Multivariable Calculus
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Lecture - 03
Limits for multivariable functions-II

Hello friends. Welcome to lecture series on multivariable calculus. In the last lecture, we have seen that; what do you mean by limits for several variable functions we have seen that if we write limit x y tending to x naught y naught f x y is equals to L.

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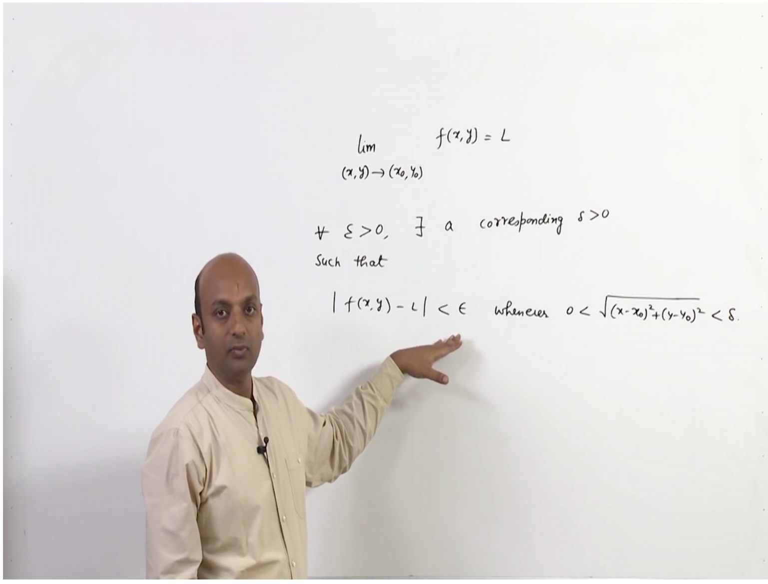

This means if this limit exist and is equal to L and it means for every epsilon greater than 0, there exist a corresponding delta greater than 0 such that such that mod of f x y minus L is less than epsilon whenever 0 less than under root x minus x naught whole square plus y minus y naught whole square is less than delta.

So, we have seen that whatever epsilon may be no matter how small how large it may be there will always exist a corresponding delta greater than 0 such that this inequality hold; that means, for every epsilon for every epsilon there will exist a delta such that a disk centered at x naught y naught of radius delta, all those x y lying in that disk will always the image of all those x lying in that disk will be contained in L minus epsilon 2 L plus epsilon and at the geometric interpretation of this definition.

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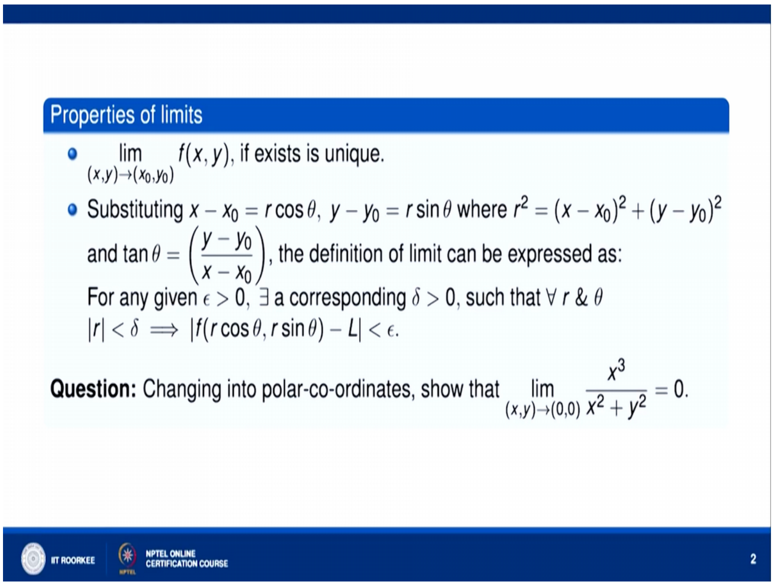

Now, let us discuss some important properties of limits the first property is limit x y tan to x naught y naught f x y if exist is always unique ok. Next is to find out the value of the limit the another method is convert Cartesian coordinate into polar coordinate system; that means, if we substitute x minus x naught equal to r cos theta y minus y naught equal to r sin theta where r square is equal to x minus x naught whole square plus y minus y naught whole square and tan theta is equal to y minus y naught upon x minus x naught that can easily be obtained. If you divide the second expression y minus y naught is equal to r cos theta and x minus x cos tan theta, then we obtain tan theta equal is equal to y minus y naught upon x minus x naught the definition of the limit can be expressed in this way.

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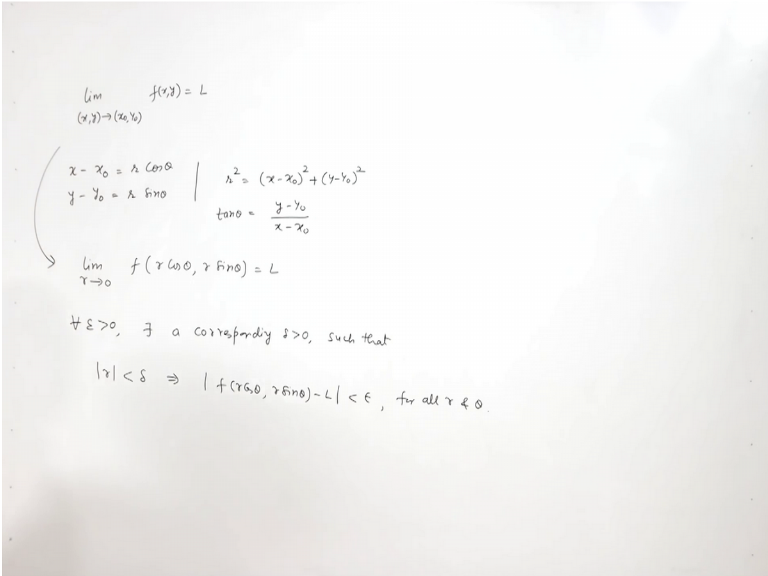

So, basically if we are having limit x y tending to x naught y naught which we have just explained you f x y equal to L.

