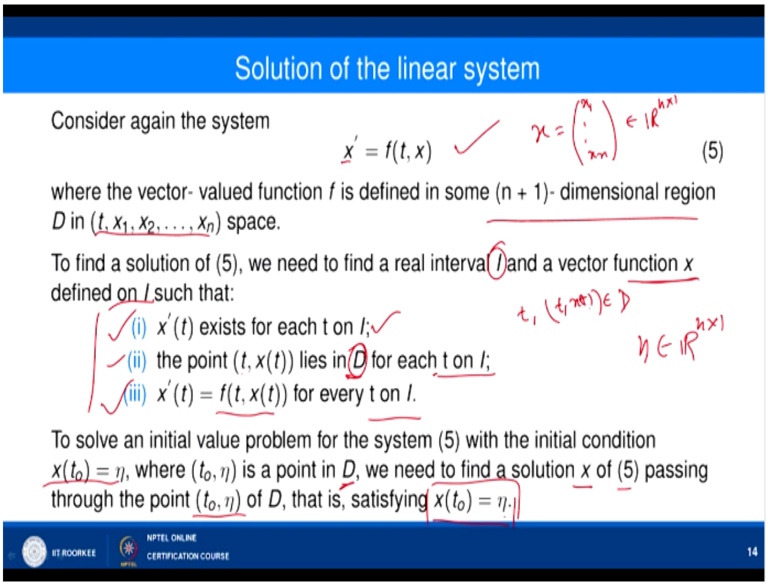
So here when we consider the vector-valued function this is denoting, this denotes the norm of vector-valued function and when we talk about the component then this represent the modulus of or absolute value of its component. So we can define norm of a to b gs ds=as we have already defined that a norm is given at the absolute sum of its component. So we can write down the modulus of a to b g1 s ds+modulus of a to b g2 s+so on modulus of a to b gn s ds.

Now this can be written as this can be bounded by<=a to b modulus of g1 s ds and similarly this can be this is bounded by a to b modulus of gn s ds. Now if we can simply write this as a to b and we can write g1 s+g2 s+and so on gn s ds and if you look at this is nothing but norm of gn. So we can write this a to b norm of gs ds. So I can write it like this. So here to prove this inequality, we are using the following definition of norm that norm of y is summation i=1 to n modulus of yi right.

Now here I am assuming that yi's are real-valued function. Now if it is complex-valued function then this modulus is supposed to be the modulus of yi t. That is here if yi are complex, then I can write this as ai+ibi then modulus of yi is nothing but the under root ai square+bi square. So here right now I am assuming that yi is a real- valued function that is why I am calling it as modulus function.

Otherwise it will give you the absolute values. So in case of complex, I can consider this modulus of yi as under root ai square+bi square. So it means that this property is very useful and will use in a later slides.

(Refer Slide Time: 19:45)



Now we define what we mean by solution of the system. So for that we consider the system x dash=f t, x where the vector-valued function f is defined in some n+1 dimensional region D in t, x1, x2, xn space and we try to define what do you mean by the solution of the system in 5. So here we define what we mean by solution of the system. For that we consider the system x dash=f t, x.

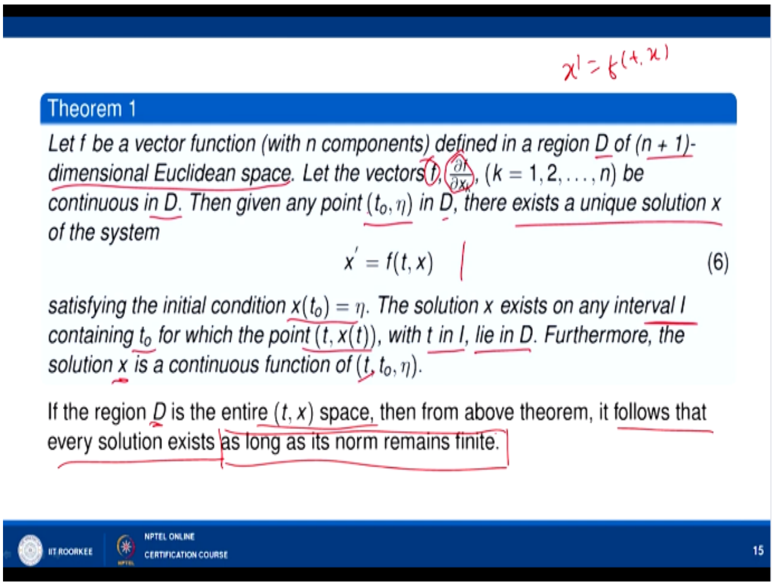
Here x is x1 to xn and it is a vector in Rn x 1 and f is a function vector-valued function defined in n+1 dimension region D in t, x1, x2, xn space. Now what you mean by solution of the system? To find out a solution of the system 5, we need to find out a real interval I. First of all, we need to find out an interval I in which this vector function x is defined and defined in a way such that x dash t exists for each t in I.