Then to find out this limit the another method is converges Cartesian coordinate into polar coordinate system. So, how can we do that you simply take x minus x naught as r cos theta y minus y naught as r sin theta where if you square an add, it is simply r square is equal to x minus x naught whole square plus y minus y naught whole square and tan theta is equal to y minus y naught upon x minus x naught. Now as r tan into 0 whatever theta may be x will tend to x naught and y will tend to y naught that is x y tan x y will tend to x naught y naught.

So; that means, these are there are 2 ways either you convert x by tan to 0 x naught y naught to x y tan to 0 and then you can convert this into polar coordinate system another way out is you simply take x minus x naught as r cos theta and y minus y naught is r sin theta,. So, now, now this limit will convert to this will convert to limit r tending to 0 because as r tend to 0 will tend to x naught y will tend to y naught that is x y will tend to x naught y naught this will be f of r cos theta r sin theta and the limit will be same L.

So, how we can define this in delta epsilon now; so, to show the existence of this limit again, we will use a concept of delta epsilon that is for every epsilon greater than 0 there will exist a corresponding delta greater than 0 such that it is mod r less than delta implies mod of r cos theta r sin theta minus L less than epsilon for all theta for all r and theta, this must hold for all are for all r and theta.

So, ah so any Cartesian coordinate; if you have a limit to find out in any Cartesian; so, either you can either you can proceed in the Cartesian way only or you can convert this into polar coordinate system to find out the limit.

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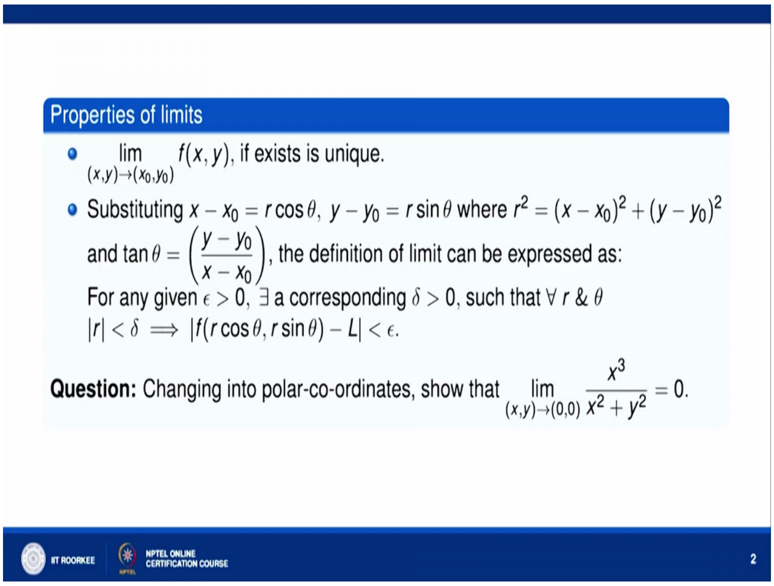

Now, for example, we have this problem changing into polar coordinate system show that limit of this is equal to 0. Now, let us try this problem. So, we were discussing about that how we can show existence of a limit by delta epsilon definition. So, let us call as at this example that is limit x y tending to 0 0 x cube upon x square plus y square and a limit is 0. Now, let us try to prove this that this limit is 0.

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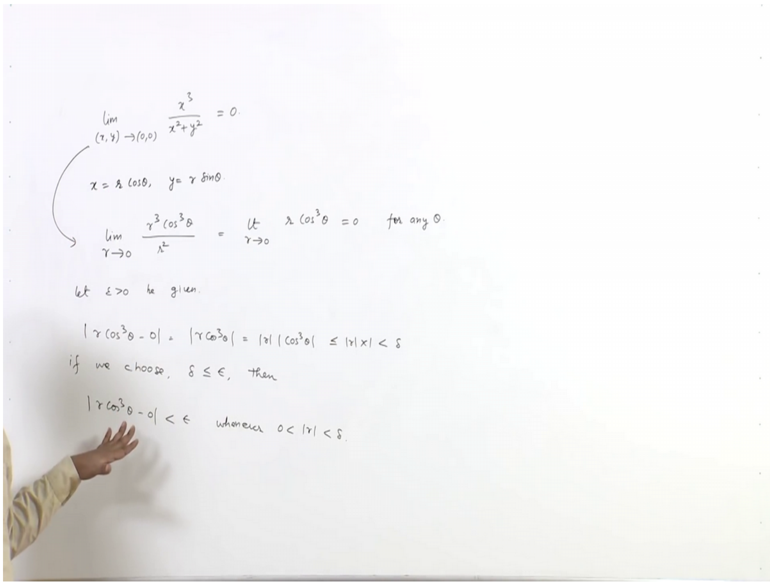

So, we have 2 ways to show this the first way is convert x and y into polar coordinate system and the second way is you can proceed by the usual Cartesian method. So, let us first try to prove it by like converting this into polar coordinate system. So, we will suppose that x equal to r cos theta and y is equal to r sin theta. Now as x y both are tending to 0 0. So, it will be possible only when r will tan to 0 ok. So, this limit we will convert into limit r tending to 0 x is r cos theta.

So, it is r cube cos cube theta. Now r square cos square plus r square sin square is r square because cos square plus sin square theta is 1. So, this is equals to limit r tending to 0 it is r cos cube theta and this is clearly 0 for any theta, you can if you take any theta the if limit r is tending to 0. This will always tend to 0. Now to show this that this is equal to 0, we will again use delta epsilon definition.

So, let epsilon get tan 0 be given. So, it is mod r cos cube theta minus 0 which is equals to mod r cos cube theta which is equals to mod r into mod cos cube theta which is less than equals to mod r into 1 because mod cos theta is always less than equal to 1. So, and this is less than delta. So, if we choose if we choose delta less than equals to epsilon, then mod of r cos cube theta minus 0 will be less than epsilon whenever 0 less than mod r less than delta. So, hence we have shown the existence of such delta for which this inequality hold hence we can say that this limit exist that is equal to 0.

Now, the same can also be proved by the by the usual like delta epsilon definition without converting this into polar coordinate if we want to prove this result without using polar coordinate then also we can do that without using polar coordinate also we can prove this limit. Let delta let epsilon 0 be given.

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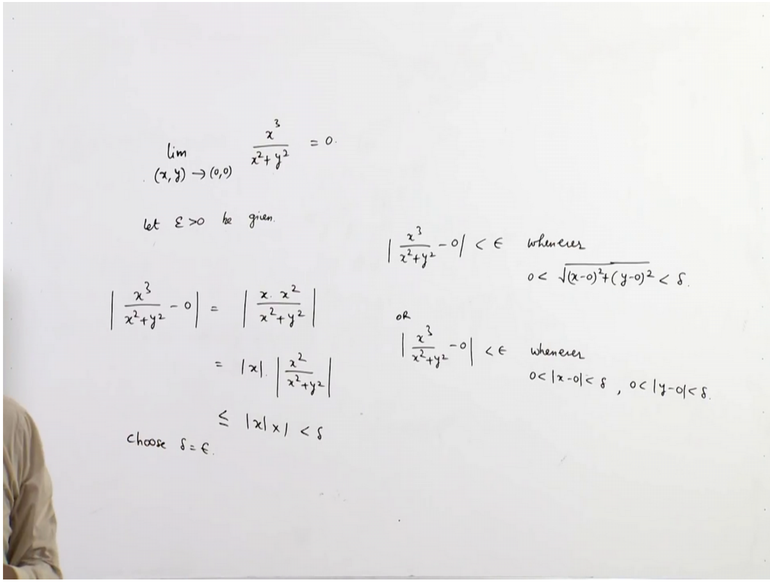

We have to show that mod of x cube upon x square plus y square minus 0 is less than epsilon whenever 0 less than under root x minus 0 whole square plus y minus 0 whole square is less than delta this, we have to prove, we have to prove that distance of such delta.

Now, we take this inequality mod x cube upon x square plus y square minus 0. This is equals to mod of x into x square upon x square plus y square. This is further equals to mod of x into mod of x square upon x square plus y square. Now x square is always less than equals to x square plus y square. So, x square upon x square plus y square is always less than equal to one and it is also non negative quantity.

So, we can say that it is less than equals to mod x into one and this mod x now you can use the other definition of, you can use the other definition of ah limit or mod x cube upon x square plus y square or if you want to use same you can use same also.

So, we can use this definition here. So, if you take this is less than delta. So, choose delta equal to epsilon then we have done. So, we can prove this existence of this limit without using polar also, but if we use polar coordinate system ah, then we can get the result easily other properties of limit is to path test for the non existence of a limit if from 2 different paths as x y; x y approaches to x naught y naught, the function f x y has different limits, then this implies limit does not exist.

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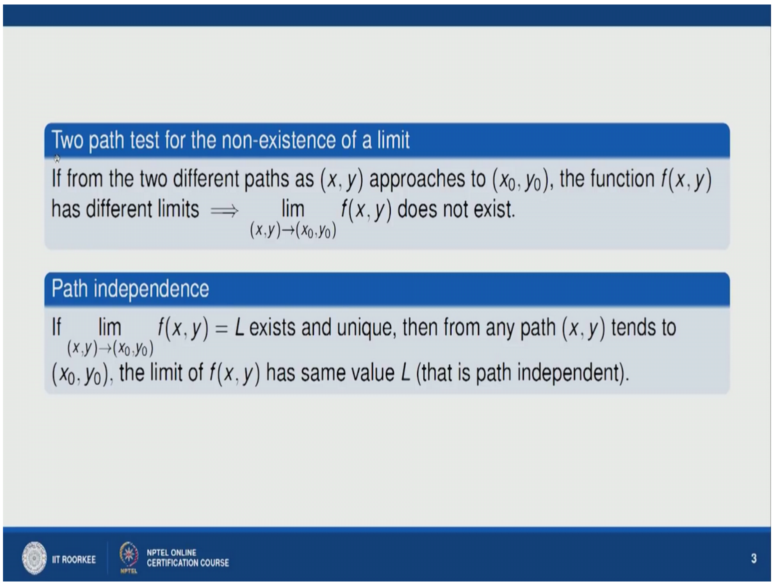

So, what does it mean; Let us see.