Second the point t, xt lies in domain D where we are talking all these things for each t on I. So for each t the tuple t, xt will lie in domain D. Third, that x dash t=f t, xt is satisfied. So first thing is that derivative exists for all t in that interval. Second for each t your t, xt will lie in your domain and third is that derivative exists and it is equal to the value of f t, xt for every t on this interval I.

If it is happening, we call the vector-valued function x as a solution of the system 5 and to solve an initial value problem for the system with initial condition x t0=eta. So if along with this equation number 5 if we have initial condition given as x t0=eta, here eta is also a vector in n x 1. So eta is also a vector of size n x 1 and we want to find out a solution of x dash=f t, x with the initial condition that x t0=eta where t0 and eta is a point in domain D.

We need to find out a solution x of 5 means this system passing through the point t0, eta of D that is x of t0 is=eta. So solution of the system is having these 3 properties and solution of the initial-value problem is in additional property that x of t0 is=eta, it means that it is passing through the point t0, eta.

(Refer Slide Time: 22:58)



Now for once we know the notation of solution what do you mean by solution, then our first thing to consider is the existence of solution, so as you remember that we have discussed the existence and uniqueness theorem in a scalar-valued case that is when we have x dash=f t, x where x is a scalar-valued function. Now in a similar way, we can define the existence and uniqueness theorem for vector-valued function also.

So let us consider the following theorem. Let f be a vector function with n components defined in a region D of n+1 dimensional Euclidean space and let the vector f and dou f/dou xk where k is running from 1 to n be continuous in a domain D. So first thing is that we have a vector-valued function f which is defined in a region D of n+1 dimensional Euclidean space and assume that f and its partial derivative up to nth order are continuous in that domain D.

Then, given any point t0, eta in D there exists a unique solution x of the system x dash=f t, x which satisfy the initial condition x t0=eta and the solution x exists on any interval I containing t0 for which the point t, xt with t in I lie in D and if you remember in scalar-valued function, this theorem is the local existence theorem. So it means that here the solution will define in a small neighborhood of t0, eta.

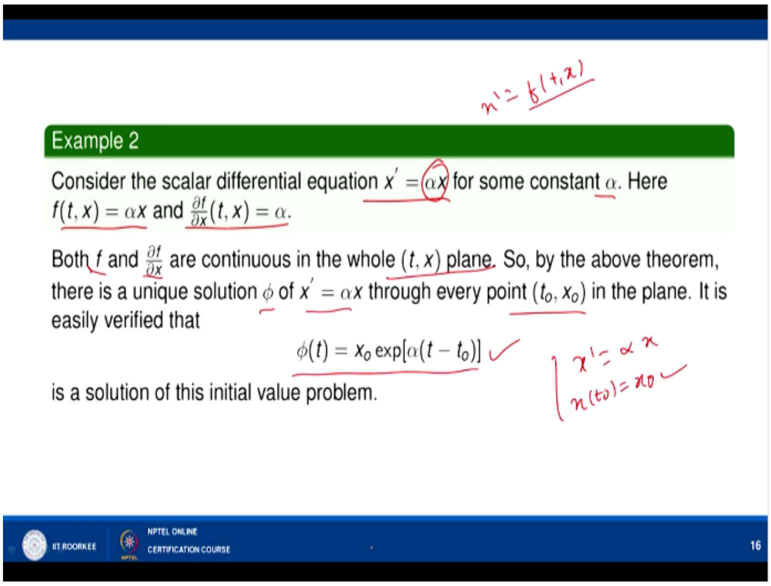
So that is why we are saying that solution will exists in an interval I containing t0 for which this point t, xt where t is in I lie in domain D. Furthermore, the solution x which we obtain is a continuous solution corresponding to t. So it means that first thing is that solution exists in a neighborhood around t0 and the obtained solution is a continuous function satisfying the equation number 6.