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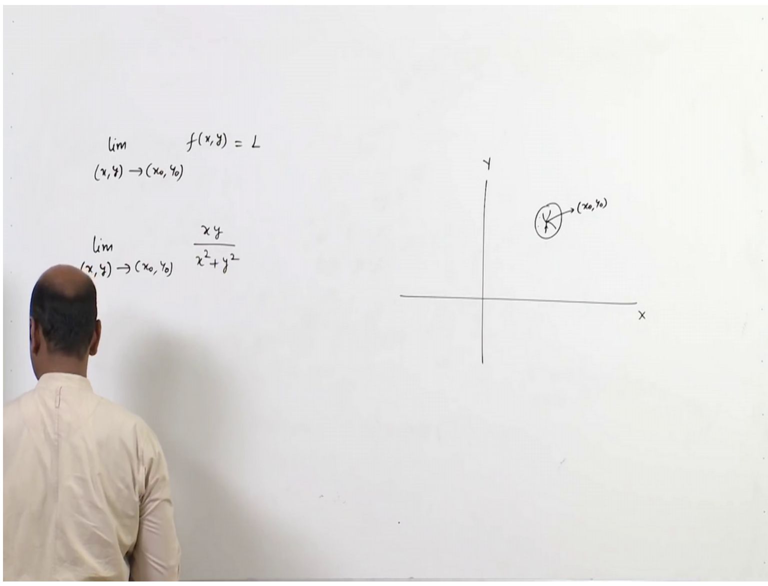

Now they are having x axis, y axis. We have a point x naught y naught ok. This point is basically x naught y naught. We take a neighbourhood of this point this point x naught y naught. Now, if you take any neighbourhood of x naught y naught. All those x y lying in this region, there are infinite paths by which this x y can approach to x naught y naught, it may be a straight line, it may be a parabolic curve it may be some other curve ok, there will be infinite paths.

Now, existence of limit means, if we follow any path from x y to x naught y naught, it must be path independent path. Independent means whatever path we follow from x y 2 x naught y naught, the value of the limit will always be unique if the if the value of the limit, if you if you are saying that limit x y tending to x naught y naught f x y is L.

This means this means; if you take a neighbourhood of x naught y naught and we are taking any x in this neighbourhood, we are infinite paths from by which x y can approach to x naught y naught it must be path independent; that means, whatever path we follow from x y to x naught y naught, the value of this limit is always; L will always remain the same because it is because limit is unique limit is always unique if it exist; that means, if from 2 different paths value of the limit are not same value a double limit we are calling a double limit, if the value of double limit are not same this means limit does not exist because if limit exist this means it must be path independent.

So, to illustrate this, let us discuss few examples the first example is limit x y tending to x naught y naught x y upon x square plus y square now 0 0 it is 0 0.

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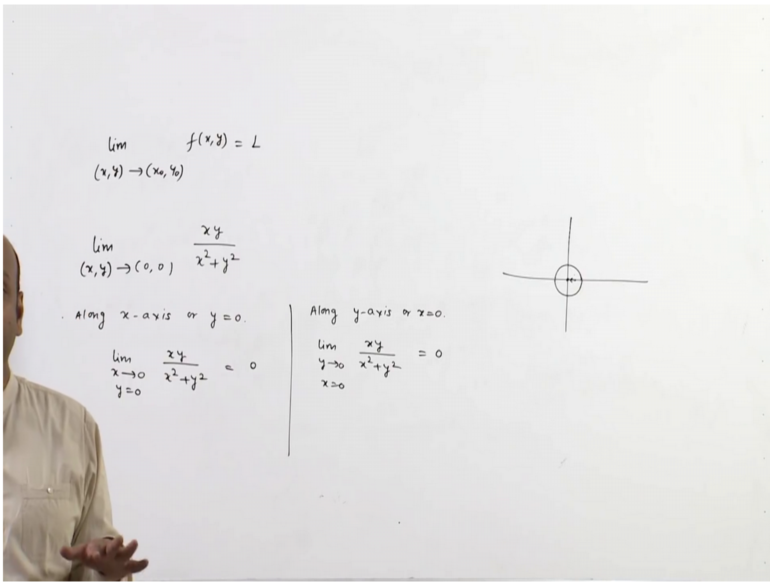

Now from x y to 0 0, they are infinite paths we can follow any path, suppose and this is origin and this is any x y this is any x y. So, we can we can move along x axis, we can move along y axis, we can move along y equal to x, we can move along y equal to 2 x, we can move along y equal to x square the infinite paths.

So, let us move along, let us move along x axis or y equal to 0. Now if we move along y equal to 0, if you move along y equal to 0 from this point to this point from this path, we are following if you follow this path, then what is the limit of this expression, there will limit x into 0 y equal to 0 and x y upon x square plus y square and it is when you substitute x equal to 0 where I substitute y equal to 0.

So, the value is 0. Now, now let us move along say y axis or x equal to 0. Now if you move along x equal to 0 it is limit y tending to 0 x y upon x square plus y square and x equal to 0 when you substitute x equal to 0 here. This is 0. Now from these 2 paths value are same; what does it mean from if from 2 paths 2 different paths value of the double limit are same it means the double limit that is this limit may or may not exist because these are only 2 paths and they are infinite paths from x y to 0 0 the infinite paths.