Now if this domain D which we are talking about is the entire t, x space then from this theorem, we can say that it follows that every solution exists as long as its norm remains finite. So it means that if this we are talking about this domain right now, now if this domain D happens to be the entire space, then for any point we can find out a domain D and our solution exists in the interval I for which the point t and xt is lying in the domain D.

Now if D is the entire space, then we can say that the solution exists using the theorem 1 as long as the norm of that solution remains finite right. So it means that this solution will exist for all t for which the solution is a bounded solution. So this gives you that if region D or domain D is the entire space, then we have to look at the region in which your solution is bounded, solution is bounded means the norm of the solution is bounded.

And we say that solution exists for all t for which your norm of xt is bounded. So let us consider example based on whatever we have discussed. So consider the example 2.

(Refer Slide Time: 26:58)

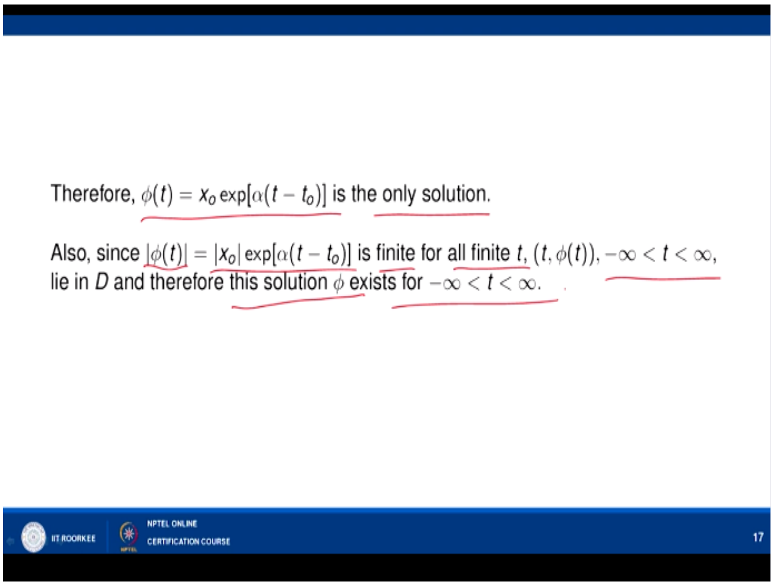


Say consider the scalar-valued function a scalar differential function x dash=alpha x for some constant alpha and here we can check that f t, x is alpha of x and dou f/dou x t, x is alpha. So it means that here x dah=f t, x and this f is continuous because it is alpha x, it is continuous for all x and the derivative of f with respect to x which is given as alpha, alpha is a constant value, so it is also continuous for all for the entire space.

So here we can say that both f and dou f/dou x are continuous in the whole t, x space. So by the above theorem, there is a unique solution phi of x dash=alpha x through every point t0, x0 in the plane. So it means that given any point t0, x0, the solution of this problem x dash=alpha x and x of t0=x0. So it means that for every initial condition, solution of this problem exists.

Now we can also verify that your solution phi t is given as x0 exponential of alpha t-t0. That is not very difficult to obtain. You can find out the solution using separation and variable method and we can say that this is one solution of the initial-value problem and since it satisfy the condition of the existence and uniqueness theorem. So we say that this is the solution of the initial-value problem.

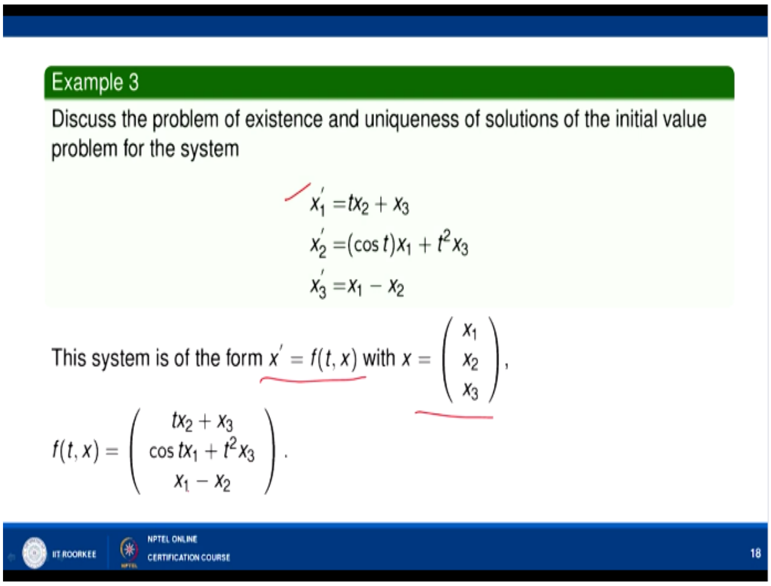
(Refer Slide Time: 28:31)