And if from to paths value are same it does not mean that the value exist value double limits exist and is equal to 0 because there may be some other path from which the value of this limit may be different for example, if you take say y equal to x if you take if you move along y equal to x along y equal to x.

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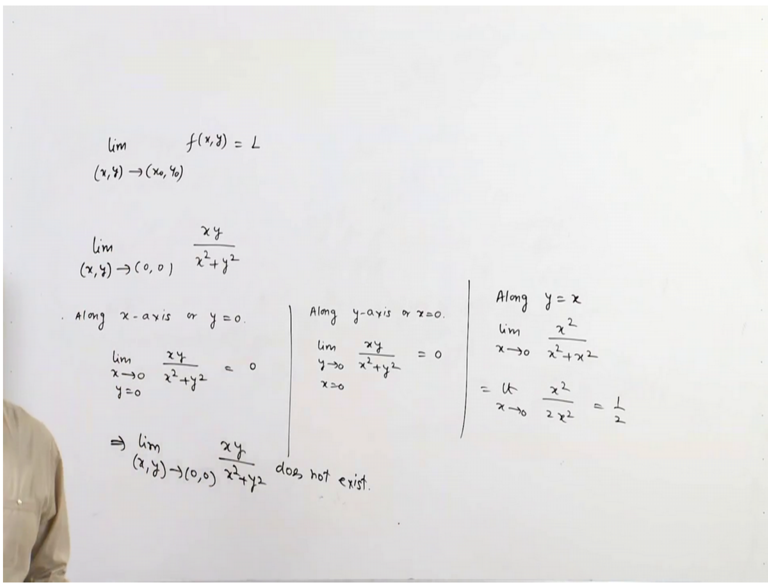

Then limit x into 0 you substitute y equal to x it is x square upon x square plus x square which is limit x into 0 x square upon 2 x square which is 1 by 2. Now from this path from this path value is 0 from this path value is 0 and from some other path value is 1 by 2.

So, values are not same values are different this means this limit does not exist. So, we can simply say this implies limit x y tending to 0 0 x y upon x square plus y square. It does not exist, why does not exist because from 2 different paths values are different now the other way out to show that limit does not exist is other way out is you take you take a path general path you take a general path you move along say y equals to m x.

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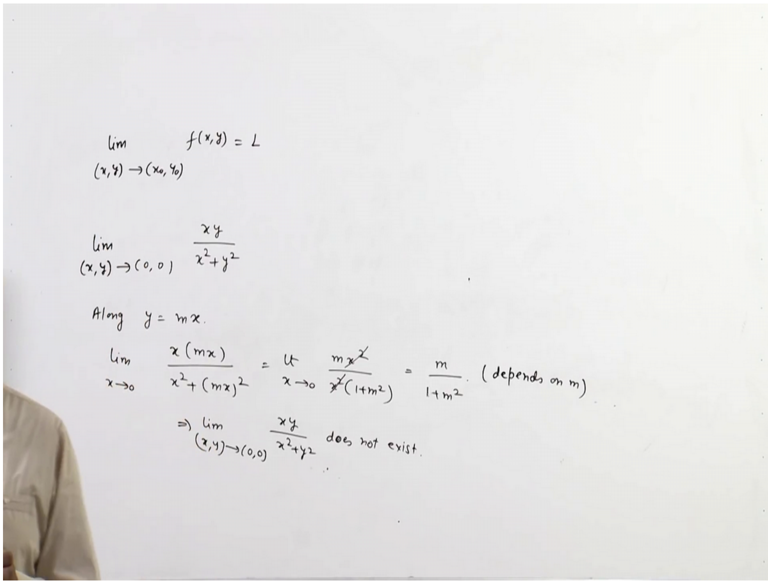

If you move along y equal to m x, this means it is limit, you substitute y equal to m x it is x into m x upon x square plus m x whole square and x is tending to 0.

Remember this that this path must path through x naught y naught must path here x naught y naught is 0 0. So, this path must path through as 0 0 whatever path we are choosing it must must path through this point now this is equals to limit x into 0, it is m x square upon x square times one plus m square x square cancel out and it is m upon one plus m square.

Now this value the limit this value comes out to be dependent on m you take different values of m say you take m equal to 1. This value is 1 by 2 you take m equal to 2, then this value is 2 upon 5.

So, for different values of m the value of the limit are different this means limit does not exist because now it is path dependent we take different paths values are different it is path dependent; however, it must if limit exist it must be path independent.

So, it depends on m this implies limit x y tend to 0 0 x y upon x square plus y square, it does not exist. So, basically 2 showed to show that limit does not exist the double limit does not exist. We have 2 ways the first way is you take 2 different paths and try to show that from 2 different paths value of the limit are different. The other way out is you try to show that it is path dependent you take some arbitrary path like y equal to m x or y equal to k x square or something and try to show that it is it depends on m or k in this way we can show that limit does not exist.

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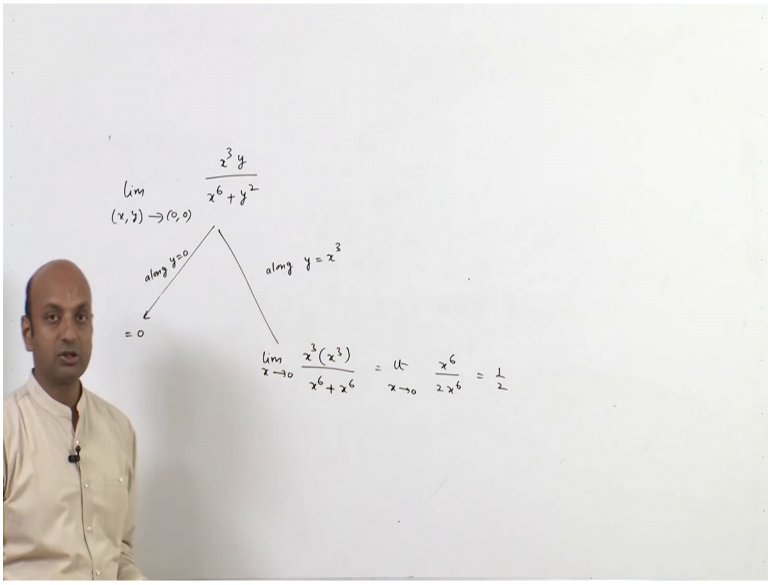

Say we have second example it is limit x y tending to 0 0 the problem is x cube y upon. Now if we move along, if we move along say y equal to 0, if you move along y equal to 0, then this value when you substitute y equal to 0 then this is clearly 0 because it when you substitute y equal to 0 here this is 0. Now you move along say y equal to x cube around this curve.

If you move along y equal to x cube, then this is nothing, but limit x tending to 0 x cube into x cube upon x to power 6 plus it is x to the power 6 which is equal to limit x tending to 0 x to the power 6 upon which is 1 by 2. So, from one path value is 0 and from other path value is 1 by 2 this means this limit does not exist.

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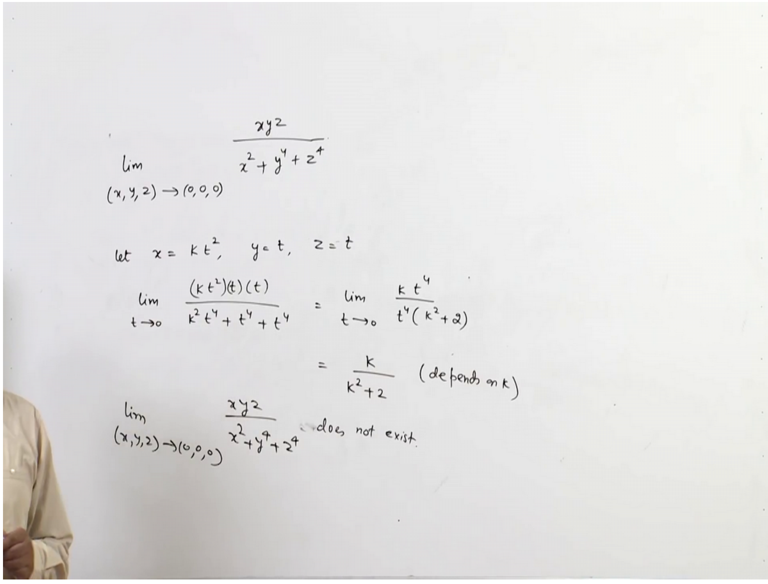

Now the next example next example is limit x y z tending to 0 0 0, it is x y z x square plus it is right to the power 4 plus z to the power 4. Now, we have to find a path such that it comes out of a path dependent to show that this limit does not exist. So, we can choose some path say, we can take, let x is equals to some k t square say y equal to y equal to say t and z equal to t where t is some parameter basically in 3 d, we are taking this curve.

Now, is substitute this it is limit x is k t square y is t and z is t and it is k square p raised to power 4 plus t raised to power 4 plus t raised to power 4 and limit t tends to 0 because as x y z all tend to 0 this will happen only when t with t tending to 0 and this is equals to limit t tending to 0. It is k into t raised to power 4 upon k plus k a square plus 2. So, this will be equal to k of k upon k square plus 2 that is depends on k it depends on k this means, this means this limit does not exist.

So, in this way we can show that double limit does not exist now if you take say if you take 4 or 5 paths and the value of the limit always come out to the same, then again this does not guarantee that the limit exist because there may be some other path by which the value of the limit comes out to be different, if we have to show the existence of a limit we have only option is delta epsilon definition, we have to show the existence of a limit using delta epsilon definition only to show that the limit does not exist, we can use this concept we can we can use 2 different path and try to show the value of limit comes out to be different or where we can try to show that it is path dependent.

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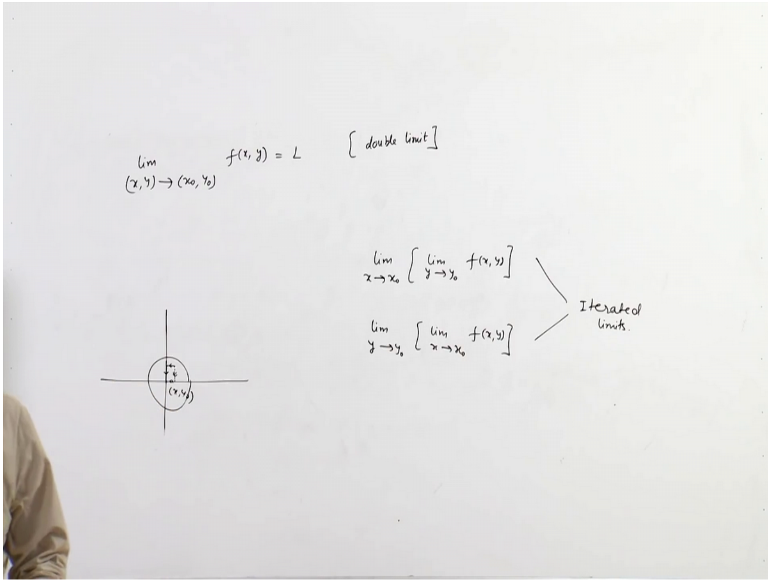