Therefore, we can say that phi t=x0 exponential of alpha t-t0 is the only solution which we are discussing. Now we say that we can find out the modulus value of phi t and you can check that it is given as modulus of x0*exponential of alpha t-t0 and if you look at this is always finite for all finite t such that t, phi t belongs to the t, x space and here t is lying between –infinity to infinity.

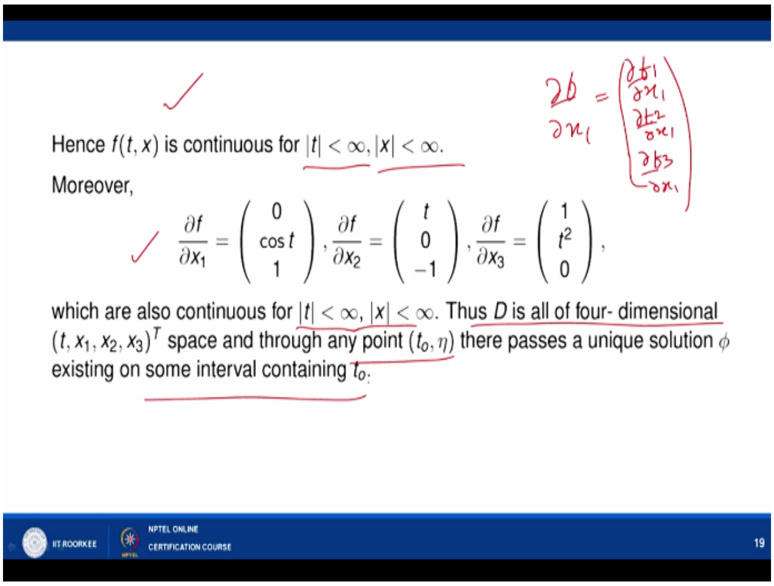
So it means that this solution phi exists for all t between –infinity to infinity. So here since this condition that f and partial derivative of f exists for all t, x, so we can say that solution will exists for all t, x for which norm of phi t is bounded and you can see that here the norm of phi t is always bounded for all finite t. So it means that solution of this problem exists for all t.

(Refer Slide Time: 29:51)



Now looking at next example and here also we want to discuss the problem of existence and uniqueness and consider the system x1 dash=tx2+x3, x2 dash=cos tx1+t square x3, x3 dash=x1-x2 and we can write down this system as x dash=f t, x with x=x1 x2 x3 and f t, x is=tx2+x3 cos tx1+t square x3 and x1-x2.

(Refer Slide Time: 30:16)



So here if you look at f t, x is continuous. If you look at here, this each component is continuous basically. So tx2+x3 is continuous, cos of tx1+t square x3 is continuous, x1-x2 is continuous. So each component is continuous, so we can say that f is continuous for all t such that modulus of t is<infinity and for all x such that modulus of x is<infinity. Now look at the partial derivative of f.

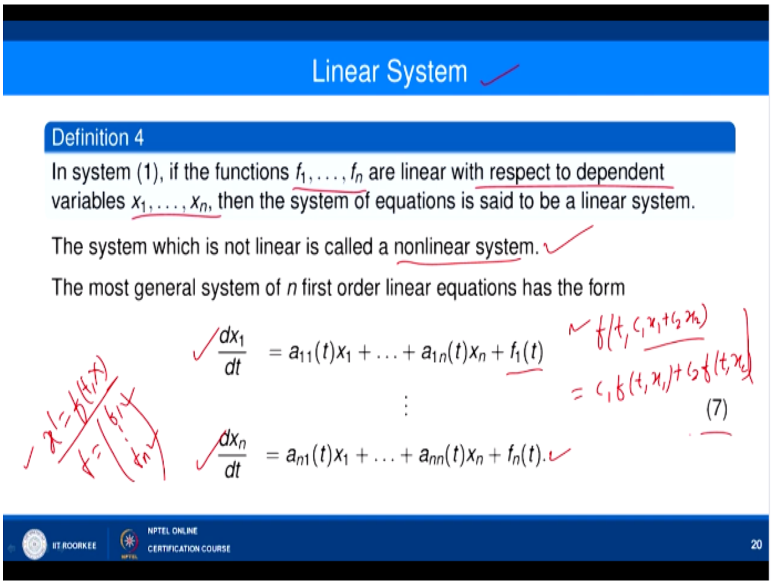
So partial derivative of f with respect to x1 is 0, cos t and 1, so that you can easily verify. You look at dou f/dou x1 means here we can write dou f1/dou x1 dou f2/dou x1 and dou f3/dou x1. So here the derivative means this, so dou f1/dou x1 is basically there is no component of x1. So you can say dou f1/dou x1 is 0 and here it is cos of t and here it is 1. So we can write down dou f/dou x1 as 0 cos t 1.