Now, we will talk about iterated limits and double limit now what does it mean you see that double limit is this thing. This is x y tending to x naught y naught f x y, suppose it exist and equal to L and we have it is it is called double limit also called the double limit and other things are iterated limit iterated limit means limit x tend to x naught limit y tend to y naught f x y or limit y tend to y naught limit x to x naught f x y, these are called iterated limits.

Now, now if you take now if you take x naught y naught here and you take a neighbourhood of this point centroid x naught y naught, you take any x y in this disk this means you first you first stage y tend to y naught keeping x constant and then you take x x tan to x naught. So, first you are taking y tend to y naught this is this is this is x naught y naught first you are taking y tending to y naught means this thing y tend into y naught, this is some point x y y tend to y naught now this now this point is this point is x naught y naught now here first x tend to x naught and then y tan to y naught. So, we come to this point.

So, this is y tend to y naught and then x tend to x naught. So, we come to this point. So, t these are these are basically 2 different paths, one path is this another path is this. Now if this limit exist and is equal to L, then the iterated limit, then the iterated limit value the iterated limit is also equal to L provided limit y tending to y naught f x y and limit actually as x naught f x y.

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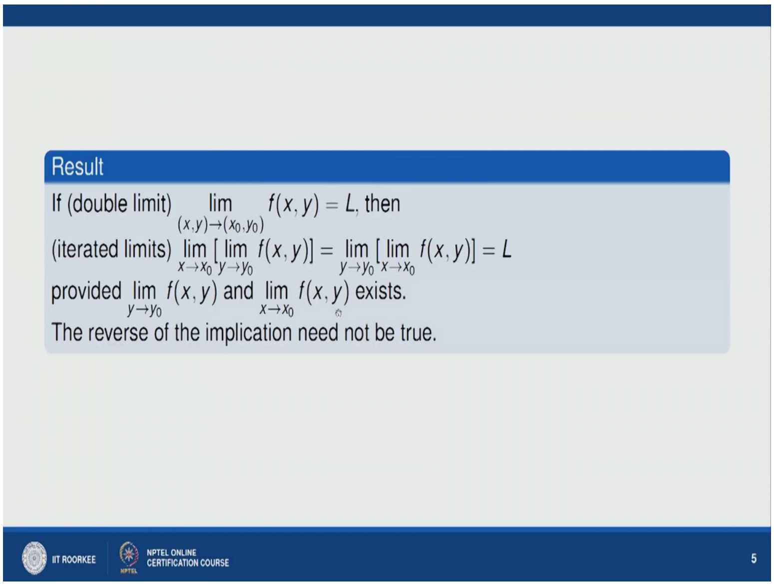

Exist if this condition hold, then only we can say that if double limit exist, then iterated limit also exist equal to l.

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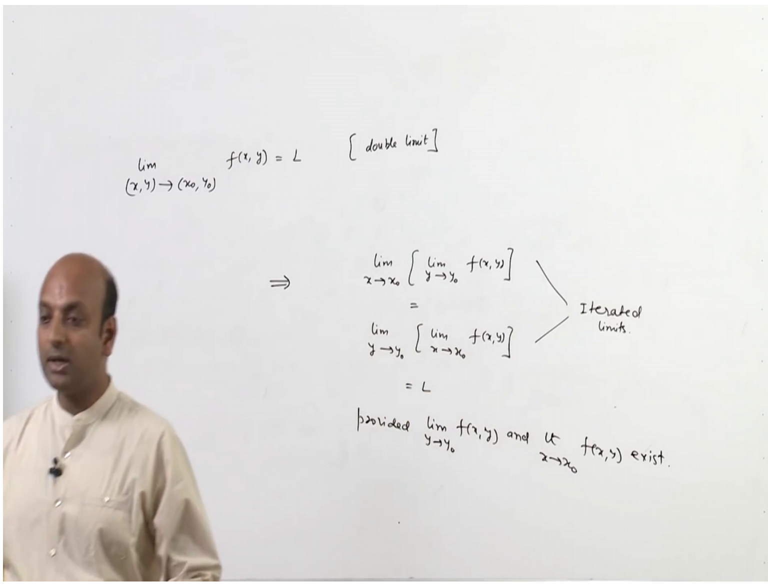

So, basically if this is equal to L, then this implies,, then this condition implies if this is equal to L that is then this condition implies that these are also equal and is equal to L provided.

Provided this inside limit exist because if this limit exist then these are basically 2 paths if this limit exist and these are basically 2 paths and if this is equal to L this means it is path independent if it is path independent then from these 2 paths also the value will be same value will be L now if we see the converse path if this is if this exist and suppose this and this are equal to L, then these are 2 only 2 paths ok, if this and if this and this limit exist then these are only 2 paths and from these 2 paths if limit comes out to be L then this double limit may or may not exist because basically these iterated limit if this limit exist are only 2 paths.