Similarly, we can find out dou f/dou x2, this is t, 0, -1 and dou f/dou x3 is=1, t square and 0. So we can say that the partial derivative corresponding to x1, x2 and x3 are also continuous for all t such that modulus of t is<infinity and for all x such that mod of x is<infinity. So it means that here that f is continuous and partial derivative is also continuous for all for the entire four-dimensional space that is t, x1, x2, x3 space and through any point t0, eta, there passes a unique solution phi existing on some interval containing t0.

So it means that here your solution will exists which passes through a given point t0, eta and containing the initial condition t0. This is regarding this example number 3. So we have discussed the existence and uniqueness theorem for some system x dash=f t, x and we have seen that solution exists provided your any interval I such that your f is continuous and partial derivative of f is continuous in that region.

And if that region happens to be the entire space, then solution will exists for all say t such that norm of that solution remains bounded and we have seen some two examples based on that. Now let us move to a particular form of f t, x such that f t, x is linear in terms of x. So now consider the concept like linear system.

(Refer Slide Time: 33:17)

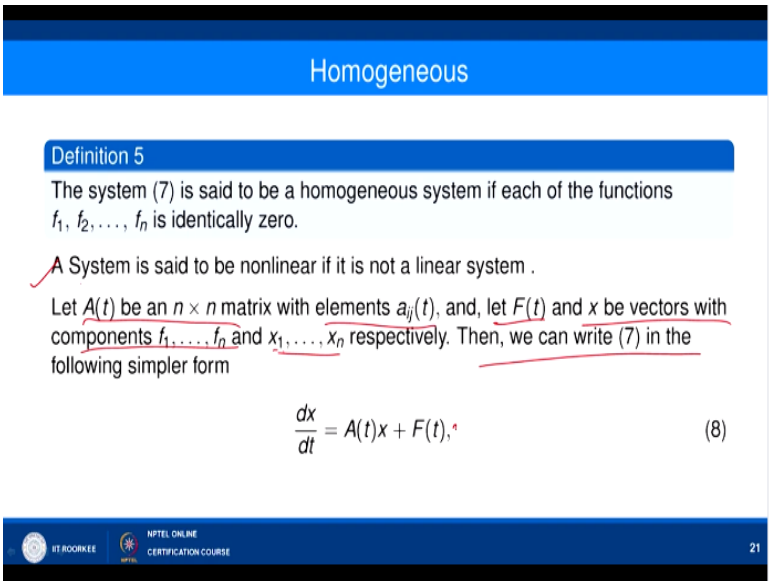


So in system 1 that is x dash=f t, x, if the function f1 to fn where f is given as f1 to say fn, if these function f1 to fn are linear with respect to dependent variable x1 to xn, then the system of equation is said to be a linear system. So it means that if we have system like this x dash=f t, x and if f is linear corresponding to x then we say this system as a linear system. What do you mean by that f is linear in terms of x?

So here you please recall the condition that f t, c1 x1+c2 x2, if we can write this as c1 f t, x1+c2 f t, x2 if it satisfy this condition, then we say that f is a linear function corresponding to the dependent variable x. So we say that if f satisfies the following condition, then we call the system of equations as a linear system of equation and the system which is not linear we call that system as a nonlinear system.

And we can check that the most general system of n first-order linear equation has the following form. That is dx1/dt=a11 t x1+a12 t x2+so on a1n t xn+f1 t and similarly the nth equation dxn/dt=an1 t x1+an2 t x2 and ann t xn+fn t. So here we can write that this is the most general system of n first-order linear differential equation and is given by equation number 7 here.

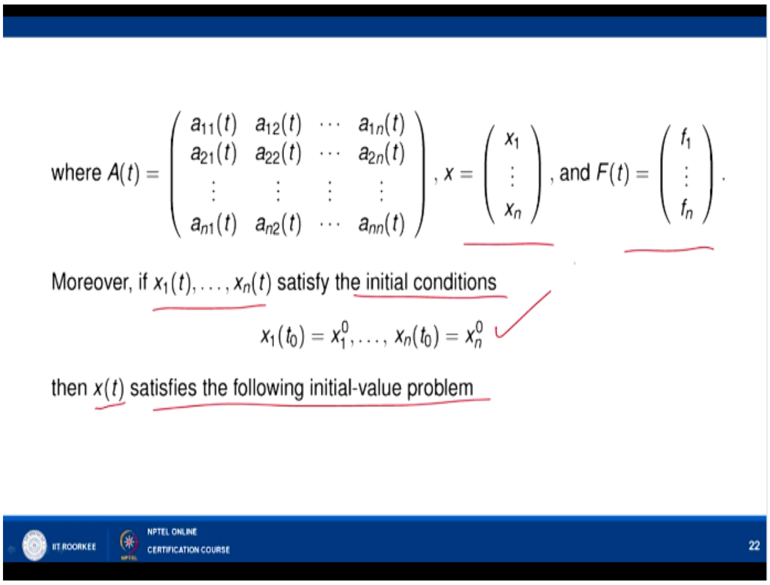
(Refer Slide Time: 35:09)



Now if in the system 7 if all these right hand side f1 t and fn t are say 0 function, then we call the system 7 as homogenous system. So it means that the functions which are not involving any dependent variable and it is a function of independent variable t such that so if these functions are 0 for all t, then we call our linear system as a homogenous linear system and a system is said to be nonhomogeneous system if it is not a homogenous system.

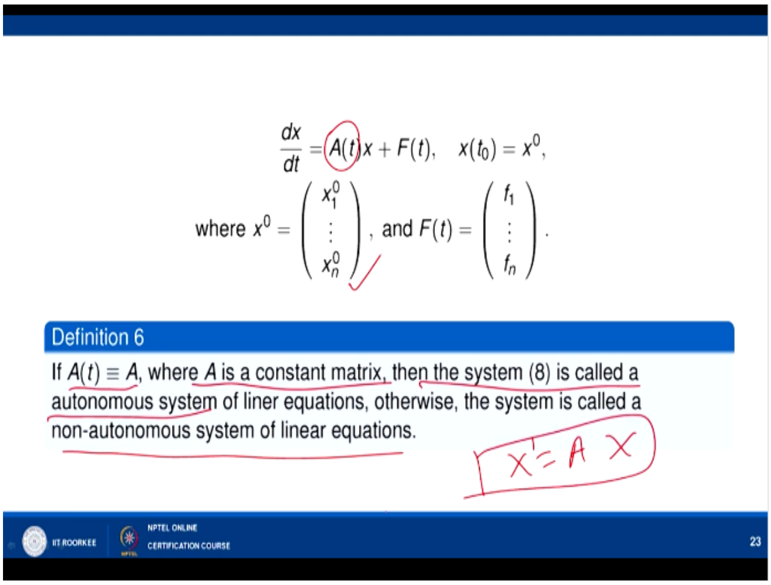
So it means that if these functions f1 to fn's are not identically 0, it means that exists one function which are say nonzero for at least few values of t. Then, we call this system 7 as a nonhomogeneous system of differential equation. Now we can find out the compact notation of equation number 7 and for that notation let us assume that At be a n x n matrix with the element aij t and let Ft and x be vectors with components f1 to fn and x1 to xn.

(Refer Slide Time: 36:48)



Then, we can write down the equation number 7 as follows dx/dt=At x+F of t where At is given as a11 t a12 t up to a1n t and so on and x is a vector defined as x1, x2, xn and Ft is a vector having component f1 to fn. Moreover, if x1 to xn t satisfy some initial conditions given at a fixed point say t0, then x1 t0=x1 0 and xn t0=xn 0, then xt satisfy the following initial-value problem.