So, let us understand this by giving some examples you see suppose you take this it is x plus y upon x minus y.

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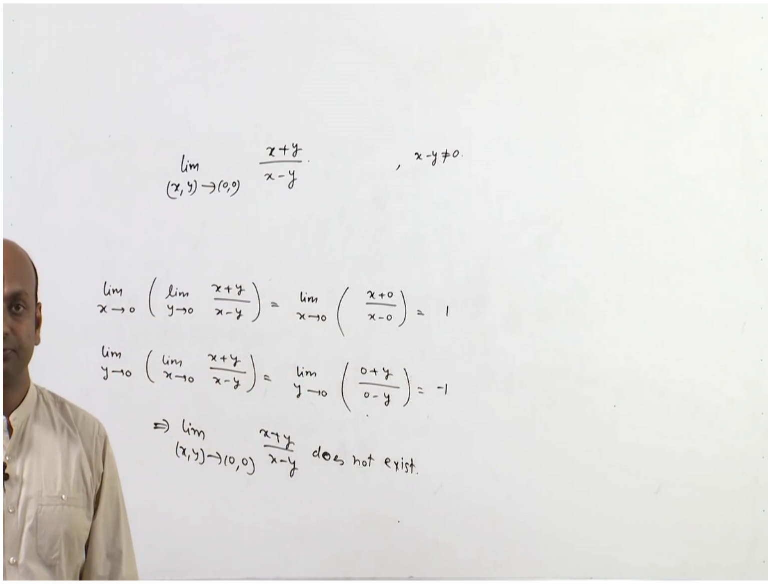

Suppose you want to compute, this limit x minus y should not equal to see. Now if you find this limit x tend to 0, limit y tending to 0 x plus y upon x minus y if you find this limit, this is this iterated limit, then this is limit x tend to 0 you simply substitute you simply tan by tend to 0 then it is x plus 0 upon x minus 0 and when you take x into 0 then this is one.

Now, you take the other iterated limit it is limit y tend to 0 it is 0 plus y upon 0 minus y and when you take y tend to 0 it is minus one. So, iterated limit exist and are not equal you see you see that this, this limit and this limit exist this limit is 1 and this limit is minus 1, this and this limit exist and iterated limit are not same this means this implies limit x y tend to 0 0 x plus y upon x minus y, this does not exist because if because if this inside limit exist, then this iterated limiter simply 2 paths and from the 2 different paths values are different this means this limit double limit does not exist now see another example it is limit x y tending to 0 0.

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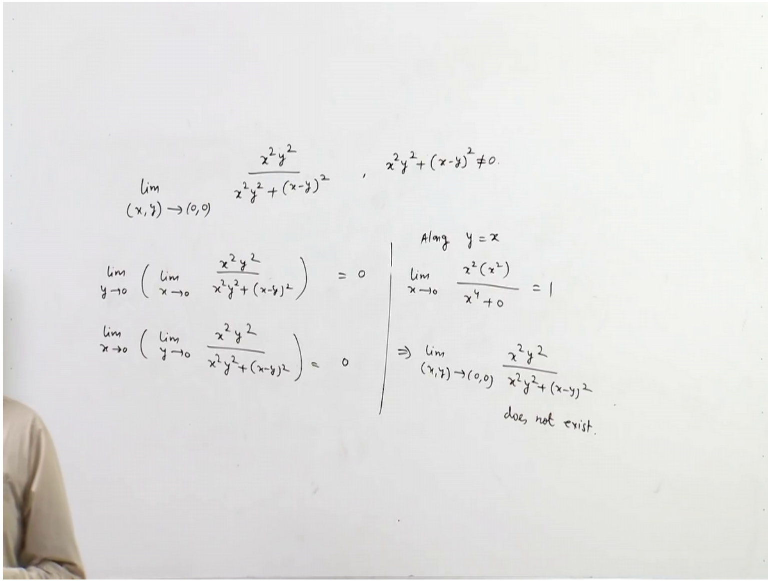

It is x square y square upon it is again x square y square plus x minus y whole to square ok, the problem is find a double limit and the iterated limit if they exist the provided denominator is not equal to 0.

Now, first you find the double limits ok. So, you take limit y tending to 0 limit x tend to 0 f x y which is x square y square upon x square y square plus x minus y whole square. Now when you put x when you take x x tend to 0, here in this expression. So, this will tend to 0, then this is simply equal to 0 ok, one can easily see that when you take x tend to 0 in this expression. So, numerator is 0. So, the value is 0 now the other iterated limit is limit x tend to 0 suppose and limit y tend to 0 x square y square upon x square y square plus x minus y whole square.

Now, when you take y tend to 0 again numerator is zero. So, this value is again 0 now these this inside limits exist and the iterated limits are same what does it mean ? What can we say about double limit from here, we can say that double limit may or may not exist because these are only 2 paths it may possible from some other path value are double limit comes out to be different from this limit from this value say if you take a path say you take a path along say take a path y equal to x, if you take a path y equal to x a here. So, we will obtain limit x tend to 0 x square x square upon x is to power 4 plus 0 which is one from this path they are getting value one and from other paths they are getting a value 0.

This means limit does not exist because there are 2 different paths from which value of the limits are different though this means a path dependent then this implies this limit does not exist. So, hence we can easily show get whether a limit exist or it does not exist to show the existence we have to go only through delta epsilon definition to show that the limit does not exist we have to we have to show that from 2 different paths values of the limit are different so.

Thank you very much.