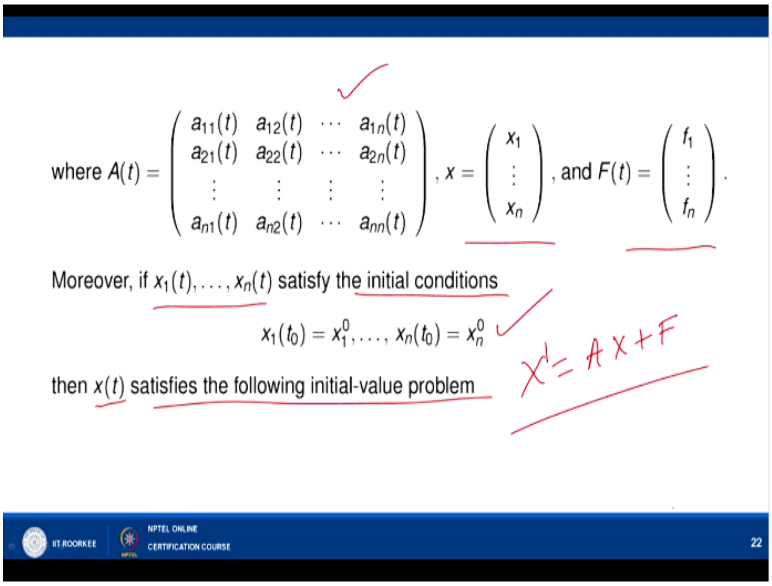
(Refer Slide Time: 37:29)



So along with the differential equation if xt satisfy the initial condition, then we can say that xt satisfy the following initial-value problem that is dx/dt=At x+F of t such that x, t0=x0 where x0 is defined as x1 0 to xn 0 and Ft is given as the vector-valued function f1 to fn. So it means that from now onwards, we use compact notation to discuss the linear system of differential equation.

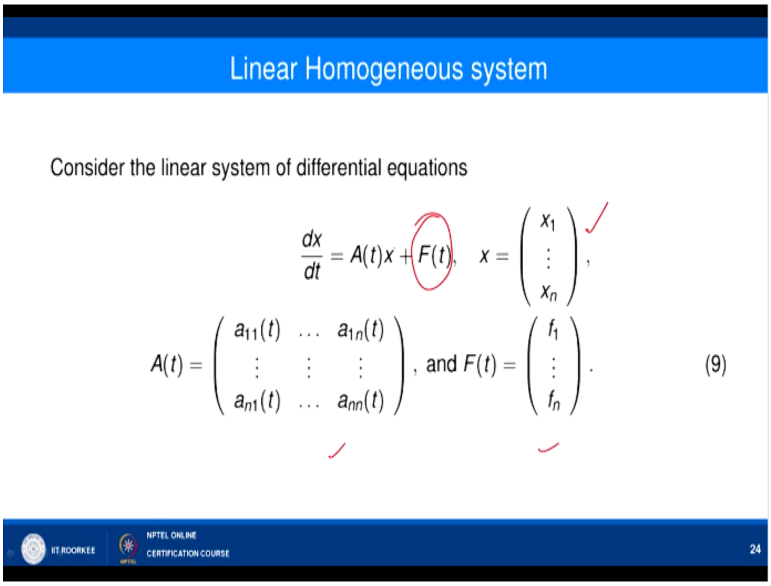
Now if the coefficient matrix that is At, A is identically=A, it means that At, A is not a function of t, it means that its components are constant values. Then, we can say that if A is a constant matrix then the system 8 is called an autonomous system of linear equation. Otherwise, the system is called a non-autonomous system of linear equation. So it means that here if we look at the coefficients here if all the coefficients are independent of t it means just a constant value, some real constant.

(Refer Slide Time: 38:41)



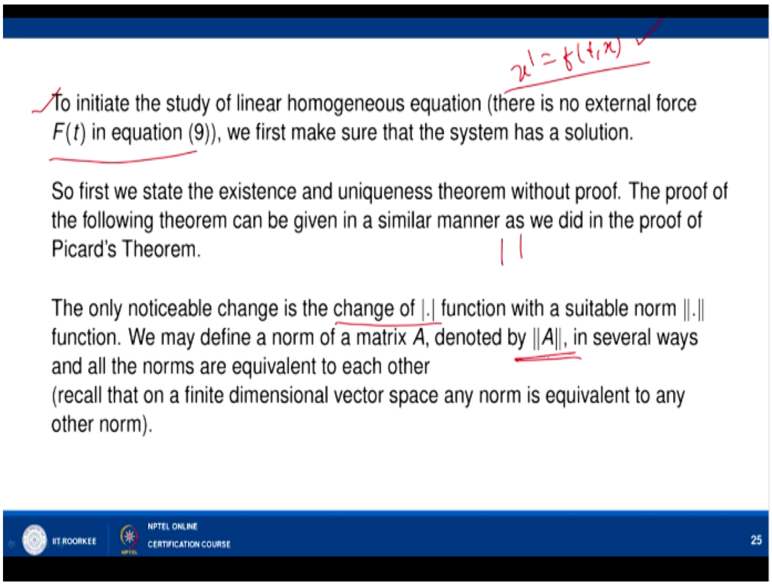
Then, your system x dash=Ax+F is known as autonomous system, otherwise it is known as non-autonomous system. So this is an example of non-autonomous system and the system x dash=Ax is said to be an autonomous system of linear equations. Now let us discuss one particular case of linear system that is linear homogenous system for which consider the linear system of differential equation dx/dt=At x+Ft where x is given as x1 to xn, At is given as aij t and Ft is given as f1 to fn.

(Refer Slide Time: 39:21)



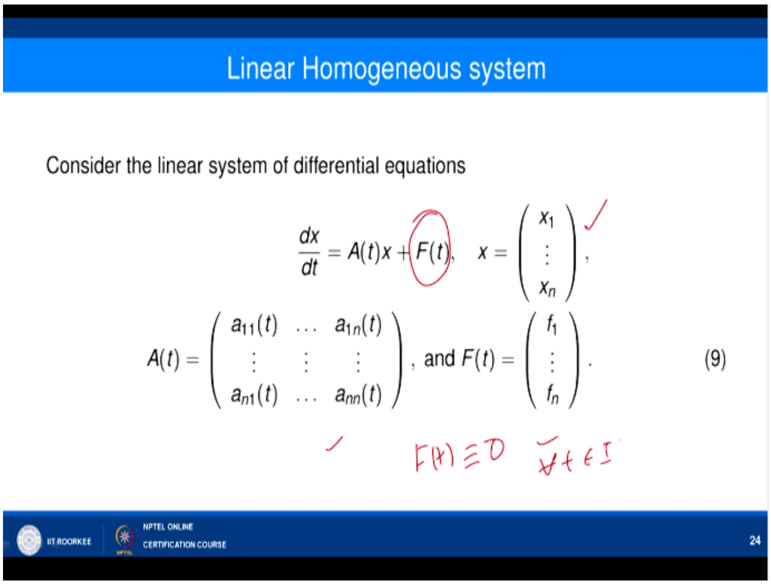
Now if you look at this Ft, if this Ft is nonzero, then nonzero vector-valued function then we call it a nonhomogeneous system and if Ft is identically=0 for all t then we call this as a linear homogenous system. So in this slide, we are assuming that Ft is identically=0 function.

(Refer Slide Time: 39:47)



So it means that to initial the study of linear homogenous equation that is there is no external force Ft in equation 9.

(Refer Slide Time: 39:57)



That is we are assuming that this Ft is identically=0 function for all t belongs to this interval I. So it means that to initiate the study of linear homogenous equation, we first make sure that the system has a solution. So first thing is that system has a solution and then we try to find out that what are the type of solution and how we can find out the properties of solutions of the homogenous equation.

So first we state the existence and uniqueness theorem without any proof and the proof of the theorem can be given in a similar manner as we did in the proof of the Picard's theorem. If you remember the scalar-valued function x dash=f t, x and under some condition on f that is if f satisfy the ellipses condition or we can say that partial derivative f are say continuous. Then, we can find out that in this system this equation has a solution and that solution is unique in some neighborhood of initial point that is t0.

So the only change in the case of vector-valued system of linear equation and scalar-valued system of linear equation is the say notation of modulus function. So in the scalar-valued function, this modulus represent the absolute difference or absolute difference between the points but now here this norm this modulus is represented by norm of the corresponding norm.

So here we can restate this theorem, existence and uniqueness theorem, the only change we are going to consider is the change in terms of distance function that is modulus function. So here in case of modulus, we are going to use the norm function. So the only noticeable change is the change of modulus function with the suitable norm function and we may define a norm of a matrix A denoted by norm of A and it can be defined in several ways and all the norms are equivalent to each other.

And there is a property of a finite dimensional vector space that whatever norm you are going to define, they are going to be equivalent norm, equivalent means it is not the same norm but the qualitative properties is same whatever norm you will choose okay. So this we are going to discuss in next lecture. So for this lecture we will stop and will discuss in next, will continue our study in next lecture. Thank you very much for listening us. Thank you.